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FIXED POINT THEOREM AND PERIODICITY THEOREM FOR DYNAMICAL SYSTEMS OF ITERATED DISCRETE LAPLACIANS ON THE PLANE LATTICE

(離散ラプラス作用素の反復力学系 (不動点定理と周期性定理))

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We study dynamical systems generated by discrete Laplacians on the plane lattice and prove a fixed point theorem for even neighborhoods and a periodicity theorem for odd neighborhoods.

1. ITERATION DYNAMICAL SYSTEM OF DISCRETE LAPLACIANS

We consider the plane lattice which is generated by two families of lines which are orthogonal to each other. The naturally defined squares are called cells of the lattice. A set of cells which are attached to the reference cell p defines a neighborhood U_p. The neighborhood is named even (or odd) if the number of the cells is even (or odd). We give several examples, some of them are well known.

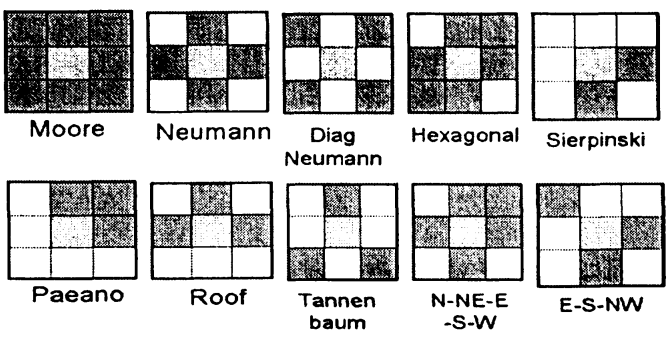
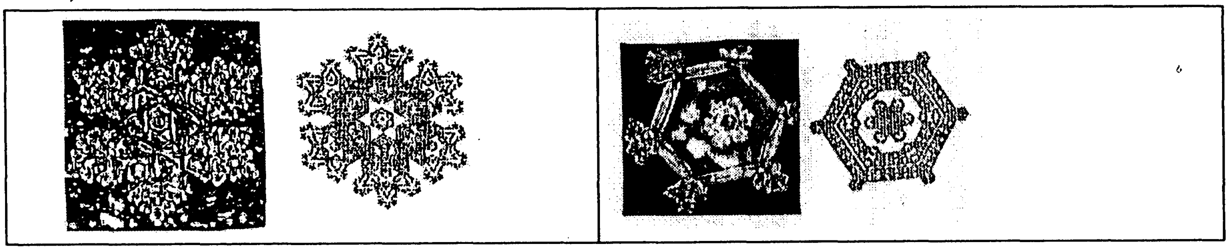


Figure 1

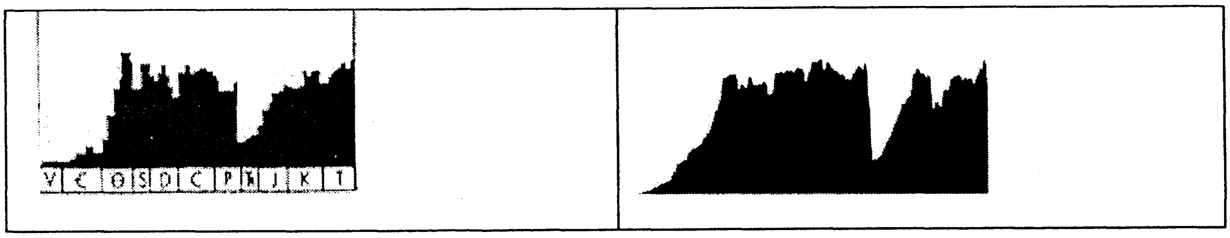
We take the set $\mathbf{F}$ of $\{0,1\}$-valued functions defined on the plane lattice U and introduce a discrete Laplacian by $\Delta_{U_p} f(p) = \sum_{q\in U_p}(f(q)-f(p)),\ \ \mathrm{mod}\,2$. For any initial function $f_0 \in \mathbf{F}$, we consider the dynamical system $\{f_n\}$, $f_n(p) = (\Delta_{U_p} f_{n-1})(p), \forall p \in U$, $(n = 1, 2, \ldots)$.

2. COMPUTER SIMULATION

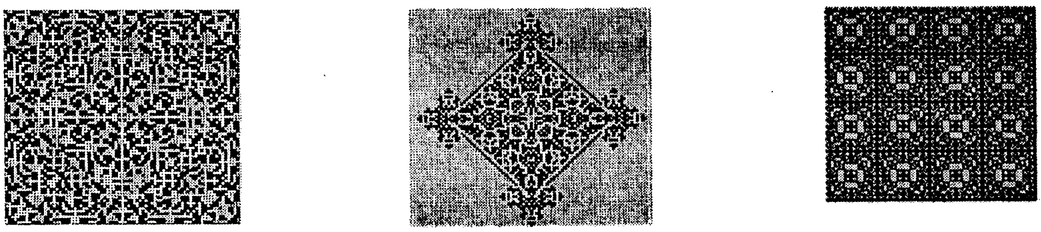
Choosing sources and neighborhoods, we can realize a wide class of phenomena by these dynamical systems. We call a point Q a source of the dynamical system if $f_n(Q) = 1$ for any n. In case that we have sources, we apply the Laplacian at all points except the sources. We give several examples of computer simulations, by plotting several f_n ([1],[2]):



Simulations of crystals of water



Evolution of an extinct animal



Generation of design patterns

Figure 2

3. MATHEMATICAL PROBLEMS OF DISCRETE LAPLACIANS

Here we recall some basic notations on dynamical systems and state assertions on mathematical structures ([1], [3]). At first we restrict ourselves to dynamical systems of periodic functions. For an integer M, which is called the size, we consider the following periodic functions:

$$F(M) = \{f \in F \mid f(x+mM, y+nM) = f(x,y), (n,m \in Z)\}$$

Choosing a neighborhood we define the discrete Laplacian respecting the periodicity and we consider the corresponding dynamical system. Hence, we understand that we consider the dynamical system on the torus with the size $M \times M$. The torus is denoted by $T(M)$. We prepare several basic notations:

(1) A dynamical system has a fixed point, if $\exists k \in N$ such that $f_n = f_k (\forall n \geq k)$

(2) A dynamical system is called periodic, if $\exists n, \exists l \in N$ such that $f_n = f_{n+kl} (\forall k \in N)$. If $n = 0$, then it is simply called periodic and if $n \neq 0$, it is called asymptotically periodic, respectively.

(3) Points $Q_j \in \{Q_1, Q_2, .., Q_k\} \subset T(M)$ are called sources of a dynamical system, if $f_n(Q_j) = 1$, for $\forall n \in N, j = 1, 2, \ldots, k$.

Conjecture ([1], [2])

We propose the following conjectures:

(1) In the case $M = 2^p$ and a single source, we have the following results:

(α) If a neighborhood is even, we see that the dynamical system has a fixed point and its fixed point can be attained after $2^{p-1} (or\ 2^p)$ steps for Moore, Hexagonal, and Neumann (resp. Sierpinski) neighborhoods.

(β) If the neighborhood is odd, we see that the dynamical system is periodic, period is depending on neighborhoods.

(2) In the case where M is odd, we see that the dynamical system is periodic in the case of a single source. We give the table of periods for some M (see Table 1).

M	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
Period	1	5	6	13	30	62	29	30	511	126	2046	2045	1021	16384	61
Recurrence point	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 1 Periods for smaller odd sizes

4. FIXED POINT THEOREM FOR EVEN NEIGHBORHOODS

Theorem I

In the case that $M = 2^p$, the neighborhood is of Sierpinski type (resp. Neumann type), and it has one source, the dynamical system has a fixed point after 2^p (resp. 2^{p-1}) steps.

Proof of the assertion for Sierpinski neighborhood

We give an idea of the proof of Theorem I in the case $p = 2$. Making an observation only in this simple case, we may understand that our assertion holds (see Figure 3).

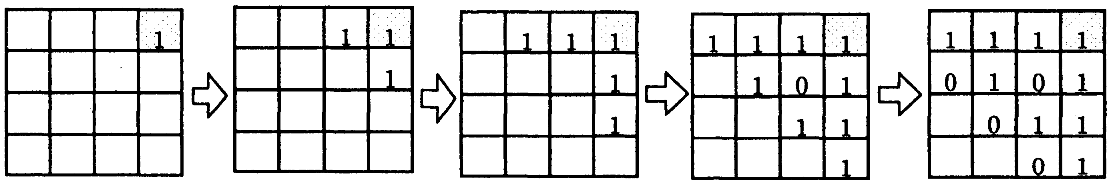


Figure 3

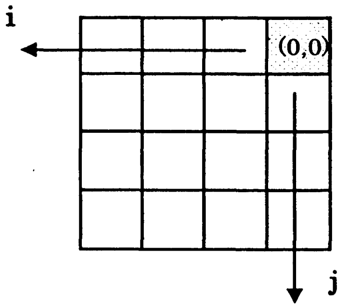


Figure 4

We introduce a coordinate system such that the origin is $(0,0)$ at the right upper corner of the rectangle as in Figure 4. We denote the support (or locus) of the n th generation by $N_n : N_n = \{(i,j) : i + j = n, i, j \geq 0\}$. We also put $M_n = \cup_{k=0}^{n} N_k$. We can prove the following proposition which proves the assertion of Theorem I in the case of general 2^p:

Proposition 1

For the dynamical system $\{f_n\}$ with the source at the origin, we see that

(1) $f_n(i,j) = f_n(j,i)$ on N_n,

(2) $f_n(n,0) = f_n(0,n) = 1, (0 \leq n \leq M-1)$

(3) $f_n(i,j) = f_n(i-1,j) + f_n(i,j-1)$ on $N_n (\text{mod}\, 2)$

(4) The Laplacian preserves the invariance on $M_n : f_{n+1|M_n} = f_n$

Remark 1

By proposition 1 we recognize the following facts: (i) The Pascal triangle mod 2 appears in the upper triangle part. (ii) At the 2^p -th step, every element in the diagonal is 1. (iii) Then the lower triangle is filled by 0 (see Figure 3).

Proof of the assertion for Neumann neighborhood

At first we give a proof of Theorem I in the case $p=2$ (see Figure 5).

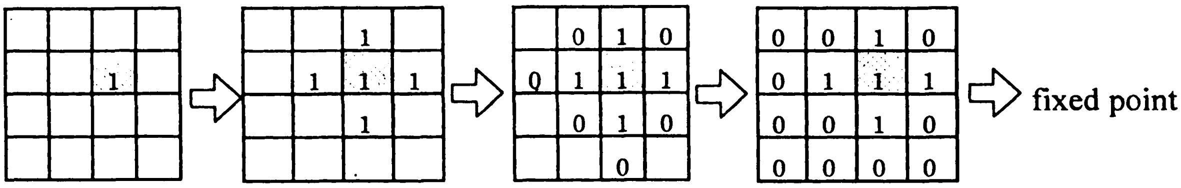


Figure 5

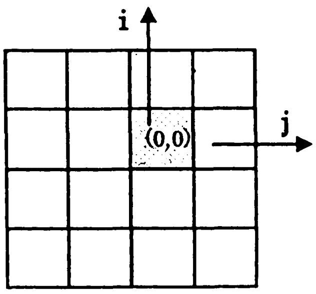


Figure 6

We introduce a coordinate system such that the origin (0,0) is centered as in the Figure 6. We denote the support (or locus) of n-th generation again by N_n: $N_n = \{(i,j) : |i+j| = n\}$. We also put $M_n = \cup_{k=0}^{n} N_k$. We can prove the following proposition which proves the assertion of Theorem I in the case of general 2^p:

Proposition 2

Let $\{f_n\}$ be a dynamical system with a source at the origin. Then we can prove the following facts for an integer n of the form $n = 2^q \, (0 \le q \le p-1)$:

(1) The Laplacian Δ maps the support of M_n to M_{n+1},

(2) The Laplacian preserves the function f_n on M_n, i.e., $f_{n+1}|_{M_n} = f_n$,

(3) $f_n(i,j) = 1$ *if* $i+j = \pm n$, and $f_n(i,j) = 0$ outside of M_n,

(4) $f_{n+1}(\pm(n+1),0) = 1$, $f_{n+1}(0,\pm(n+1)) = 1$ on N_{n+1}.

Remark 2

The condition (2) in proposition 2 is called monotonic increasing condition. We can prove the same assertion under this condition.

Remark 3

In [3], the concept of the "symmetric matrix" is introduced for a discrete Laplacian and the basic matrix theory with binary values {0,1} is developed. Also its dynamical system is considered. The comparison theorems on the fixed points and periodicity between these operators and the original discrete Laplacian might be interesting topics and should be considered.

Remark 4

In [4], using the concept of characteristic polynomials which are considered in [5], the case of 1-dimensional lattice can be transported to the plane lattice and it is proved that the period of Neumann neighborhood is identical with that of Moor neighborhood.

5. PERIODICITY THEOREM FOR ODD NEIGHBORHOODS

Theorem II

In the case that $M = 2^p$, the neighborhood is of Peano type, Roof type or Tannenbaum type and it has one source, the dynamical system is periodic and its period is 2^p (resp. 2^{p-1}).

Proof

We give the proof for the Peano neighborhood. The proofs for the other cases are similar. We illustrate the idea of the proof in the case p=2 (see Figure 7).

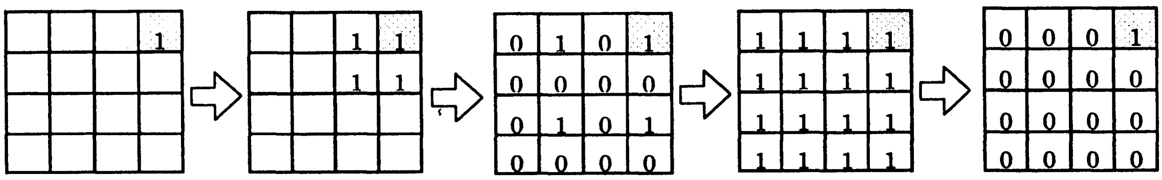


Figure 7

We choose a local coordinate system as in the case of Sierpinski neighborhood. Then we can prove the assertion of Theorem II by the following proposition:

Proposition 3

We denote the square domain of size $2^k (= m)$ with the origin at a corner by $T(m)$. We consider a dynamical system $\{f_n\}$ with a source at the origin. Then we can prove the following assertions for an integer m of a form $m = 2^q \, (0 \le q \le p-1)$:

(1) The Laplacian Δ maps the support of M_m to M_{m+1},
(2) $f_m (m = 2^k)$ is harmonic on $T(m)$, i.e., $\Delta f_{m-1} |_{T(m)} = 0$,

Remark 5

The condition (2) in proposition 3 is called harmonic monotonic increasing condition. We can prove the same assertion under this condition.

REFERENCES

[1] Y. Aiba, K. Maegaito and O. Suzuki (2006). Iteration dynamical systems of discrete Laplacian on the plane lattice (I) (Basic properties and computer simulations). *17-th International Conference on the Applications of Computer Science and Mathematics in Architecture and Civil Engineering,* K. Gürlebeck and C. Könke (eds.), Weimar, Germany, 2006 (ISSN 1611-4085).

[2] Y. Makino, C. Hadlich, K. Guerlebeck, A. Kimura, and O. Suzuki: Iteration dynamical systems of discrete Laplacians on the plane lattice (Its mathematical structure and computer simulations of designs):Report on Mathematical Sciences of Kyoto University, vol.1552,(2007),107-116

[3] C. Hadlich: Eine Anwendung finiter Differenzenoperatoren auf die Theorie dynamischer Systeme, Diplomarbeit, Bauhaus-Universität Weimar (2007)

[4] Xing, Li: Erzeugung zweidimensionaler zellulärer Automaten durch diskrete Laplace-Operatoren, Diplomarbeit, Bauhaus-Universität Weimar (2009)

[5] O. Martin, A. M. Odlyzko, and S. Wolfram. Algebraic Properties of Cellular Automata. 1984.