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Fractional Calculus and Gamma Function

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Abstract

The following variable change formulae of fractional order and logarithm of differentiation are shown.

$$\frac{d^a}{dx^a}|_{x=e^t} = e^{-at}(\frac{\Gamma(1+X)}{\Gamma(1+X-a)}|_{X=\frac{d}{dt}}),$$
$$\log(\frac{d}{dx})|_{x=e^t} = -t+\frac{d}{dX}(\log(\Gamma(1+X))|_{X=\frac{d}{dt}}.$$

As an application, we show the group generated by 1-parameter groups $\{\frac{d^a}{dx^a}|a\in\mathbb{R}\}$ and $\{x^a|a\in\mathbb{R}\}$ is the crossed product $\mathbb{R}\ltimes G_\Gamma^\natural$. Here $G_\Gamma^\natural$ is the abelian group generated by $F_b^a(s)=\dfrac{\Gamma(1+s+a)}{\Gamma(1+s+b)}$, $a,b\in\mathbb{R}$ by multiplication. $c\in\mathbb{R}$ acts on F_b^a as the translation τ_c:$\tau_cF_b^a(x)=F_b^a(x+c)=F_{b+c}^{a+c}$.

1 Introduction

Fractional calculus (fractional order indefinite integral and differentiation) was already considered by Leibniz. Its frist application is Abel's study of the following dynamical problem: Find the curve $F(x)$ when the required total time $f(x)$ for a particle falling down along this curve is given.

$F(x)$ should satisfy

$$f(x)=\int_0^x\frac{\sqrt{1+F'(t)^2}}{\sqrt{2g(x-t)}}dt.$$

Since $I^nf=\dfrac{1}{(n-1)!}\displaystyle\int_0^x(x-t)^{n-1}f(t)dt$ is the n-th order indefinite integral of f, the above integral can be regarded as the half order indefinite integral of $\sqrt{1+F'(x)^2}$. In fact, this equation is solved by using this argument.

Traditionally, fractional calculus is used to analyse phenomena having singularities of type x^a. Fractional order differentiation is a nonlocal opertor. Recently, owing to this property, fractional calculus is used to study effects of memories of Brownian motion, which is thought to be useful in mathematical finance.

As for recent studies on applications of fractional calculus, we refer [1],[2],[6],[7],[8],[9],[11] and [15]. Besides these studies, Prof.Nakanishi suggested to use fractional calculus to the study of deformation of canonical commutation relation (CCR) ([4],[12],cf.[13]).

In this paper, we show by the variable change $x = e^t$, fractional Euler differentiation $x^a \frac{d^a}{dx^a}$ is written as follows;

$$\mathfrak{d}_{a,t} = x^a \frac{d^a}{dx^a}|_{x=e^t} = \frac{\Gamma(1+X)}{\Gamma(1+X-a)}|_{X=\frac{d}{dt}}.$$

This is a continuous extension of the formula

$$x^n \frac{d^n}{dx^n}|_{x=e^t} = \frac{d}{dt}(\frac{d}{dt}-1)\cdots(\frac{d}{dt}-n+1).$$

$\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ is a 1-parameter group. Its generating operator is the logarithm of differentiation $\log(\frac{d}{dx})$;

$$\log(\frac{d}{dx})f(x) = -(\log x + \gamma)f(x) - \int_0^x \log(x-t)\frac{df(t)}{dt}dt.$$

Here γ is the Euler constant. As for logarithm of differentiation, we have

$$\mathfrak{d}_{\log,t} = \big(\log(\frac{d}{dx}) + \log x\big)|_{x=e^t} = \big(\frac{d}{dX}\log(\Gamma(1+X))\big)|_{X=\frac{d}{dt}}.$$

Note. For the simplicity, we use $\mathfrak{d}_a$ and $\mathfrak{d}_{\log}$ instead of $\mathfrak{d}_{a,t}$ and $\mathfrak{d}_{\log,t}$, in the rest.

As an application, we show the group $G^{\natural}_{\log}$ generated by 1-parameter groups $\{x^a|a \in \mathbb{R}\}$ and $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ is the crossed product $\mathbb{R} \ltimes G^{\natural}_{\Gamma}$ of $\mathbb{R}$. Here the abelian group $G^{\natural}_{\Gamma}$ is generated by the functions

$$F^a_b(x) = \frac{\Gamma(1+x+a)}{\Gamma(1+x+b)}, \quad a, b \in \mathbb{R},$$

by multiplication. The action of $c \in \mathbb{R}$ to $F^a_b(x)$ is the translation τ_c: $\tau_c F^a_b(x) = F^a_b(x+c)(= F^{a+c}_{b+c}(x))$.

For the convenience of readers, brief review of fractional calculus and logarithm of differentiation together with a proof of the variable change formula of $\log(\frac{d}{dx})$ (Prop.1, (4)) are given in §2. §3 proves variable change formula of fractional Euler differentiation (Th.1, (6)). As an application of (6) and (4), formal adjoint of fractional Euler differentiation is studied in §4. §5 deals with alternative definitions of fractional calculus. (4) and (6) suggest there might exist infinite order differential operator expressions of fractional order and logarithm of differentiations. Such expressions are given in [4] as applications of Leibniz rules and reviewed in §6 (Th.2, (10),(11)). We can regard $\frac{d^a}{dx^a}$ and x^a as deformed annihilation and creation operators acting on suitable Hilbert space. This is investigated in [4] and reviewed in §7. Corresponding discussions for $\log(\frac{d}{dx})$ and $\log x$ are given in §8 and higher commutation relations in the Lie algebra $\mathfrak{g}_{\log}$ generated by $\log(\frac{d}{dx})$ and $\log x$ are given as an application of (4) (Prop.3. cf.[4]). $G^\natural_{\log} = \mathbb{R} \ltimes G^\natural_\Gamma$ is the main part of the target of the exponential map from $\mathfrak{g}_{\log}$. As the preliminary of the study of structures of $G^\natural_{\log}$ and $G_{\log}$, the target of the exponential map from $\mathfrak{g}_{\log}$, we study Laplace transformations of $\mathfrak{d}_a$ and $\mathfrak{d}_{\log}$ in §9. This section also contains an alterantive proof of (6). Then we study strucures of $G^\natural_{\log}$ and $G_{\log}$ in §10, the last section.

Acknowledgement. Our original proof of (6) is based on (4) and stated in §9. Then we discovered simple proof of (6) which is stated in §3. Prof. Nakanishi also discoverd same simple proof of (6) simultaneously.

2 Review on fractional calculus

Definition 1. *Let a be a positive real number. We define the a-th order indefinite integral (from 0) by*

$$I^a f(x) = \frac{1}{\Gamma(a)} \int_0^x (x-t)^{a-1} f(t)dt. \quad (1)$$

Note. If a is a complex number with positive real part, then we can define a-th order indefinite integral by the same formula.

There are two kinds of definitions of frational order differentation:

$$\begin{aligned} \frac{d^{n-a} f(x)}{dx^{n-a}} &= \frac{d^n}{dx^n} I^a f(x), \quad 0 < a < 1, \\ \frac{d^{n-a} f(x)}{dx^{n-a}} &= I^a (\frac{d^n f}{dx^n})(x). \end{aligned}$$

The frist is called Riemann-Liouville fractional derivative and the second is called Caputo (or Riesz-Caputo) fractional derivative (cf.[1],[6]). They are different. But if we replace f be f_+;

$$f_+(x) = \begin{cases} f(x), \; x > 0, \\ 0, \; x \leq 0, \end{cases}$$

then this ambiguity is resolved. Because we have

$$I^a f(x) = \frac{1}{\Gamma(a)} (x^a)_+ * f_+, \quad f * g = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

As a price, we need to replace the constant function 1 by Y, the Heaviside function. The range of fractional differentiation needs to involve distribution;

$$\frac{df_+}{dx} = \frac{df}{dx} + f(0)\delta,$$

where δ is the Dirac function and $f(0)$ means $\lim_{x\downarrow 0} f(x)$.

If we take the space of Mikusinski's operators (cf.[10]) as the domain of fractional order differentiations, $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$, $\frac{d^{-a}}{dx^{-a}} = I^a$, becomes a 1-parameter group.

Definition 2. *We say the generating operator of the* 1*-parameter group* $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ *to be the logarithm of differentiation* $\log(\frac{d}{dx})$.

Explicitly, $\log(\frac{d}{dx})$ is given by

$$\log(\frac{d}{dx})f(x) = -(\log x + \gamma)f(x) - \int_0^x \log(x-t)\frac{df(t)}{dt}dt.$$

Here γ is the Euler constant and $\frac{df}{dx}$ means $\frac{df_+}{dx}$.

By the variable change $t = xs$, we have

$$I^a x^c = \frac{x^{c+a}}{\Gamma(a)}\int_0^1 (1-s)^{a-1}s^c ds = \frac{\Gamma(1+c)}{\Gamma(1+c+a)}x^{a+c}.$$

Hence we have

$$\frac{d^a}{dx^a}x^c = \frac{\Gamma(1+c)}{\Gamma(1+c-a)}x^{c-a}. \tag{2}$$

Here, we assume both of $1+c$ and $1+c-a$ are not 0 or negative integer.

(2) shows if a is not an integer, then $\frac{d^a}{dx^a}1 = \frac{1}{\Gamma(1-a)}x^{-a} \neq 0$.

Note. Since $\frac{1}{\Gamma(1+x)} = 0$, if x is a negative integer, $\frac{d^a}{dx^a}x^{a-n}$ vanishes if n is an integer. But in this case, we regard x^{a-n} is defined on $\mathbb{R}$. If we consider fractional derivatives are defined only for the functions on $\{x|x > 0\}$, then x^{a-n} is repalced to x_+^{a-n}. In this case, we have

$$\frac{d^a}{dx^a}x^{a-1} = \Gamma(a)\delta \neq 0, \quad 0 < a < 1,$$

etc.. Here δ is the Dirac function.

As for logarithm of differentiation, we have

$$\log(\frac{d}{dx})x^c = -\big(\log x + \gamma - \sum_{n=1}^{\infty} \frac{c}{n(n+c)}\big)x^c,$$
$$\log(\frac{d}{dx})x^n = -\big(\log x + \gamma - (1 + \frac{1}{2} + \cdots + \frac{1}{n})\big)x^n.$$

Especially, we have $\log(\frac{d}{dx})1 = -(\log x + \gamma) \neq 0$. We also have

$$\log(\frac{d}{dx})(\log x)^n = -(\log x + \gamma)(\log x)^n + + \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1} n! \zeta(n-k+1)}{k!} (\log x)^k. \quad (3)$$

Introducing the operator

$$\mathfrak{d}_{\log} = -\gamma + \sum_{n=1}^{\infty} (-1)^{n-1} \zeta(n+1) \frac{d^n}{dt^n},$$

we have

$$\log(\frac{d}{dx}) f(\log x) = (-t + \mathfrak{d}_{\log}) f(t)|_{t=\log x},$$

if $f(t)$ is a power series of t. Since

$$\log(\Gamma(1+X)) = -\gamma X + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} X^n,$$

we obtain

Proposition 1. *We can write*

$$\mathfrak{d}_{\log} = \big(\frac{d}{dX} \log(\Gamma(1+X))\big)|_{X=\frac{d}{dt}}. \quad (4)$$

3 Fractional Euler derivtive

Frational Euler derivative $x^a \frac{d^a}{dx^a}$ satisfies

$$x^a \frac{d^a}{dx^a}\big(x^b \frac{d^b}{dx^b}\big) = x^b \frac{d^b}{dx^b}\big(x^a \frac{d^a}{dx^a}\big).$$

But $\{x^a \frac{d^a}{dx^a} | a \in \mathbb{R}\}$ is not a 1-parameter group. Because

$$x^a \frac{d^a}{dx^a}(x^b \frac{d^b}{dx^b}) \neq x^{a+b} \frac{d^{a+b}}{dx^{a+b}}.$$

On the other hand, since $\frac{d}{da}(x^a \frac{d^a}{dx^a}) = \log x \cdot x^a \frac{d^a}{dx^a} + x^a \log(\frac{d}{dx}) \frac{d^a}{dx^a}$, we have

$$\frac{d}{da}(x^a \frac{d^a}{dx^a})|_{a=0} = \log(\frac{d}{dx}) + \log x. \tag{5}$$

Therefore we may say $\log(\frac{d}{dx}) + \log x$ is the logarithm of Euler differentiation.

By (2), we have

$$(x^a \frac{d^a}{dx^a}) x^c = \frac{\Gamma(1+c)}{\Gamma(1+c-a)} x^c.$$

Hence x^c is an eigenfunction of $\frac{d^a}{dx^a}$ if both of $1+c$ and $1+c-a$ are not equal to 0 or negative integer. x^c also satisfies

$$(\log(\frac{d}{dx}) + \log x) x^c = (-\gamma + \sum_{n=1}^{\infty} \frac{c}{n(n+c)}) x^c.$$

Theorem 1. *By the variable change* $\log x = t$, *we have*

$$(x^a \frac{d^a}{dx^a})|_{x=e^t} = \frac{\Gamma(1+X)}{\Gamma(1+X-a)}|_{X=\frac{d}{dt}}. \tag{6}$$

Proof. Since we have

$$\frac{\Gamma(1+c)}{\Gamma(1+c-a)} e^{ct} = (\frac{\Gamma(1+X)}{\Gamma(1+X-a)}|_{X=\frac{d}{dt}}) e^{ct},$$

we obtain

$$(x^a \frac{d^a}{dx^a} x^c)|_{x=e^t} = (\frac{\Gamma(1+X)}{\Gamma(1+X-a)}|_{X=\frac{d}{dt}}) e^{ct}.$$

Therefore, if $f(x)$ allows Taylor expansion, or more generally, if $f(x) = \int_{-\infty}^{\infty} x^s F(s) ds$, then

$$(x^a \frac{d^a}{dx^a} f(x))|_{x=e^t} = (\frac{\Gamma(1+X)}{\Gamma(1+X-a)}|_{X=\frac{d}{dt}}) f(e^t).$$

Hence we have Theorem.

Note. If a is not a natural number, then $\dfrac{\Gamma(1+X)}{\Gamma(1+X-a)}$ allows Taylor expansion at the origin;

$$\frac{\Gamma(1+X)}{\Gamma(1+X-a)} = c_0 + c_1 X + c_2 X^2 + \cdots .$$

In this case, (6) means

$$\frac{\Gamma(1+X)}{\Gamma(1+X-a)}|_{X=\frac{d}{dt}} = c_0 + c_1\frac{d}{dt} + c_2\frac{d^2}{dt^2} + \cdots .$$

Hence if the convergence radius of $\sum_{n=0}^{\infty} c_n X^n$ is r, then $\mathfrak{d}_a$ can act on finte exponetial type functions f which satisfy estimate $|f(t)| \leq Me^{q|t|}$, $q < r$, for some $M > 0$.

If $a = n$, a natural number, then

$$\begin{aligned} \Gamma(1+X) &= \Gamma(1+(X-n)+n) \\ &= X(X-1)\cdots(X-n+1)\Gamma(1+X-n). \end{aligned}$$

Hence we have the well known formula

$$(x^n\frac{d^n}{dx^n})|_{x=e^t} = \frac{d}{dt}(\frac{d}{dt}-1)\cdots(\frac{d}{dt}-n+1).$$

Note. Usually, this formula is shown by using $[\dfrac{d^m}{dx^m}, x] = m\dfrac{d^{m-1}}{dx^{m-1}}$ which provides

$$x^n\frac{d^n}{dx^n} = \prod_{m=0}^{n-1}(x\frac{d}{dx}-m).$$

The proof as a corollary of (6) is simpler than this proof.

Since

$$\frac{d}{da}(\frac{\Gamma(1+X)}{\Gamma(1+X-a)}) = \frac{\Gamma'(1+X-a)\Gamma(1+X)}{\Gamma(1+X-a)^2},$$

we have

$$(\log(\frac{d}{dx}) + \log x)|_{x=e^t} = \frac{\Gamma'(1+X)}{\Gamma(1+X)}|_{X=\frac{d}{dt}}.$$

Therefore, (4) follows from (6).

Note. (3) is shown directly from the definition of $\log(\dfrac{d}{dx})$. But by using (4), we get (3) easily. Therefore above alternative proof of (4) also provides simpler proof of (3).

4 Formal adjoints of fractional Euler diferentiations

If f, g belong to $W^k(\mathbb{R})$, the Sobolev k-space on $\mathbb{R}$, and $\lim_{|t|\to\infty} \frac{d^m f}{dt^m}(t) = 0$, $0 \le m \le k-1$, then we have

$$(\frac{d^k f}{dt^k}, g) = (f, (-1)^k \frac{d^k g}{dt^k}), \quad (f, g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt.$$

Hence we obtain

Proposition 2. *Let $\mathfrak{d}_a^*$ and $\mathfrak{d}_{\log}^*$ be*

$$\begin{aligned} \mathfrak{d}_a^* &= \frac{\Gamma(1-X)}{\Gamma(1-X-a)}|_{X=\frac{d}{dt}}, \\ \mathfrak{d}_{\log}^* &= (-\frac{d}{dX}\log(\Gamma(1-X)))|_{X=\frac{d}{dt}} = \frac{\Gamma'(1-X)}{\Gamma(1-X)}|_{X=\frac{d}{dt}}. \end{aligned}$$

Then we have

$$(\mathfrak{d}_a f, g) = (f, \mathfrak{d}_a^* g), \quad (\mathfrak{d}_{\log} f, g) = (f, \mathfrak{d}_{\log}^* g), \tag{7}$$

if f, g belong to $W^k(\mathbb{R})$ for all k and

$$\lim_{|x|\to\infty} \frac{d^k f}{dt^k}(t) = 0, \ k = 0, 1, 2, \dots .$$

By this Proposition, we may say $\mathfrak{d}_a^*$ and $\mathfrak{d}_{\log}^*$ to be formal adjoints of $\mathfrak{d}_a$ and $\mathfrak{d}_{\log}$. But to consider formal adjoints of fractional Euler differentiations, some remarks are necessary.

Fractiona differentition does not map polynomials to polynomials. So we can not consider fractional differentiation to be an operator of $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ = \{x | x > 0\}$. But fractional Euler differentiations map polynomials to polynomials and power serieses of convergence radius r to power serieses of same convergence radius. Hence we can regard Euler differntiations to be operaotrs of $L^2(\mathbb{R}_+)$. Since

$$\int_0^{\infty} f(x)\overline{g(x)}dx = \int_{\infty}^{\infty} f(e^t)\overline{g(e^t)}e^t dt,$$

by the variable change $\log x = t$, $L^2(\mathbb{R})$ does not mapped to $L^2(\mathbb{R})$, but mapped to

$$L^2(\mathbb{R}, e^t dt) = \{f | \int_{-\infty}^{\infty} |f|^2 e^t dt < \infty\}.$$

As an operator of $L^2(\mathbb{R}, e^t dt)$, formal adjoint of $\mathfrak{d}_a$ is not $\mathfrak{d}_a^*$, but $e^{-t}\mathfrak{d}_a^* e^t$, where $e^{\pm t}$ are considered to be linear operators by multiplication. Although $e^{-t}\frac{d^k}{dt^k}e^t$ is a constant coefficients differential opearotr, $e^{-t}\mathfrak{d}_a e^t$ is not a differential operator. Because $\mathfrak{d}_a$ is an infinite order differential operator. In fact, if we use Laplace transformation and adapt arguements of §9, §10, we have

$$e^{-t}\mathfrak{d}_a^* e^t = \frac{\Gamma(-X)}{\Gamma(-X-a)}|_{X=\frac{d}{dt}}.$$

Since $\Gamma(-X)$ has a pole of order 1 at $X = 0$, this shows $e^{-t}\mathfrak{d}_a e^t$ is not a differential operator, but sum of the indefinite integral operator and infinite order constant coefficients differential opertor.

Note. As for $\log(\frac{d}{dx}) + \log x$, we have same conclusion. In this case, we have

$$e^{-t}\mathfrak{d}_{\log}^* e^t = \frac{\Gamma'(-X)}{\Gamma(-X)}|_{X=\frac{d}{dt}}.$$

5 Alternative definitions of fractional calculus

We have derived (4) from (6). Conversely, we can derive (6) from (4). For this purpose, first we state alternative definitions of fractional calculus.

Definition by differences.

Let τ_h: $\tau_h f(x) = f(x+h)$ be the translation operator. As the operator on the space of power serieses, we have

$$\tau_h = e^{h\frac{d}{dx}} = 1 + \sum_{n=1}^{\infty} \frac{h^n}{n!}\frac{d^n}{dx^n}.$$

Hence we have $\tau_h^a = \tau_{ah}$ and $\log(\tau_h) = h\frac{d}{dx}$. We set

$$\begin{aligned}
\frac{\mathrm{d}^a}{\mathrm{d}x^a} f(x) &= \lim_{h\to 0} h^{-a}(\tau_h - \mathrm{I})^a f(x), \\
\log(\frac{\mathrm{d}}{\mathrm{d}x}) f(x) &= \lim_{h\to 0}(\log(\mathrm{I} - \tau_h) + \log h) f(x),
\end{aligned}$$

where I is the identity; $\mathrm{I}f = f$. Here, we define

$$\begin{aligned}
(\tau_h - \mathrm{I})^a &= \tau_{ah}(\mathrm{I} - \tau_{-h})^a \\
&= \tau_{ah}\big(\mathrm{I} + \sum_{n=1}^{\infty} \frac{(-1)^n a(a-1)\cdots(a-n+1)}{n!}\tau_{-nh}\big), \\
\log(\tau_h - \mathrm{I}) &= \log(\tau_h) + \log(\mathrm{I} - \tau_{-h}) = h\frac{d}{dx} + \sum_{n=1}^{\infty}\frac{\tau_{-nh}}{n}.
\end{aligned}$$

By definitions, $(\tau_h - \mathrm{I})^a f(x)$ and $\log(\tau_h - \mathrm{I})f(x)$ are finite sums if $f(x) = 0$, $x < 0$.

By definition, we have

$$e^{a\frac{\mathrm{d}}{\mathrm{d}x}} = \frac{\mathrm{d}^a}{\mathrm{d}x^a}.$$

Since $\tau_h = e^{h\frac{d}{dx}}$, we have

$$\log(\tau_h - \mathrm{I}) = h\frac{d}{dx} + \log h + \log(\frac{d}{dx}) + \log(G(h\frac{d}{dx})),$$

where $G(X) = \dfrac{1-e^{-X}}{X}$. Hence we have

$$\log(\frac{\mathrm{d}}{\mathrm{d}x}) = \log(\frac{d}{dx}).$$

Therefore we obtain $\dfrac{\mathrm{d}^a}{\mathrm{d}x^a} = \dfrac{d^a}{dx^a}$. But the classes of functions on which above definitions work, are not known.

Direct proof of the formula $e^{a\log(\frac{d}{dx})} = \dfrac{d^a}{dx^a}$.

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a holomorpphic function at the origin. We define its Borel transformation $\mathcal{B}[f]$ by

$$\mathcal{B}[f](z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n = \frac{1}{2\pi} \oint e^{\frac{z}{\zeta}} \frac{f(\zeta)}{\zeta} d\zeta.$$

Borel transformation is linear and satisfies

$$\frac{d}{dz}\mathcal{B}[f(\zeta)] = \mathcal{B}[\zeta^{-1}f(\zeta)], \quad \mathcal{B}[fg] = \mathcal{B}[f]\sharp\mathcal{B}[g],$$

where $u\sharp v = \dfrac{d}{dx}\displaystyle\int_0^x u(x-t)v(t)dt$. Inverse Borel transformation is given by

$$\mathcal{B}^{-1}[f](x) = \int_0^{\infty} e^{-t} f(xt)dt.$$

Since $\mathcal{B}^{-1}[\log\zeta] = \log z - \gamma$, $\mathcal{B}^{-1}[\zeta^a] = \Gamma(1+a)z^a$ and $\mathcal{B}^{-1}[\delta] = z^{-1}$, we define

$$\mathcal{B}[\log\zeta] = \log z + \gamma, \quad \mathcal{B}[\zeta^a] = \frac{z^a}{\Gamma(1+a)}.$$

Since we have

$$\lim_{\epsilon\to 0}\mathcal{B}[\log(\zeta+\epsilon)] = \log z + \gamma, \quad \lim_{\epsilon\to 0}\mathcal{B}[(\zeta+\epsilon)^a] = \frac{z^a}{\Gamma(1+a)},$$

only on $\{z|\Re z > 0\}$, the domain of these extended Borel transformation shoud be functions on $\{z|\Re z > 0\}$ or $\mathbb{C} \setminus \{x|x < 0\}$.

It is known ([3], cf.[12])

$$e^{\sharp t \log x} = \frac{e^{-\gamma t}}{\Gamma(1+t)} x^t, \quad e^{\sharp f} = \sum_{n=0}^{\infty} \frac{f^{\sharp n}}{n!},$$

where $f^{\sharp n} = \overbrace{f\sharp \cdots \sharp f}^{n}$. Hence main properties of Borel transformation are conserved in this extended Borel transformation ([3]). We define

$$\begin{aligned} \frac{d^a}{dz^a}\mathcal{B}[f(\zeta)](z) &= \mathcal{B}[\zeta^{-a} f(\zeta)](z), \\ \log(\frac{d}{dz})\lfloor[f(\zeta)](z) &= -\mathcal{B}[\log \zeta f(\zeta)](z). \end{aligned}$$

Then we have $e^{a\log(\frac{d}{dz})} = \frac{d^a}{dz^a}$ in one hand, and

$$-\mathcal{B}[\log \zeta f(\zeta)] = -(\gamma u + \int_0^z \log(z-t)\frac{du}{dt}dt),$$

if $u = \mathcal{B}[f]$, on the other hand. Hence we have $e^{a\log(\frac{d}{dx})} = \frac{d^a}{dx^a}$.

6 Leibniz rules and infinite order differential operator expressions

If g satisfies suitable condition, *e.g.*, g is a Gevrey class of index $\alpha < 1$;

$$|\frac{d^n}{dx^n} g(x)| < M_x (n!)^\alpha.$$

then by integration by parts, we have

$$I^1(fg) = (I^1 f)g - (I^2 f)g' + \cdots + (-1)^{n-1}(I^n f)g^{(n-1)} + \cdots,$$

We set $f_a(t) = \frac{(x-t)^{a-1}}{\Gamma(a)} f(t)$, where x is a parameter. Then we have

$$\begin{aligned} (I^n f_a)(x) &= \frac{1}{(n-1)!\Gamma(a)} \int_0^x (x-t)^{n+a-2} f(t)dt \\ &= \frac{\Gamma(n+a-1)}{(n-1)!\Gamma(a)} I^{n+a-1} f(x). \end{aligned}$$

Hence we obtain

$$\begin{aligned} I^a(fg) &= (I^a f)g - a(I^{a+1} f)g' + \cdots + \\ &\quad +(-1)^{n-1}\frac{a(a+1)\cdots(a+n-1)}{n!}(I^{a+n-1} f)g^{(n-1)} + \cdots. \end{aligned}$$

Then, by using $\frac{d^m}{dx^m}I^a = \frac{d^{m-a}}{dx^{m-a}}$, we obtain

$$\begin{aligned}\frac{d^a}{dx^a}(fg) &= \frac{d^a f}{dx^a}g + a\frac{d^{a-1}f}{dx^{a-1}}\frac{dg}{dx} + \cdots \\ &\quad + \frac{a(a-1)\cdots(a-n+1)}{n!}\frac{d^{a-n}f}{dx^{a-n}}\frac{d^n g}{dx^n} + \cdots . \end{aligned} \tag{8}$$

Taking $\frac{d}{da}$ of this formula, and tends a to 0, we have

$$\log(\frac{d}{dx})(fg) = \big(\log(\frac{d}{dx})f\big)g + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(I^n f)\frac{d^n g}{dx^n}. \tag{9}$$

Leibniz rules (8) and (9) are not symmetric in f and g, unless a is an integer (cf.[12]). These formulae valid for continuous f and Gevrey class function g of index $\alpha < 1$. But by these asymmetry, we obtain the following infinite order differential opearator expressions of $\frac{d^a}{dx^a}$, a is not a positive integer, and $\log(\frac{d}{dx})$ ([4]).

Theorem 2. *Let f be a Gevrey class function of index $\alpha < 1$, then*

$$\frac{d^a f}{dx^a} = \frac{x^{-a}}{\Gamma(1-a)}(f + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}a}{(n-a)n!}x^n\frac{d^n f}{dx^n}), \tag{10}$$

$$\log(\frac{d}{dx})f = -(\log x + \gamma)f + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\cdot n!}x^n\frac{d^n f}{dx^n}. \tag{11}$$

Proof. Since $f(x) = 1\cdot f(x)$ and

$$\frac{d^{a-n}}{dx^{a^n}}1 = \frac{1}{\Gamma(n+1-a)}x^{n-a}, \quad \log(\frac{d}{dx})1 = -(\log x + \gamma),$$

we have Theorem by (8) and (9).

Note. (10) have no meanings if a is a natural number. But we have

$$\lim_{a\to m}\frac{x^{-a}}{\Gamma(1-a)}(f + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}a}{(n-a)n!}x^n\frac{d^n f}{dx^n}) = \frac{d^m f}{dx^m},$$

if $x \neq 0$.

Problem. Derive (6) and (4) directly from (10) and (11).

(10) and (11) show fractional order and logarithm of differentiations can not be defined as a 1-valued operator for functions on $\mathbb{R}$. As for functions on $\mathbb{R}$, we need to use

$$\frac{d^a}{dx^a{}_\pm} f(x) = \frac{x_\pm^{-a}}{\Gamma(1-a)}(f(x)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-a)n!}x^n\frac{d^n f(x)}{dx^n}), \quad (12)$$

$$\log_\pm(\frac{d}{dx})f(x) = -(\log_\pm x+\gamma)f(x)+\sum_{n=1}^{\infty}\frac{(^1)^{n-1}}{n\cdot n!}x^n\frac{d^n f(x)}{dx^n}. \quad (13)$$

Here, we set

$$x_\pm^a = \begin{cases} x^a, & x>0, \\ e^{\pm i\pi a}|x|^a, & x<0, \end{cases} \qquad \log_\pm x = \begin{cases} \log x, & x>0, \\ \log|x| \pm \pi i, & x<0. \end{cases}$$

But fractional Euler differentations and $\log(\frac{d}{dx}) + \log x$ can be defined for functions on $\mathbb{R}$ (or on $\mathbb{C}$).

7 Commutation relations and deformed CCR

By (8) and (9), regarding x to be a multiplication operator, we have

$$[\frac{d^a}{dx^a}, x] = a\frac{d^{a-1}}{dx^{a-1}}, \quad [\log(\frac{d}{dx}), x] = I^1.$$

Note. It is known $[F(\frac{d}{dx}), x] = F'(\frac{d}{dx})$, where

$$F(\frac{d}{dx}) = \sum_{n=0}^{\infty} c_n \frac{d^n}{dx^n}, \quad F(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Above formulae show this formula is valid for $F(x) = x^a$ and $F(x) = \log x$.

These formulae are not fit to regard fractional order or logarithm of differentiations as deformed annihilation operators. As for $\frac{d^a}{dx^a}$, $0 < a < 1$, appropriate space of the domain to investigate deformation of CCR is

$$\mathrm{H}_a = \{\sum_{n=1}^{\infty} c_n x^{an-1} | \sum_{n=1}^{\infty} |c_n|^2 < \infty\},$$

([4]. As for deformation of Heisenberg algebras, we refer [13]).

In this definition, $a \neq 0$ is arbitray and H_1 is the Hardy space H, when it is considered to be a complex Hilbert space. But in the rest, we assume

$0 < a < 1$. In the later discussions, there are no essential differences between H_a as a real Hilbert space and as a complex Hilbert space. Inner product (f, g), $f, g \in \mathrm{H}_a$ is defined by

$$(f, g) = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} f(z)\overline{g(z)} d\theta.$$

Here $z = re^{i\theta}$ is the compelx variable, and the integral is taken on $-\pi/a < \theta < \pi/a$, which is regarded as a cover of the unit circle $\{z||z| = 1\}$. By this inner product, H_a is a Hilbert space and $\{x^{an-1}|n \in \mathbb{N}\}$ is a complete ortho-normal basis of H_a. The map ρ_a:$\rho_a(x^{an-1}) = x^{n-1}$ defines an isometry from H_a to H.

x^a does not belong to H_a. But as the multiplication operator, we can define x^a; $x^a(x^{an-1}) = x^{a(n+1)-1}$. The adjoint $x^{a\dagger}$ is x^{-a}; $x^{-a}(x^{an-1}) = x^{a(n-1)-1}$, $n \geq 2$. Since $\displaystyle\int_{-\pi/a}^{\pi/a} x^{an-1}\overline{x^{-1}} d\theta = 0$, it should be $x^{-1} = 0$ as an operator on H_a. Hence we set

$$x^{-a}(x^{a-1}) = 0.$$

As an operator on H_a, we set

$$\frac{d^a}{dx^a} x^{a-1} = 0.$$

Owing to this definition, we do not have $(\frac{d^a}{dx^a})^m = \frac{d^{am}}{dx^{am}}$ in general. But by this definition, to set

$$e_{a,t}(x) = \sum_{n-1}^{\infty} \frac{t^{n-1}}{\Gamma(an)} x^{an-1} = x^{a-1} E_{a,a}(tx^a),$$

where $E_{\alpha,\beta}(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha + \beta)}$ is the generalized Mittag-Leffler function (cf.[5]), we have

$$\frac{d^a}{dx^a} e_{\alpha,t}(t) = t e_{\alpha,t}(x).$$

This solution suggests appropriate boundary condition of the equation $\dfrac{d^a y}{dx^a} = \lambda y$ is

$$y(e^{-\pi i/a}) = te^{2\pi i/a} y(e^{\pi i/a}), \quad E_{a,a}(\lambda) = -t.$$

Of course, these are only special results. As for generalities of fractional differntial equations, we refer [14] (cf.[11]).

Let $A_{a,\pm}$ be diagonal form operators on H_a defined by

$$A_{a,+}x^{an-1} = \frac{\Gamma(a(n+1))}{\Gamma(an)}x^{an-1}, \quad A_{a,-}x^{an-1} = \frac{\Gamma(an)}{\Gamma(a(n-1))}x^{an-1}.$$

Here, we consider $A_{a,-}x^{a-1} = 0$. Then we have

$$\frac{d^a}{dx^a} = x^{-a}A_{a,-} = A_{a,+}x^{-a}.$$

Hence $\frac{d^a}{dx^a}^{\dagger} = A_{a,-}x^a = x^a A_{a,+}$ and

$$[\frac{d^a}{dx^a}, x^a] = A_{a,+} - A_{a,-}.$$

We set $C_a = A_{a,+} - A_{a,-}$. It is a p-Schatten class operator (cf.[16]) if $p > \frac{1}{1-a}$. We have $\lim_{a\to 1} \rho_a C_a \rho_a^{-1} = \mathrm{I}$ by the strong topology of operators. But $\{\rho_a C_a \rho_a^{-1} | a > 0\}$ does not converge by the uniform topology.

We regard $\frac{d^a}{dx^a}$ and x^a as deformed annihilation and creation operators. Then the Lie algebra $\mathfrak{g}_a$ generated by $\frac{d^a}{dx^a}$ and x^a is a projective limit of nilpotent Lie algebras, and its higher order elements belong to the Schatten ideal ([4]). Precisely saying $\mathfrak{g}_a$ is a real Lie algebra if we consider H_a to be a real Hilbert space and complex Lie algebra if we consider H_a to be a complex Hilbert space. If necessary we fix $\mathfrak{g}_a$ as a real Lie algebra and denote $\mathfrak{g}_{a,\mathbb{C}}$ the complex Lie algebra $\mathfrak{g}_a \otimes \mathbb{C}$.

Note. We do not consider topology of $\mathfrak{g}_a$. Hence an element u of $\mathfrak{g}_a$ is written as the form

$$u = c_0 X_0 + \sum_{k=1}^{m} c_k \overbrace{[X_{k_1}, \cdots [X_{k_k}}^{k}, X_{k_{k+1}}] \cdots],$$

where X_0, X_{i_j} are either $\frac{d^a}{dx^a}$ or x^a.

By the variable change $\log x = t$, we have $\frac{d^a}{dx^a} = e^{-at}\mathfrak{d}_a$, $x^a = e^{at}$. Hence $\mathfrak{g}_a$ is isomorphic to the Lie algebra generated by $e^{-at}\mathfrak{d}_a$ and e^{at}. Therefore we have

$$C_a|_{x=e^t} = e^{-at}\mathfrak{d}_a e^{at} - \mathfrak{d}_a.$$

Since $e^{-at}\frac{d^n}{dt^n}e^{at} = \sum_{k=0}^{n} \frac{a^k n!}{k!(n-k)!}\frac{d^{n-k}}{dt^{n-k}}$ and $|a| < 1$, C_a is changed to an infinite order constant coefficients differential operator by the variable change $\log x = t$.

8 Lie algebra generated by logarithm of differentiation and $\log x$

As for $\log(\frac{d}{dx})$, appropriate domain to investigate deformed canonical commutation relation should be spanned by $\{(\log x)^n | n = 0, 1, \dots\}$. We set

$$\mathrm{H}_{\log} = \{\sum_{n=0}^{\infty} c_n (\log x)^n | \sum_{n=0}^{\infty} |c_n|^2 < \infty\}.$$

If we take x to be a complex variable and set $x = e^{i\theta}$, $-\pi < \theta < \pi$, then $\mathrm{H}_{\log}$ is isometric to $W^{1/2}[-\pi, \pi]$, the Sobolev 1/2-space on $[-\pi, \pi]$. But we do not use this identification in the rest.

We can not regard $\log(\frac{d}{dx})$ to be a deformed annihilation operator. But we can regard $\mathcal{R} = \log(\frac{d}{dx}) + \log x + \gamma$ and $\log x$ to be deformed annihilation operator and creation operator.

Definition 3. *We denote the Lie algebra generated by* $\log(\frac{d}{dx})$ *and* $\log x$ *by* $\mathfrak{g}_{\log}$ *and the Lie algebra generated by* $\mathcal{R}$ *and* $\log x$ *by* $\mathfrak{g}_{\mathcal{R}}$.

By definitions, they are isomorphic and

$$\mathfrak{g}_{\log} \oplus \mathbb{R}\mathrm{I} = \mathfrak{g}_{\mathcal{R}} \oplus \mathbb{R}\mathrm{I}.$$

As for commutation relation, we have

$$[\log(\frac{d}{dx}), \log x] = [\mathcal{R}, \log x].$$

To compute higher commutation relations in $\mathfrak{g}_{\log}$, it is convenient to use variable change $\log x = t$. Then $\mathfrak{g}_{\log}$ is isomorphic to the Lie algebra generated by $-t + \mathfrak{d}_{\log}$ and t. Since $\mathfrak{d}_{\log} = \big(\frac{d}{dX} \log(\Gamma(1+X))\big)|_{X=\frac{d}{dt}}$, we have

$$\begin{aligned} & \overbrace{[t, [\cdots .}^{m}[t, \mathfrak{d}_{\log}] \cdots] = \big(\frac{d^{m+1}}{dX^{m+1}} \log(\Gamma(1+X))\big)|_{X=\frac{d}{dt}} \\ = \ & (-1)^m (m+1)! \zeta(m+2) I + \sum_{k=1}^{\infty} (-1)^{k+m} \frac{(k+m)!}{k!} \zeta(k+m+1) \frac{d^k}{dt^k}. \end{aligned}$$

We set $\mathfrak{d}_{\log} = \mathfrak{d}_{\log,0}$ and

$$\mathfrak{d}_{\log,m} = \big(\frac{d^{m+1}}{dX^{m+1}} \log(\Gamma(1+X))\big)|_{X=\frac{d}{dt}}.$$

Then we obtain (cf.[4])

Proposition 3. *Regarding* $\mathfrak{g}_{\log}$ *to be a Lie algebra generated by* t *and* $\mathfrak{d}_{\log}$, *its element* u *is written uniquely in the form*

$$u = at + c_0\mathfrak{d}_{\log,0} + c_1\mathfrak{d}_{\log,1} + \cdots + c_m\mathfrak{d}_{\log,m}. \tag{14}$$

Especially, $[u, v] = 0$, *if* $u = \sum_k c_k\mathfrak{d}_{\log,k}$ *and* $v = \sum_k c'_k\mathfrak{d}_{\log,k}$.

In the original $\mathfrak{g}_{\log}$, we have

$$\overbrace{[\log x, [\cdots, [}^{m}\log x, \log(\frac{d}{dx})]\cdots] = (-1)^m(m+1)!\zeta(m+2)\mathrm{I} + N_{\log,m}.$$

Here $N_{\log,m}$ is a generalized nilpotent operator: $N_{\log,m}(\log x)^k = 0$, $k \leq m$. Special values of the ζ-function appeared in this formula come from the Taylor expansion of $\log(\Gamma(1+x))$. So may not be interesting.

$\mathfrak{g}_{\log}$ is a projective limit of nilpotent Lie algebras. If the exponential map is defined for $\mathfrak{g}_{\log}$, the image of $\mathfrak{g}_{\log}$ should contain the group generated by the 1-parameter groups $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ and $\{x^a|a \in \mathbb{R}\}$.

Definition 4. *We denote the group generated by the* 1*-parameter groups* $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ *and* $\{x^a|a \in \mathbb{R}\}$ *by* $G^\natural_{\log}$.

We can clarify the structure of $G^\natural_{\log}$ as an application of (4) and (6). By using Laplace transformation and Proposition 3, it is shown the target of the exponential map from $\mathfrak{g}_{\log}$ is generated by $G^\natural_{\log}$ and $e^{a\frac{d^{m+1}}{ds^{m+1}}\log(\Gamma(1+s)))}$, $a \in \mathbb{R}$, $m = 0, 1, \ldots$.

Definition 5. *The group generated by* $G^\natural_{\log}$ *and* $e^{a\frac{d^{m+1}}{ds^{m+1}}}$, $a \in \mathbb{R}$, $m = 0, 1, \ldots$ *is denoted by* $G_{\log}$.

We can clarify the structure of $G_{\log}$ as applications of (4),(6) and Proposition 3. For this purpose, first we study Laplace transformation of $\mathfrak{d}_{\log}$, *etc.*, which also provides an alternative proof of (6).

9 Laplace transformations of $\mathfrak{d}_{\log}$ and $\mathfrak{d}_a$

By the variable change $\log x = t$, the domain $\{x|x > 0\}$ is mapped to $\{-\infty < t < \infty\}$. So we need to use birateral Laplace transformation. For the convenience, we set

$$\mathcal{L}[f](t) = \int_{-\infty}^{\infty} e^{st}f(s)ds.$$

If f is a rapidly decreasing function, then

$$t\int_{-\infty}^{\infty} e^{st}f(s)ds = \int_{-\infty}^{\infty}(\frac{d}{ds}e^{st})f(s)ds = -\int_{-\infty}^{\infty} e^{st}f'(s)ds.$$

Hence we have

$$(-t+\mathfrak{d}_{\log})\mathcal{L}[f(s)](t) = \mathcal{L}[(\frac{d}{ds}+\frac{\Gamma'(1+s)}{\Gamma(1+s)})f(s)](t).$$

Solution of the equation

$$\frac{dY(s)}{ds}+\frac{\Gamma'(1+s)}{\Gamma(1+s)}Y(s) = \lambda Y(s),$$

is given by

$$Y(s) = C\frac{e^{\lambda s}}{\Gamma(1+s)}, \quad C \in \mathbb{C}.$$

Hence if the inverse Laplace transformation of

$$\frac{e^{\lambda s}}{\Gamma(1+s)} = e^{(\gamma+\lambda)s}\prod_{n=1}^{\infty}((1+\frac{s}{n})e^{-\frac{s}{n}},$$

exists, we obtain solutions of the equation $(-t+\mathfrak{d}_{\log})U(t) = \lambda U(t)$.

An alternative proof of (6).

By using above solution $Y(s)$, we can derive (6) from (4) as follows: Since $e^{a\log(\frac{d}{dx})} = \frac{d^a}{dx^a}$, if

$$\log(\frac{d}{dx})u_\lambda(x) = \lambda u(x),$$

then

$$\frac{d^a}{dx^a}u_\lambda(x) = e^{a\lambda}u_\lambda(x).$$

To set $u_\lambda(\exp t) = U_\lambda(t)$, by the variable change $\log x = t$, we have

$$e^{-at}\mathfrak{d}_a U_\lambda(t) = e^{a\lambda}U_\lambda(t).$$

If $\mathcal{L}[f] = g$ is sufficiently smooth, then we have

$$\mathcal{L}[e^{at}f(t)](s) = e^{-a\frac{d}{ds}}g(s) = g(s-a). \tag{15}$$

By this formula, we obtain

$$(\mathfrak{d}_a)|_{\frac{d}{dt}=s}Y_\lambda(s) = Y_\lambda(s-a).$$

Since $Y_\lambda(s) = e^{\lambda s}(\Gamma(1+s))^{-1}$, we have

$$(\mathfrak{d}_a)|_{\frac{d}{dt}=s} = \frac{\Gamma(1+s)}{\Gamma(1+s-a)}.$$

Therefore (6) follows from (4).

Problem. By (6) and (4), we have

$$\frac{d^a}{dx^a}|_{x=e^t} = e^{-at}\mathfrak{d}_a, \quad \log(\frac{d}{dx})|_{x=e^t} = -t+\mathfrak{d}_{\log}.$$

Can we show directly $e^{a(-t+\mathfrak{d}_{\log})} = e^{-at}\mathfrak{d}_a$?

10 Structures of groups $G^{\natural}_{\log}$ and $G_{\log}$

(8) shows the commutation relation of x^a and $\frac{d^b}{dx^b}$ is not simple. But by using variabel change $\log x = t$ and Laplace transformation, we can investigate on such commutation relations.

For the convenience, we set

$$F^a_b(x) = \frac{\Gamma(1+x+a)}{\Gamma(1+x+b)}.$$

For example, we have

$$\mathfrak{d}_a = F^0_{-a}(X)]|_{X=\frac{d}{dt}}, \quad \mathcal{L}[\mathfrak{d}_a](x) = F^0_{-a}(x).$$

By definition, we have $F^a_b(x)F^c_d(x) = F^c_d(x)F^a_b(x)$ and

$$F^a_a(x) = 1, \quad F^a_b(x)F^b_c(x) = F^a_c(x).$$

Definition 6. *We denote $G^{\natural}_{\Gamma}$, the group generated by $\{F^a_b(x)|a, b \in \mathbb{R}\}$ by multiplication.*

By definition, $G^{\natural}_{\Gamma}$ is homomorphic to the quotient group of the free abelian group

$$\mathbb{Z}_{\mathbb{R}^2} = \sum_{(a,b)\in\mathbb{R}^2} \oplus\mathbb{Z}(a,b),$$

by the relations

$$(a,a) = 1, \quad (a,b) + (b,c) = (a,c).$$

By (11), as an operator, $\mathcal{L}[\exp(at)] = \tau_{-a}$. Hence as an operator, we have

$$\mathcal{L}[\frac{d^a}{dx^a}|_{x=e^t}] = \tau_a F^0_{-a}.$$

Regarding F^a_b to be an operator by multiplication, the commutation relation between τ_c and F^a_b is

$$\tau_c F^a_b = F^{a+c}_{b+c}\tau_c, \quad F^a_b\tau_c = \tau_c F^{a-c}_{b-c}. \tag{16}$$

By the variable change $\log x = t$ and Laplace transformation, $G^{\natural}_{\log}$ is isomorphic to the group generated by $\{\tau_c|c \in \mathbb{R}\}$ and $\{\tau_a F^0_{-a}|a \in \mathbb{R}\}$. By (16), we obtain

$$\begin{aligned} & \tau_{c_1}\tau_{a_1}F^0_{-a_1}\cdots\tau_{c_n}\tau_{a_n}F^0_{-a_n} \\ = \ & \tau_{a_1+\cdots+a_n+c_1+\cdots+c_n}F^{-(a_2+\cdots+a_n+c_2+\cdots+c_n)}_{-a_1-(a_2+\cdots+a_n+c_2+\cdots+c_n)}\cdots F^0_{-a_n}. \end{aligned}$$

Therefore we have

Theorem 3. $G_{\log}^{\natural}$ *is the crossed product*

$$G_{\log}^{\natural} \cong \mathbb{R} \ltimes G_{\Gamma}^{\natural}, \tag{17}$$

where $c \in \mathbb{R}$ *acts on* $G_{\Gamma}^{\natural}$ *as the translation* τ_c.

Corollary $G_{\log}^{\natural}$ *is a solvable group of solvable length* 2.

Proof. Since we have

$$\tau_c F_{b_1}^{a_1} \cdots F_{b_n}^{a_n} = F_{b_1+c}^{a_1+c} \cdots F_{b_n+c}^{a_n+c} \tau_c,$$

$G_{\Gamma}^{\natural}$ is a normal subgroup of $G_{\log}^{\natural}$. Then by (16), we have

$$G_{\log}^{\natural} / G_{\Gamma}^{\natural} \cong \mathbb{R}. \tag{18}$$

Since $G_{\Gamma}^{\natural}$ is an abelian group, we have Corollary.

Note 1. Since

$$F_b^a \tau_c F_a^b = \tau_c F_{b+c}^{a+c} F_a^b \notin \{\tau_a | a \in \mathbb{R}\},$$

$\{\tau_a | a \in \mathbb{R}\}$ is not a normal subgroup of $G_{\log}^{\natural}$.

Note 2. Let $G_{\Gamma, r_1, \ldots, r_k}$ be the subgroup of $G_{\log}^{\natural}$ generated by $\{F_b^a | a-b \in \{r_1, \ldots, r_k\}\}$. Then $G_{\Gamma, r_1, \ldots, r_k}$ is also a normal subgroup of $G_{\log}^{\natural}$.

By Proposition 3, regarding $\mathfrak{g}_{\log}$ geenrated by t and $\mathfrak{d}_{\log}$, elements obtained by higher commution relations are linear combinartion of

$$\mathfrak{d}_{\log,m} = \Big(\frac{d^{m+1}}{dX^{m+1}} \log(\Gamma(1+X))\Big|_{X=\frac{d}{dt}}.$$

We denote Laplace transfromation of $\mathfrak{d}_{\log,m}$ by $g_m(s)$; $g_m(s) = \frac{d^{m+1}}{ds^{m+1}} \log(\Gamma(1+s))$, and set

$$G_m(s) = e^{g_m(s)}, \quad G_{m,h}(s) = G_m(h+s).$$

By definition, we have

$$G_{m,h}(s)^a = e^{a g_m(s+h)} = e^{a \frac{d^{m+1}}{ds^{m+1}} \log(\Gamma(1+s+h))}.$$

By the map

$$\prod_{i=1}^k G_{m,h_i}^{a_i} \to \sum_{i=1}^k a_i \langle h_i \rangle,$$

we can identify the group $G_{\log,m}$, generated by $G_m(s)$ and $\mathbb{R}$, which act as tranlations, by the multiplication, and $\mathbb{R}^{\mathbb{R}}$, the $\mathbb{R}$-vector space having elements of $\mathbb{R}$ as the basis. For the convenience, we denote $\mathbb{R}^{\mathbb{R}}_m$, the vector space $\mathbb{R}^{\mathbb{R}}$ identified to $G_{\log,m}$.

We set $G^{\flat}_{\Gamma} = \prod_{m=0}^{\infty} G_m$. Then

$$G^{\flat}_{\Gamma} \cong \sum_{m=0}^{\infty} \oplus \mathbb{R}^{\mathbb{R}}_m.$$

Definition 7. *We set*

$$G_{\Gamma} = G^{\natural}_{\Gamma} \times G^{\flat}_{\Gamma}. \tag{19}$$

Note. Since G_{Γ} is an abelian group, we may write $G_{\Gamma} = G_{\Gamma} \oplus G^{\flat}_{\Gamma}$.

Since elements of G_{Γ} are (analytic) functions on $\mathbb{R}$, elements of $\mathbb{R}$ act on G_{Γ} as translations. We set

$$G_{\log} = \mathbb{R} \ltimes G_{\Gamma}. \tag{20}$$

By definition, we can define the exponential map $\exp : \mathfrak{g}_{\log} \to G_{\Gamma}$ by $\exp(u) = e^u$, regarding $\mathfrak{g}_{\log}$ is generated by t and $\mathfrak{d}_{\log}$ and the map is defined on its images by Laplace transforamtion. Therefore we obtain

Theorem 4. *We can take $G_{\log}$ as the target of the exponential map from $\mathfrak{g}_{\log}$. It is a solvable group of derived solvable length* 2.

Proof. Since we have $\exp(\mathfrak{g}_{\log}) \subset G_{\log}$ and

$$G_{\log}/G_{\Gamma} \cong \mathbb{R},$$

we have Theorem, because G_{Γ} is an abelian group.

Note 1. In these discussions, we regard $\mathfrak{g}_{\log}$ to be a real Lie algebra. As for complex Lie algebra $\mathfrak{g}_{\log} = \mathfrak{g}^{\mathbb{C}}_{\log}$, we have same resuluts, replacing $G^{\natural}_{\Gamma}$ the group geenrated by $\{F^a_b(z) | a, b \in \mathbb{C}\}$, which is denoted by $G^{\natural,\mathbb{C}}_{\Gamma}$, *etc.*, and set

$$G^{\natural,\mathbb{C}}_{\log} = \mathbb{C} \ltimes G^{\natural,\mathbb{C}}_{\Gamma},$$

etc..

Note 2. We do not consider topologies of $G_{\log}$, *etc.*. Our realization of $G_{\log}$ is not so useful to the study of its algebraic structure. But to give good toplogies to $\mathfrak{g}_{\log}$ and $G_{\log}$ so that the completion of $\mathfrak{g}_{\log}$ becomes the Lie algebra of the completion of $G_{\log}$, our realizations must be useful.

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