

Remarks on deformation quantization - quantization of the twistor space -

Naoya MIYAZAKI

Department of Mathematics, Keio University,
Yokohama, 223-8521, JAPAN

Abstract: This article is an announcement of a paper entitled “On deformation quantization of the twistor space and star exponentials” [16]. Here we study a deformation of the twistor space $\mathbb{C}P^3$. After stating noncommutative, associative product $\#$ on a twistor space, we also compute star exponentials of quadratic polynomials on them.

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1 Introduction

It is well-known that (super) twistor spaces themselves are very interesting objects to study, and besides, they give excellent and practical view points to study the Yang-Mills theory, eg. the ADHM-construction of instanton solutions, and the Atiyah-Ward correspondence, see [8, 32] in for details.

In this article, we are concerned with deformation quantization of a twistor space. Deformation quantization introduced in [1], is a fruitful approach to developing quantum theory in a purely algebraic framework, and was also a prototype for noncommutative calculus on noncommutative spaces (cf. [1, 2, 3, 7, 9, 11, 12, 14, 15, 19, 20, 21, 22, 23, 24, 25, 26, 27, 30, 34]).

We believe that these new features with techniques which are employed in the development of the argument of deformation quantization will provide a new approach to noncommutative nonformal calculus which also plays a pivotal role in geometric quantization (cf. [33]), strict deformation quantization, theory of operator algebra (cf. [17]) and (geometrically) asymptotic analysis (cf. [5]).

In this article, we are not concerned with the delicate issues associated with convergence of deformation quantization and nonformal calculus.

The first purpose of this article is to give an even-even (to an ordinary Poisson structure direction¹) deformation quantization of twistor space $\mathbb{C}\mathbb{P}^3$, and showing the existence of globally well-defined noncommutative, associative product $\#$ on the twistor space.

The second purpose is to compute star exponentials with respect to star product $\#$ of quadratic polynomials with respect to homogeneous coordinate of twistor space $\mathbb{C}\mathbb{P}^3$ of double fibrations (cf. Fig.1 below) which appears in describing the twistor space, and then, to show that the star exponentials give transcendental elements on the twistor space.

Main Theorem (cf. [16]) Consider the following diagram **Fig.1**:

$$((x^{\alpha,\dot{\alpha}}), [\pi_1 : \pi_2]) \in M := \mathbb{C}^4 \times \mathbb{C}\mathbb{P}^1$$

$$\begin{array}{ccc} & & \\ & \swarrow & \searrow \\ \Pi_1 & & \Pi_2 \\ & \swarrow & \searrow \\ ([z_1 : \dots : z_4]) \in \mathbb{C}\mathbb{P}^3 & & (x^{\alpha,\dot{\alpha}}) \in \mathbb{C}^4 \end{array}$$

where $x^{\alpha,\dot{\alpha}}$ are even variables, we set

$$\begin{aligned} (x^{\alpha,\dot{\alpha}}) &:= (x^{1,\dot{1}}, x^{1,\dot{2}}, x^{2,\dot{1}}, x^{2,\dot{2}}), \\ ([z_1 : \dots : z_4]) &:= ([x^{\alpha,\dot{1}}\pi_\alpha : x^{\alpha,\dot{2}}\pi_\alpha : \pi_1 : \pi_2]). \end{aligned}$$

Here we use Einstein's convention (we will often omit \sum unless there is a danger of confusion). We call $([z_1 : \dots : z_4])$ the homogeneous coordinate system of $\mathbb{C}\mathbb{P}^3$.

1. The relations² ($\dot{\alpha}, \dot{\beta} = \dot{1}, \dot{2}$)

$$[z^{\dot{\alpha}}, z^{\dot{\beta}}] = \hbar D^{\alpha\dot{\alpha},\beta\dot{\beta}} \pi_\alpha \pi_\beta, \quad (1)$$

¹More precisely, deformation quantization to the direction of the holomorphic Poisson structure.

²Here $[,]$ denotes the commutator bracket.

where $z^{\dot{1}} := z_1$, $z^{\dot{2}} := z_2$, give a globally defined noncommutative associative product $\#$ on $\mathbb{C}\mathbb{P}^3$, where $(D^{\alpha\dot{\alpha},\beta\dot{\beta}})$ is a skew symmetric matrix.

2. Let $A[Z]$ be a homogeneous polynomial³ of $z^{\dot{1}} = z_1 = x^{\alpha,\dot{1}}\pi_\alpha$, $z^{\dot{2}} = z_2 = x^{\alpha,\dot{2}}\pi_\alpha$ with degree 2. Then a star exponential function $e_{\#}^{\frac{1}{\hbar}A[Z]}$ gives a “function” on $\mathbb{C}\mathbb{P}^3$.

We hope that the results above will shed a light on the study of deformation theory of the Atiyah-Ward correspondence and the Ward transform.

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2 Deformation quantization

2.1 Definition

In the 1970’s, supported by the mathematical developments, Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer considered quantization as a deformation of the usual commutative product of classical observables into a noncommutative associative product which is parametrized by the Planck constant \hbar and satisfies the correspondence principle.

In the approach above, the precise definition of the space of quantum observables and star product⁴ is given in the following way(cf. [1]):

Definition 2.1 A star product of Poisson manifold (M, π) is a product $*$ on the space $C^\infty(M)[[\hbar]]$ of formal power series of parameter \hbar with coefficients in $C^\infty(M)$, defined by

$$f * g = fg + \hbar\pi_1(f, g) + \cdots + \hbar^n\pi_n(f, g) + \cdots, \quad \forall f, g \in C^\infty(M)$$

³In our situation, it should be regarded as an $\mathcal{O}_{\mathbb{C}\mathbb{P}^3}(2)$ -sheaf cohomology class.

⁴In the present paper, we use this notion in a quite different situation, i.e., in holomorphic categories.

satisfying

1. $*$ is associative,
2. $\pi_1(f, g) = \frac{1}{2\sqrt{-1}}\{f, g\}$,
3. each π_n ($n \geq 1$) is a $\mathbb{C}[[\hbar]]$ -bilinear and bidifferential operator, where $\{, \}$ is the Poisson bracket defined by the Poisson structure π .

A deformed algebra (resp. a deformed algebra structure) is called a *star algebra* (resp. a *star product*).

2.2 Existence of formal deformation quantization I (Omori-Maeda-Yoshioka quantization)

As to a symplectic manifold (M, ω) , DeWilde-Lecomte [2], Omori-Maeda-Yoshioka [26] found the method of construction for formal deformation quantization by patching work of the Weyl algebra bundle with suitable conditions.

2.3 Existence of formal deformation quantization II (Fedosov quantization)

Fedosov [3] found a geometric method of formal deformation quantization of a symplectic manifold via adjusting the *canonical connection of the jet bundle* so that it is compatible with fibre-wise Moyal-Weyl product on the symmetric tensor algebra of the cotangent bundle of (M, ω) .

Let (M, ω) be a symplectic and ∇^{symp} a symplectic connection. Set

$$\delta^{-1}(\nu^m Z^\alpha dz^\beta) = \begin{cases} \sum_{i=1}^{2n} dz_i \iota_{Z_i} \nu^m Z^\alpha dz^\beta & (|\alpha| + |\beta| \neq 0), \\ 0 & (|\alpha| + |\beta| = 0), \end{cases} \quad (2)$$

where ι is a inner product. We may write $\nabla^F|_{W_M} = \nabla^{symp} - \delta + r$, where W_M is the Weyl algebra bundle on M , and r is a 1-form with $\Gamma(W_M)$ coefficient. Then as in [3], r satisfies the following equation

$$\delta r = R_\omega + \nabla^{symp} r + \frac{1}{2\nu}[r, r], \quad (3)$$

where R_ω is a curvature of symplectic connection. Or equivalently r satisfies

$$r = \delta^{-1} \left\{ (\nabla^{symp} + \frac{1}{2\nu} [r, r]) + R_\omega \right\}, \quad (4)$$

under the assumptions $\deg r \geq 2$, $\delta^{-1}r = 0$, $r_0 = 0$. Set r_k is the degree k term of r . Since it is easy to verify that this equation can be solved by recursively in the following way

$$\begin{aligned} r_3 &= \delta^{-1} R_\omega, \\ r_{n+3} &= \delta^{-1} \left(\nabla^{symp} r_{n+2} + \frac{1}{\nu} \sum_{l=1}^{k-1} r_{l+2} * r_{k+2-l} \right). \end{aligned} \quad (5)$$

The connection obtained as above is called the *Fedosov connection*.

Theorem 2.2 *Restriction of fiber-wise Moyal-Weyl product into the space of parallel sections with respect to the Fedosov connection gives a formal deformation quantization on a symplectic manifold.*

2.4 Existence of formal deformation quantization III (L_∞ -algebras as an exhibition for Kontsevich's formality theorem)

As to general Poisson manifolds, Kontsevich [7] established the *formality theorem*. Roughly speaking, he considered the Batalin-Vilkovisky-Maurer-Cartan equation ⁵ in the category of L_∞ -algebras.

We review the basics of formal deformation quantization for readers. See [3, 7] for details.

Let $V = \bigoplus_{k \in \mathbb{Z}} V^k$ be a graded vector space, and $[1]$ a shift-functor, that is, $V[1]^k = V^{k+1}$. $V[1] = \bigoplus_k V[1]^k$ is called a shifted graded vector space of V . We set $C(V) = \bigoplus_{n \geq 1} Sym^n(V)$ where

$$Sym^n(V) = T^n(V) / \{ \cdots \otimes (x_1 x_2 - (-1)^{k_1 k_2} x_2 x_1) \otimes \cdots; x_i \in V^{k_i} \}.$$

This space has a coproduct $\Delta : C(V) \rightarrow C(V) \otimes C(V)$ defined in the following way:

$$\begin{aligned} & \Delta(x_1 \cdots x_n) \\ &= \sum_{k=1}^{n-1} \frac{1}{k!(n-k)!} \sum_{\sigma \in S_n} \text{sign}(\sigma; x_1 \cdots x_n) \\ & \quad \times (x_{\sigma(1)} \cdots x_{\sigma(k)}) \otimes (x_{\sigma(k+1)} \cdots x_{\sigma(n)}), \end{aligned}$$

⁵For example, it is well-known that the Maurer-Cartan equation appears in geometry of connection.

where $\text{sign}(\sigma; x_1 \cdots x_n)$ is defined by

$$x_{\sigma(1)} \cdots x_{\sigma(n)} = \text{sign}(\sigma; x_1 \cdots x_n) x_1 \cdots x_n.$$

This coproduct is coassociative, i.e. $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$. We denote $k_1 + k_2 + \cdots + k_n$ by $\text{deg}(x_1 \cdots x_n)$, where $(x_i \in V^{k_i})$.

Definition 2.3 A map $f : C(V_1) \rightarrow C(V_2)$ is called a coalgebra homomorphism if (1) $\Delta \circ f = (f \otimes f) \circ \Delta$, (2) f preserves the grading.

The coderivation is defined in the following way.

Definition 2.4 A map $\ell : C(V) \rightarrow C(V)$ is called a coderivation if the following properties are satisfied: (1) ℓ is an odd vector field of degree +1, (2) $(\ell \hat{\otimes} \text{id} + \text{id} \hat{\otimes} \ell) \circ \Delta = \Delta \circ \ell$, where $(\text{id} \hat{\otimes} \ell)(x \otimes y) = (-1)^{\text{deg } x} x \otimes \ell(y)$.

We also use the following notation: Set $f^{(n)} = p \circ f|_{\text{Sym}^n(V_1)} : \text{Sym}^n(V_1) \rightarrow V_2$, and $\ell^{(n)} = p \circ \ell|_{\text{Sym}^n(V_1)} : \text{Sym}^n(V_1) \rightarrow V_2$, where $p =$ canonical projection : $C(V_2) \rightarrow V_2$.

Under the above notation, L_∞ -algebras and L_∞ -morphisms are defined in the following way:

Definition 2.5 An L_∞ -algebra is a pair (V, ℓ) , where V is a graded vector space and ℓ is a coderivation on the graded coalgebra $C(V)$, such that $\ell^2 = 0$.

Definition 2.6 An L_∞ -morphism F_* between two L_∞ -algebras (V_1, ℓ_1) and (V_2, ℓ_2) is a coalgebra homomorphism such that $\ell_2 \circ F_* = F_* \circ \ell_1$.

Remark (example) If $\ell = \ell^{(1)} + \ell^{(2)}$, and $d = \ell^{(1)}$, $[x, y] = (-1)^{\text{deg } x - 1} \ell^{(2)}(x, y)$, then $\ell^2 = 0$ if and only if

$$\begin{aligned} d^2 &= 0, \\ d[x, y] &= [dx, y] + (-1)^{\text{deg } x - 1} [x, dy], \\ [[x, y], z] &+ (-1)^{(x+y)(z+1)} [[z, x], y] + (-1)^{(y+z)(x+1)} [[y, z], x] = 0, \end{aligned}$$

that is, (V, ℓ) is a graded differential Lie algebra.

We next recall examples which play important roles in Kontsevich's formality theorem.

Differential Graded Lie algebra of T_{poly} -fields

Let M be a smooth manifold. Set $T_{poly}(M) = \bigoplus_{k \geq -1} \Gamma(M, \wedge^{k+1} TM)$, and let $[\cdot, \cdot]_S$ be the *Schouten* bracket:

$$\begin{aligned} & [X_0 \wedge \cdots \wedge X_m, Y_0 \wedge \cdots \wedge Y_n]_S \\ := & \sum_{i,j} (-1)^{i+j+m} [X_i, Y_j] \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{Y}_j \wedge \cdots, \end{aligned}$$

where $X_i, Y_i \in \Gamma(M, TM)$. Then, the triple

$$(T_{poly}(M)[[\hbar]], d := 0, [\cdot, \cdot] := [\cdot, \cdot]_S)$$

forms a differential graded Lie algebra. It is well-known that for any bivector $\pi \in \Gamma(M, \wedge^2 TM)$, π is a Poisson structure if and only if

$$[\pi, \pi]_S = 0. \quad (6)$$

Differential Grade Lie algebra of D_{poly} -fields

Let (A, \bullet) be an associative algebra and set $C(A) := \bigoplus_{k \geq -1} C^k$, $C^k = Hom(A^{\otimes k+1}, A)$. For $\varphi_i \in C^{k_i}$ ($i = 1, 2$), we set

$$\begin{aligned} & \varphi_1 \circ \hat{\varphi}_2(a_0 \otimes a_1 \otimes \cdots \otimes a_{k_1+k_2}) \\ := & \sum_{i=0}^k (-1)^{ik_2} \varphi_1(a_0 \otimes \cdots \otimes a_{i-1} \otimes \varphi_2(a_i \otimes \cdots \otimes a_{i+k_2}) \otimes a_{i+k_2+1} \otimes \cdots \otimes a_{k_1+k_2}). \end{aligned}$$

Then the *Gerstenhaber* bracket is defined in the following way:

$$[\varphi_1, \varphi_2]_G = \varphi_1 \circ \hat{\varphi}_2 - (-1)^{k_1 k_2} \varphi_2 \circ \hat{\varphi}_1 \quad (7)$$

and Hochschild coboundary operator $\delta = \delta_\bullet$ with respect to \bullet is defined by $\delta_\bullet(\varphi) = (-1)^k [\bullet, \varphi]$ ($\varphi \in C^k$). Then it is known that the triple

$$(C(A), d := \delta_\bullet, [\cdot, \cdot] := [\cdot, \cdot]_G)$$

is a differential graded Lie algebra.

Let M be a smooth manifold. Set $\mathcal{F} := C^\infty(M)$, and $D_{poly}(M)^n(M)$ equals a space of all multidifferential operators from $\mathcal{F}^{\otimes n+1}$ into \mathcal{F} . Then

$$D_{poly}(M)[[\hbar]] := \bigoplus_{n \geq -1} D_{poly}^n(M)[[\hbar]]$$

is a subcomplex of $C(\mathcal{F}[[\hbar]])$.

Furthermore, the triple $(D_{poly}(M)[[\hbar]], \delta, [\cdot, \cdot]_G)$ is a differential graded Lie algebra.

Proposition 2.7 *Let B be a bilinear operator and $f \star g = f \cdot g + B(f, g)$. Then the product \star is associative if and only if B satisfies*

$$\delta.B + \frac{1}{2}[B, B]_G = 0. \quad (8)$$

Next we recall the moduli space $\mathcal{MC}(C(V[1]))$. For $b \in V[1]$, set $e^b := 1 + b + \frac{b \otimes b}{2!} + \dots \in C(V[1])$.

Definition 2.8 $\ell(e^b) = 0$ is called a Batalin-Vilkovisky-Maurer-Cartan equation, where ℓ is a L_∞ -structure.

Using this equation, we define the moduli space as follows:

Definition 2.9

$$\widehat{\mathcal{MC}}(C(V[1])) := \{b; \ell(e^b) = 0\}, \quad (9)$$

$$\mathcal{MC}(C(V[1])) := \widehat{\mathcal{MC}}(C(V[1])) / \sim, \quad (10)$$

where V stands for $T_{poly}(M)[[\hbar]]$ (i.e. T -poly vector fields), and $D_{poly}(M)[[\hbar]]$ (i.e. D -poly vector fields), and \sim means the gauge equivalence⁶.

With these preliminaries, we can state precise version of the formality theorem:

Theorem 2.10 *There exists a map \mathcal{U} such that*

$$\mathcal{U} : \mathcal{MC}(C(T_{poly}(M)[[\hbar]][1])) \cong \mathcal{MC}(C(D_{poly}(M)[[\hbar]][1])).$$

As a biproduct, we have

Theorem 2.11 *For any Poisson manifold (M, π) there exists a formal deformation quantization (i.e. noncommutative associative product (say Kontsevich's star product) on $C^\infty(M)[[\hbar]]$).*

In the proof of the formality theorem, Kontsevich constructed the map denoted by \mathcal{U} which seems to be deeply depending on the combinatorial methods based on the Feynman diagram which was, may be, inspired by the pioneer works by Dirac.

⁶Strictly speaking, as for formal Poisson bivectors, $\pi_1(\hbar) \sim \pi_2(\hbar)$ if there exists a formal vector field $D \in \mathfrak{X}(M)[[\hbar]]$ such that $\exp \hbar D \circ \pi_1(\hbar) = \pi_2(\hbar) \circ (\exp \hbar D \otimes \exp \hbar D)$

2.5 Sketch of the proof of the first assertion 1 in main result

We can consider formal deformation quantization with respect to an even-even part direction (an ordinary Poisson structure direction⁷) of it.

Theorem 2.12 ([16]) *For the double fibration Fig.1, the relations ($\dot{\alpha}, \dot{\beta} = \dot{1}, \dot{2}$) below*

$$[z^{\dot{\alpha}}, z^{\dot{\beta}}] = \hbar D^{\alpha\dot{\alpha}, \beta\dot{\beta}} \pi_{\alpha} \pi_{\beta}, \quad (11)$$

where $z^{\dot{1}} := z_1$, $z^{\dot{2}} := z_2$, give a globally defined noncommutative associative product ⁸ $\#$ on $\mathbb{C}\mathbb{P}^3$, where $(D^{\alpha\dot{\alpha}, \beta\dot{\beta}})$ is a skew symmetric matrix.

Remark. Normalizing the above relations, our product is closely related to the algebra obtained in [6]. For other approaches to the problem of deformation quantization of complex projective spaces, see also [1, 6, 7, 18, 19, 31].

Proof We give two proofs of this statement.

(I) In our situation, $D^{\alpha\dot{\alpha}, \beta\dot{\beta}} \pi_{\alpha} \pi_{\beta} \partial_{\dot{\alpha}} \wedge \partial_{\dot{\beta}}$ gives a holomorphic Poisson structure on the projective space ⁹. Since for any Poisson manifold has a formal deformation on it, as seen in the previous subsection 2.4, we have the assertion.

(II) Second proof is more direct and referent formula enables us to compute star exponentials explicitly. First we remark that Weyl type star product means the following product:

$$f(Z) * g(Z) = f(Z) \exp\left[\frac{\mu}{2} \overleftarrow{\partial}_{Z_{\dot{\alpha}}} \Lambda^{\dot{\alpha}, \dot{\beta}} \overrightarrow{\partial}_{Z_{\dot{\beta}}}\right] g(Z), \quad (12)$$

that is, the Moyal type product formula, where $Z = (Z_1, \dots, Z_{2n})$ and $\mu = -\sqrt{-1}\hbar$. Then we have the following.

⁷More precisely, deformation quantization to the direction of the holomorphic Poisson structure.

⁸More precisely, it gives a globally defined noncommutative associative product on the structure sheaf $\mathcal{O}_{\mathbb{C}\mathbb{P}^3}[[\mu, \mu]]$, $\mu = -\sqrt{-1}\hbar$.

⁹Note that this Poisson structure is not the Fubini-Study fundamental form.

Proposition 2.13 *Suppose the assumption below:*

$$\begin{aligned} & \overleftarrow{\partial}_{Z^{\dot{\alpha}_1}} \Lambda^{\dot{\alpha}_1, \dot{\beta}_1} \overrightarrow{\partial}_{Z^{\dot{\beta}_1}} \cdots \overleftarrow{\partial}_{Z^{\dot{\alpha}_k}} \Lambda^{\dot{\alpha}_k, \dot{\beta}_k} \overrightarrow{\partial}_{Z^{\dot{\beta}_k}} \\ &= \overleftarrow{\partial}_{Z^{\dot{\alpha}_1 \dots \dot{\alpha}_k}} \Lambda^{\dot{\alpha}_1, \dot{\beta}_1} \cdots \Lambda^{\dot{\alpha}_k, \dot{\beta}_k} \overrightarrow{\partial}_{Z^{\dot{\beta}_1 \dots \dot{\beta}_k}}. \end{aligned} \quad (13)$$

Then, the Weyl type star product gives a noncommutative, associative product. Hence, it gives a star product.

In order to realize the noncommutative, associative product, we use the Weyl type star product (i.e. Weyl ordering)¹⁰: For abbreviation, we set a matrix

$$\hat{\Lambda} := \left[\frac{2}{\sqrt{-1}} D^{\alpha\dot{\alpha}, \beta\dot{\beta}} \pi_\alpha \pi_{\dot{\beta}} \right]_{\dot{\alpha}, \dot{\beta}}, \quad (14)$$

and then $\hat{\Lambda}$ is a skew symmetric matrix.

Proposition 2.14 *The coefficients of $\hat{\Lambda}$ depend on the variables on the base manifold in our case. However, $\hat{\Lambda}$ satisfies the above assumption (13).*

Combining these Propositions 2.13 and 2.14 completes the proof of Theorem 2.12, thus the first assertion 1 of main theorem. \square

3 Sketch of the proof of the second assertion 2 in main result

In this section, we would like to compute star exponentials for quadratic polynomials with the form $f(Z) = g(t)e^{\frac{1}{\mu}Q[Z](t)}$ under a quite general setting more than settings of [9, 10, 18, 20, 21, 22, 24, 25].

We begin this section with remarking that we can demonstrate our computation of star product under a slightly general setting with the **assumption** above as seen in the previous subsection: Let $Z = {}^t(Z^1, \dots, Z^{2n})$, $A[Z] := {}^tZAZ$, where $A \in \text{Sym}(2n, \mathbb{R})$, i.e. A is a $2n \times 2n$ -real symmetric matrix. In order to compute the star exponential function $e_*^{\frac{1}{\mu}A[Z]}$ with

¹⁰It is well-known that under the suitable conditions, Kontsevich's star product reduces the Moyal type product.

respect to the Moyal type product formula, we treat the following evolution equation:

$$\partial_t F = \frac{1}{\mu} A[Z] * F, \quad (15)$$

with an initial condition

$$F_0 = e^{\frac{1}{\mu} B[Z]}, \quad (16)$$

where $B \in \text{Sym}(2n, \mathbb{R})$, $\mu = -\sqrt{-1}\hbar$.

As seen above, our setting¹¹ is rather different from the situations considered in the article [10] by Maillard, in [9, 18, 20, 21, 22, 24, 25] by OMMY, and in the book [18] entitled *Physics in Mathematics*, Univ. Tokyo Press by Omori (see also [28]¹²). However, to compute star exponentials, we can use similar methods employed in the articles and book above, as will be seen below:

Under the assumption $F(t) = g \cdot e^{\frac{1}{\mu} Q[Z]}$ ($g = g(t)$, $Q = Q(t)$), we would like to find a solution of the equations (15) and (16).

Direct computations give

$$\begin{aligned} \text{L.H.S. of (15)} &= g' e^{\frac{1}{\mu} Q[Z]} + g \frac{1}{\mu} Q'[Z] e^{\frac{1}{\mu} Q[Z]}, \\ \text{R.H.S. of (15)} &= \frac{1}{\mu} A[Z] * F \\ &\stackrel{(12)}{=} \frac{1}{\mu} A[Z] \cdot F + \frac{i\hbar}{2} \Lambda^{i_1 j_1} \partial_{i_1} \frac{1}{\mu} A[Z] \cdot \partial_{j_1} F \\ &\quad - \frac{\hbar^2}{2 \cdot 4} \Lambda^{i_1 j_1} \Lambda^{i_2 j_2} \partial_{i_1 i_2} \frac{1}{\mu} A[Z] \partial_{j_1 j_2} F \end{aligned} \quad (17)$$

where $A = (A_{ij})$, $\Lambda = (\Lambda^{ij})$ and $Q = (Q_{ij})$. Comparing the coefficient of μ^{-1} gives

$$Q'[Z] = A[Z] - 2^t A \Lambda Q[Z] - Q \Lambda A \Lambda Q[Z]. \quad (18)$$

Applying Λ by left and setting $q := \Lambda Q$ and $a := \Lambda A$, we easily obtain

$$\Lambda Q' = \Lambda A + \Lambda Q \Lambda A - \Lambda A \Lambda Q - \Lambda Q \Lambda A \Lambda Q$$

¹¹i.e. deformation quantization of the structure sheaf $\mathcal{O}_{\mathbb{C}P^3}$ to the direction of holomorphic Poisson structure.

¹²Quillen's method is very useful to compute superconnection character forms and supertrace of Dirac-Laplacian heat kernels (cf. [4, 13])

$$\begin{aligned}
&= (1 + \Lambda Q)\Lambda A(1 - \Lambda Q) \\
&= (1 + q)a(1 - q).
\end{aligned} \tag{19}$$

As to the coefficient of μ^0 , we have

$$\begin{aligned}
g' &= \frac{1}{2}\Lambda^{i_1j_1}\Lambda^{i_2j_2}A_{i_1i_2}gQ_{j_1j_2} \\
&= -\frac{1}{2}\text{tr}(aq) \cdot g.
\end{aligned} \tag{20}$$

Thus

Theorem 3.1 *The equation (15) is rewritten by*

$$\partial_t q = (1 + q)a(1 - q), \tag{21}$$

$$\partial_t g = -\frac{1}{2}\text{tr}(aq) \cdot g. \tag{22}$$

In order to solve the equations (21) and (22), we now recall the ‘‘Cayley transform.’’

Proposition 3.2 *Set $C(X) := \frac{1-X}{1+X}$ if $\det(1 + X) \neq 0$. Then*

1. $X \in sp_\Lambda(n, \mathbb{R}) \iff \Lambda X \in \text{Sym}(2n, \mathbb{R})$,
and then $C(X) \in Sp_\Lambda(n, \mathbb{R})$, where

$$\begin{aligned}
Sp_\Lambda(n, \mathbb{R}) &:= \{g \in M(2n, \mathbb{R}) \mid {}^t g \Lambda g = \Lambda\}, \\
sp_\Lambda(n, \mathbb{R}) &:= \text{Lie}(Sp_\Lambda(n, \mathbb{R})).
\end{aligned}$$

2. $C^{-1}(g) = \frac{1-g}{1+g}$, (the ‘‘inverse Cayley transform’’).
3. $e^{2\sqrt{-1}a} = c(-\sqrt{-1} \tan(a))$.
4. $\log a = 2\sqrt{-1} \arctan(\sqrt{-1}C^{-1}(g))$.
5. $\partial_t q = (1 + q)a(1 - q) \iff \partial_t C(q) = -2aC(q)$.

Proof *Direct computations show these assertions. \square*

Solving the above equation 5 in Proposition 3.2, we have

$$C(q) = e^{-2at}C(b),$$

where $b = \Lambda B$ and then

$$q = C^{-1}(e^{-2at} \cdot C(b)) = C^{-1}(C(-\sqrt{-1} \tan(\sqrt{-1}at) \cdot C(b))).$$

Hence, according to the inverse Cayley transform, we can get Q in the following way.

Proposition 3.3

$$Q = -\Lambda \cdot C^{-1}(C(-\sqrt{-1} \tan(\sqrt{-1}\Lambda At)) \cdot C(\Lambda B)). \quad (23)$$

Next we compute the amplitude coefficient part g . Solving

$$g' = -\frac{1}{2}Tr(aq) \cdot g \quad (24)$$

gives

Proposition 3.4

$$g = \det^{-\frac{1}{2}} \left(\frac{e^{at}(1+b) + e^{-at}(1-b)}{2} \right). \quad (25)$$

Setting $t = 1$, $a = \Lambda A$ and $b = 0$, we get

Theorem 3.5

$$e_{\#}^{\frac{1}{2}A[Z]} = \det^{-\frac{1}{2}} \left(\frac{e^{\Lambda A} + e^{-\Lambda A}}{2} \right) \cdot e^{\frac{1}{\mu} \left(\frac{\Lambda^{-1}}{\sqrt{-1}} \tan(\sqrt{-1}\Lambda A) \right) [Z]}. \quad (26)$$

Combining Theorems 2.12, 3.5 with sheaf cohomology of projective space, we have the following (cf. [16]).

Theorem 3.6 *Assume that $\Lambda := \hat{\Lambda}$ and $A[Z]$ a homogeneous polynomial of $z^1 = x^{\alpha,1}\pi_{\alpha}$, $z^2 = x^{\alpha,2}\pi_{\alpha}$ with degree 2. Then a star exponential function $e_{\#}^{\frac{1}{2}A[Z]}$ gives a cohomology class of $\mathbb{C}\mathbb{P}^3$ with coefficients in a suitable sheaf.*

This completes the proof of main theorem. \square

4 Concluding remarks

In this article, we are mainly concerned with only typical twistor space. However, we believe that these arguments can be extended to a certain class of Lie tensor contact manifolds in the sense of [29]. We also remark that we can deform the super twistor spaces to odd-odd direction and then obtain non-anti-commutative products (cf. [16] and [31]).

References

- [1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization I*, Ann. of Phys. 111 (1978), 61-110.
- [2] M. De Wilde and P. B. Lecomte, *Existence of star products and formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys. 7 (1983).
- [3] B. V. Fedosov, *A simple geometrical construction of deformation quantization*, Jour. Diff. Geom. 40 (1994), 213-238.
- [4] K. Gomi and Y. Terashima, *Chern-Weil construction for twisted K-theory*, preprint.
- [5] L. Hörmander, *The Weyl calculus of general pseudodifferential operators*, Commun. Pure. Appl. Math. 32 (1979), 359-443.
- [6] A. Kapustin, A. Kuznetsov and D. Orlov, *Noncommutative instantons and twistor transform*, hep-th/0002193, Commun. Math. Phys. 221 (2001) 385-432.
- [7] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66, (2003), 157-216.
- [8] C. LeBrun, Y. S. Poon and R. O. Wells, Jr., *Projective embedding of complex supermanifolds*, Commun. Math. Phys. 126 (1990) 433-452.
- [9] Y. Maeda, N. Miyazaki, H. Omori and A. Yoshioka, *Star exponential functions as two-valued elements*, Progr. Math. 232 (2005), 483-492, Birkhäuser.

- [10] J. M. Maillard, *Star exponential for any ordering of the elements of the inhomogeneous symplectic Lie algebra*, Jour. Math. Phys. vol.45, no. 2, (2004), 785-794.
- [11] N. Miyazaki, *On a certain class of oscillatory integral transformations which determine canonical graphs*, Japanese Jour. Math. 24, no.1 (1998) 61-81.
- [12] N. Miyazaki, *Formal deformation quantization and Index theorem*, unpublished, (2003), in Japanese.
- [13] N. Miyazaki, *Examples of groupoid*, Proceedings of the international Sendai-Beijing joint workshop 2005, 97-108, World Scientific.
- [14] N. Miyazaki, *Lifts of symplectic diffeomorphisms of a Weyl algebra bundle with Fedosov connection*, IJGMMP. Vol. 4, No. 4 (2007) 533-546.
- [15] N. Miyazaki, *A Lie Group Structure for Automorphisms of a Contact Weyl Manifold*, Progr. Math. 252 (2007), 25-44, Birkhäuser.
- [16] N. Miyazaki, *On deformation quantization of twistor spaces and star exponentials*, in preparation.
- [17] G. J. Murphy, *C*-algebras and operator theory*, Academic Press, (1990)
- [18] H. Omori, *Physics in Mathematics*, Univ. Tokyo Press, (2004), in Japanese.
- [19] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Poincaré-Cartan class and deformation quantization of Kähler manifolds*, Commun. Math. Phys. 194 (1998), 207-230.
- [20] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Star exponential functions for quadratic forms and polar elements*, Contemporary Mathematics. 315 (Amer. Math. Soc.), (2002), 25-38.
- [21] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Strange phenomena related to ordering problems in quantizations*, Jour. Lie Theory vol. 13, no 2 (2003), 481-510.

- [22] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Expressions of Algebra Elements and Transcendental Noncommutative Calculus*, Noncommutative Geometry and Physics, (2005), World Scientific.
- [23] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Noncommutative Minkowski Space and Transcendental Calculus*, Progress of Theoretical Physics Supplement No.171 (2007), 184-195.
- [24] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Convergent star product on Fréchet-Poisson algebras of Heisenberg type*, Contemporary Mathematics 434 (Amer. Math. Soc.), (2007), 99-123.
- [25] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *A new nonformal noncommutative calculus: Associativity and finite part regularization*, Astérisque 321 (2008), 267-297.
- [26] H. Omori, Y. Maeda and A. Yoshioka, *Deformation quantization and Weyl manifolds*, Adv. Math. 85 (1991), 224-255.
- [27] H. Omori, Y. Maeda and A. Yoshioka, *Global calculus on Weyl manifolds*, Japanese. Jour. Math. 17 (2), (1991), 57-82.
- [28] D. Quillen, *Superconnection character forms and the Cayley transform*, Topology, 27 (2), (1988), 211-238.
- [29] H. Sato, *Lie tensor contact manifolds*, preprint.
- [30] D. Sternheimer, *Deformation quantization twenty years after*, AIP Conf. Proc. 453 (1998), 107-145.
- [31] T. Taniguchi and N. Miyazaki, *On non(anti)commutative super twistor spaces*, to appear.
- [32] R. S. Ward and R. O. Wells, Jr., *Twistor geometry and field theory*, Cambridge Monographs on Math. Phys. 1990.
- [33] N. Woodhouse, *Geometric quantization*, Clarendon Press, (1980), Oxford.
- [34] A. Yoshioka, *Contact Weyl manifold over a symplectic manifold*, in “Lie groups, Geometric structures and Differential equations”, Adv. Stud. Pure Math. 37(2002), 459-493.