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SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS AND THEIR ASSOCIATED NAMBU VECTOR FIELDS

NOBUTADA NAKANISHI

Department of Mathematics, Gifu Keizai University,
5-50 Kitagata, Ogaki-city, Gifu, 503-8550, Japan
e-mail address: nakanisi@gifu-keizai.ac.jp

ABSTRACT. From a point of view of Nambu-Poisson geometry, we consider the condition when the associated Lagrange vector field with a given system of ordinary differential equations becomes a Nambu vector field. As a result, we know that this condition is deeply related to Jacobi's last multiplier.

1. INTRODUCTION

Let $(\mathbb{R}^n, \eta)$ be the *standard Nambu-Poisson manifold.* Here η is the standard Nambu-Poisson structure, which is written as $\eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$ for the standard coordinates $x_1, \cdots, x_n$ of $\mathbb{R}^n$. Let $\Omega = dx_1 \wedge \cdots \wedge dx_n$ be the standard volume form on $\mathbb{R}^n$. Then η defines *Nambu bracket* $\{g_1, g_2, \cdots, g_n\}$ for any $g_1, g_2, \cdots, g_n \in C^\infty(\mathbb{R}^n)$ by $\{g_1, g_2, \cdots, g_n\} = \eta(dg_1, dg_2, \cdots, dg_n)$.

Since Nambu bracket is nothing but the Jacobian of n functions $g_1, \cdots, g_n$, we can define a *Nambu vector field* $X_{g_1 \wedge \cdots \wedge g_{n-1}}$ by

$$X_{g_1 \wedge \cdots \wedge g_{n-1}}(g) = \{g, g_1, \cdots, g_{n-1}\}, \tag{1}$$

for any $g \in C^\infty(\mathbb{R}^n)$.

Now let us consider the following system of ordinary differential equations on $\mathbb{R}^n$:

$$\frac{dx_1}{f_1} = \frac{dx_2}{f_2} = \cdots = \frac{dx_n}{f_n} = dt, \tag{2}$$

where each f_i is a given function of $x_1, x_2, \cdots, x_n$. If there exist $n-1$ functions $H_1, H_2, \cdots, H_{n-1}$ of $x_1, x_2, \cdots, x_n$ such that

$$\frac{dx_i}{dt} = f_i = \{x_i, H_1, H_2, \cdots, H_{n-1}\}, \tag{3}$$

for $i = 1, 2, \cdots, n$, then (2) (or (3)) is called *a Nambu system.* In this case, it is easy to see that each H_j is time-independent.

Let $X = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ be the associated vector field of (2). S.Codriansky *et al.* [1] considered the following problem: Under what conditions does X become a Nambu vector field? P.Morando [5] studied the same problem as ours from the viewpoint of differential geometry.

If X is a Nambu vector field, the divergence of X is clearly 0 with respect to Ω. And this is a necessary condition for X to be a Nambu vector field. This condition is called *Liouville condition* for X. Later on as one of our main results, we will show that there exists a function A such that the following system:

$$\frac{dx_1}{Af_1} = \frac{dx_2}{Af_2} = \cdots = \frac{dx_n}{Af_n} = \frac{dt}{A} \tag{4}$$

becomes a Nambu system even if (2) is *not* a Nambu system. Put $Y = \sum_{i=1}^{n} Af_i \frac{\partial}{\partial x_i}$. Since Y is a Nambu vector field, its divergence vanishes. Hence a function A becomes a *Jacobi's last multiplier.* For details of Jacobi's last multipliers, and for related topics, see for example, M. Crâşmăreanu [2] and M. C. Nucci and P. G. L. Leach [7].

Another main result is to show that there are no non-trivial Nambu vector fields for certain autonomous linear differential equations. This is a generalization of the result of S.Codriansky *et al.* [1].

The set of Nambu vector fields is contained in the Lie algebra $\mathcal{L}$ of infinitesimal automorphisms of Nambu-Poisson structure, but it does not become a subspace of $\mathcal{L}$.

2. Nambu-Poisson geometry

Though we should consider the problems stated in the Introduction on a *general* Nambu-Poisson manifold, here we will confine ourselves to the standard Nambu-Poisson manifold by taking into account Theorem 2.1 (the local structure theorem). The details will be given at the end of this section.

Let us survey Nambu-Poisson geometry quickly. (See, for example, N. Nakanishi [6].) Let M be a smooth m-dimensional manifold and $C^\infty(M)$ the algebra of real-valued C^∞-functions on M. We denote by $\Gamma(\Lambda^n TM)$ the space of sections from M to $\Lambda^n TM$. Each element of $\Gamma(\Lambda^n TM)$ is simply called *n-vector.* Then each n-vector η defines a bracket of functions $g_i \in C^\infty(M)$ by

$$\{g_1, \cdots, g_n\} = \eta(dg_1, \cdots, dg_n).$$

This bracket also defines the vector field $X_{g_1\wedge\cdots\wedge g_{n-1}}$ by

$$X_{g_1\wedge\cdots\wedge g_{n-1}}(g) = \{g, g_1, \cdots, g_{n-1}\}, \quad g \in C^\infty(M).$$

Let $Q = \sum f_{i_1} \wedge \cdots \wedge f_{i_{n-1}}$ be an element of the space $\Lambda^{n-1}C^\infty(M)$. Then a vector field X_Q is also defined by the same manner as $X_{g_1\wedge\cdots\wedge g_{n-1}}$. Such a vector field X_Q is called *a Hamiltonian vector field.* By abuse of language, we also denote by $\mathcal{H}$ the space of Hamiltonian vector fields.

Definition 2.1. *An element η of $\Gamma(\Lambda^n TM), n \geq 3$, is called a Nambu -Poisson structure of order n if η satisfies*

$$L_{X_{g_1\wedge\cdots\wedge g_{n-1}}}\eta = 0,$$

for any Hamiltonian vector field $X_{g_1\wedge\cdots\wedge g_{n-1}}$. And a pair (M, η) is called a Nambu-Poisson manifold. The space of infinitesimal automorphisms of η is written as $\mathcal{L}$. It is clear that $\mathcal{H}$ is an ideal of $\mathcal{L}$.

This definition was proposed by L. Takhtajan [9] in 1994. If $n = 2$, this is nothing but the definition of usual Poisson structure. (See, for example, [10]).

Definition 2.2. *If Q is a monomial, say, $Q = g_1 \wedge \cdots \wedge g_{n-1}$, then $X_Q = X_{g_1 \wedge \cdots \wedge g_{n-1}}$ is called a Nambu vector field, and each function g_i is called a Hamiltonian. The set of Nambu vector fields is a subset of $\mathcal{H}$, but it is not a subspace of $\mathcal{H}$.*

In studying the geometry of Nambu-Poisson manifolds, the following theorem, which is called "local structure theorem" is fundamental. (See [3], [6].) Let $\eta(x) \neq 0, x \in M$. Then η is said to be *regular* at x, and x is called *a regular point.*

Theorem 2.1. *If η is a Nambu-Poisson structure of order $n \geq 3$, then for any regular point x, there exists a coordinate neighbourhood U with local coordinates $(x_1, \cdots, x_n, x_{n+1}, \cdots, x_m)$ around x such that*

$$\eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$$

on U, and vice versa.

The most typical example of a Nambu-Poisson structure is

$$\eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$$

defined on $\mathbb{R}^m$, and it is called *the standard Nambu-Poisson structure.* The above theorem means that a Nambu-Poisson manifold is locally considered to be the standard Nambu-Poisson manifold $(\mathbb{R}^m, \eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n})$.

If $m > n$, a vector field $X = \sum_{i=1}^m h_i \frac{\partial}{\partial x_i}$ with $h_k \neq 0$ for some $n+1 \leq k \leq m$ does not become a Nambu vector field. In fact, suppose that X would be a Nambu vector field: $X = X_{g_1 \wedge \cdots \wedge g_{n-1}}$. Then for $k \geq n+1$,

$$X(x_k) = \{x_k, g_1, \cdots, g_{n-1}\} = \frac{\partial(x_k, g_1, \cdots, g_{n-1})}{\partial(x_1, \cdots, x_n)} = 0.$$

On the other hand, $X(x_k) = h_k \neq 0$. Hence this is the contradiction.

Therefore from now on we mainly consider the case $(\mathbb{R}^n, \eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n})$, because this is the only meaningful case, when we study whether a given vector field is a Nambu vector field or not.

3. Results

Now we give a generalization of the results of S.Codriansky *et al.* [1]. Let us consider an n-th order autonomous differential equation:

(5) $$x^{(n)} = F(x, x', x'', \cdots, x^{(n-1)}).$$

Put $x_k = x^{(k-1)}$. Then (5) is rewritten as follows:

(6) $$x_1' = x_2, x_2' = x_3, \cdots, x_n' = F(x_1, x_2, \cdots, x_n),$$

or

(7) $$\frac{dx_1}{x_2} = \frac{dx_2}{x_3} = \cdots = \frac{dx_n}{F} = dt.$$

The associated vector field X is given by

(8) $$X = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + \cdots + F \frac{\partial}{\partial x_n}.$$

If X satifies the Liouville condition, F must depend only on $x_1, \cdots, x_{n-1}$. Moreover we assume here that F is a non-zero linear function, so F is of the following form:

$$F = a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1}, \quad a_1, \cdots, a_{n-1} \in \mathbb{R}. \tag{9}$$

So from now on we study the following equation:

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_3} = \cdots = \frac{dx_n}{a_1x_1 + \cdots + a_{n-1}x_{n-1}}. \tag{10}$$

Then the characteristic equation of (5) is written as

$$r^n - b_{n-1}r^{n-2} - \cdots - b_2r - b_1 = 0. \tag{11}$$

Let $r_i (1 \leq i \leq l)$ be the distinct roots of the characteristic equation (11). Then the general solution of the differential equation (10) is given by the linear combination of n linearly independent solutions $\alpha_1, \alpha_2, \cdots, \alpha_n$. Each of them has the form $t^{k_i}e^{r_it}, (0 \leq k_i \leq s_i)$. Here $s_i + 1$ is the multiplicity of r_i. Another expression of x is as follows:

$$x = x_1 = c_{11}\alpha_1 + c_{12}\alpha_2 + \cdots + c_{1n}\alpha_n = \sum_{i=1}^{l} P_i(t)e^{r_it}, \tag{12}$$

where c_{1j} are constants and each $P_i(t)$ is a polynomial of degree s_i and $n = s_1 + s_2 + \cdots + s_l + l$.

Once x_1 is given by (12), we can calculate $x_2, \cdots, x_n$ one after another. Each x_j is given by

$$x_j = c_{j1}\alpha_1 + c_{j2}\alpha_2 + \cdots + c_{jn}\alpha_n. \tag{13}$$

Hence by solving these equations with respect to α_j, we know that each α_j should be expressed as a homogeneous linear function L_j of variables $x_1, x_2, \cdots, x_n$. Using the relations among $\alpha_1, \cdots, \alpha_n$, we can eliminate time-variable t and we obtain $(n-1)$ time-independent integrals. Then we use them to define $(n-1)$ Hamiltonians $H_1, H_2, \cdots, H_{n-1}$. Note that each $H_j(x_1, \cdots, x_n)$ is a function of these combinations of $L's$.

The following lemma was first proved for the case of a linear vector field (8) satisfying the condition (9), and after that H.Suzuki proved for a general homogeneous linear vector field. The proof of the following lemma is due to H.Suzuki [8].

Lemma 3.1. *Let X be a homogeneous linear vector field. If X is a Nambu vector field with Hamiltonians $H_1, H_2, \cdots, H_{n-1}$, and if we write it by*

$$X = X_{H_1 \wedge H_2 \wedge \cdots \wedge H_{n-1}}, \tag{14}$$

then there exist $n-2$ homogeneous linear functions $\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{n-2}$ and a homogeneous quadratic function $\tilde{H}_{n-1}$ such that $X = X_{\tilde{H}_1 \wedge \tilde{H}_2 \wedge \cdots \wedge \tilde{H}_{n-1}}$.

Proof. Put $w = i(X)\Omega$. Then w is a homogeneous linear $(n-1)$ form by our assumption, where Ω is the standard volume form of $\mathbb{R}^n$. Decompose each H_i as follows:

$$H_i = H_i^{(0)} + H_i^{(1)} + \cdots,$$

where $H_i^{(k)}$ denotes a homogeneous polynomial of degree k. The constant term of w, which is denoted by $w^{(0)}$, is given by

$$w^{(0)} = dH_1^{(1)} \wedge dH_2^{(1)} \wedge \cdots \wedge dH_{n-1}^{(1)},$$

and since $w^{(0)} = 0$, we know that $dH_1^{(1)}, dH_2^{(1)}, \cdots, dH_{n-1}^{(1)}$ are linearly dependent. Thus without loss of generality, we can write $dH_{n-1}^{(1)}$ as follows:

$$dH_{n-1}^{(1)} = c_1 dH_1^{(1)} + c_2 dH_2^{(1)} + \cdots + c_{n-2} dH_{n-2}^{(1)},$$

where $c_1, c_2, \cdots, c_{n-2}$ are constants. Put $\bar{H} = H_{n-1} - c_1 H_1 - c_2 H_2 - \cdots - c_{n-2} H_{n-2}$, then $w = dH_1 \wedge dH_2 \wedge \cdots \wedge dH_{n-2} \wedge d\bar{H}$ and $\bar{H}$ has no homogeneous linear part. Hence if we put $\tilde{H}_1 = H_1^{(1)}, \tilde{H}_2 = H_2^{(1)}, \cdots, \tilde{H}_{n-2} = H_{n-2}^{(1)}$, and $\tilde{H}_{n-1} = \bar{H}^{(2)}$, then $d\tilde{H}_1 \wedge d\tilde{H}_2 \wedge \cdots \wedge d\tilde{H}_{n-1}$ is equal to the linear part of $w = dH_1 \wedge dH_2 \wedge \cdots \wedge dH_{n-2} \wedge d\bar{H}$. Recall that w itself is a homogeneous linear $n-1$ form. Thus we have $w = d\tilde{H}_1 \wedge d\tilde{H}_2 \wedge \cdots \wedge d\tilde{H}_{n-1}$, and this means that $X = X_{\tilde{H}_1 \wedge \tilde{H}_2 \cdots \wedge \tilde{H}_{n-1}}$. □

Recall that X satisfies $\mathrm{div}(X) = 0$. Our first problem is: Under what conditions can we find Hamiltonians $H_1, \cdots, H_{n-1}$ so that X satisfies $X = X_{H_1 \wedge \cdots \wedge H_{n-1}}$?

First in the case of $n = 2, 3$, we will try to find Nambu vector fields. If $n = 2$, the differential equation is given by

$$\frac{dx_1}{x_2} = \frac{dx_2}{a_1 x_1} = dt. \tag{15}$$

Since $X = x_2 \frac{\partial}{\partial x_1} + a_1 x_1 \frac{\partial}{\partial x_2}$, we can easily find a Hamiltonian $H = \frac{1}{2}(x_2^2 - a_1 x_1^2)$, and it holds that $X = X_H$.

The case of $n = 3$ was investigated in [1]. The differential equation and the associated vector field are given by

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_3} = \frac{dx_3}{a_1 x_1 + a_2 x_2} = dt, \tag{16}$$

$$X = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + (a_1 x_1 + a_2 x_2) \frac{\partial}{\partial x_3}. \tag{17}$$

Suppose that $X = X_{H_1 \wedge H_2}$. By Lemma 3.1, we can assume

$$H_1 = c_{11} x_1 + c_{12} x_2 + c_{13} x_3, \tag{18}$$

$$H_2 = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_1 x_3 + c_4 x_2^2 + c_5 x_2 x_3 + c_6 x_3^2. \tag{19}$$

Since $\frac{dH_1}{dt} = 0$, we have

$$c_{11} + c_{13} a_2 = 0, \quad c_{12} = 0, \quad c_{13} a_1 = 0. \tag{20}$$

If $c_{13} = 0$, we would have $H_1 = 0$. But this is not the case, so we must have $c_{13} \neq 0$, and we have $a_1 = 0$. If we take $c_{11} = 1$, then we obtain $H_1 = x_1 - \frac{x_3}{a_2}$.

Similarly, since H_2 is also time-independent, we have

$$2c_1 + c_3 a_2 = 0, \quad c_2 = 0, \quad c_2 + c_5 a_2 = 0, \tag{21}$$

$$c_3 + 2c_4 + 2c_6 a_2 = 0, \quad c_5 = 0. \tag{22}$$

So if we take $c_1 = 0$ and $c_4 = \frac{a_2}{2}$, we obtain $H_2 = \frac{1}{2}(a_2 x_2^2 - x_3^2)$. ($H_2$ is also obtained directly from (16) since we already know that $a_1 = 0$.) As the result, in the case of $n = 3$, we must have $a_1 = 0$ and $X = X_{H_1 \wedge H_2}$.

Next as a generalization of the results of S.Codriansky *et al.* [1], we show that there does not exist a Nambu vector field if $n \geq 4$.

Theorem 3.2. *For the system of autonomous ordinary differential equations*

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_3} = \cdots = \frac{dx_n}{a_1x_1 + \cdots + a_{n-1}x_{n-1}} = dt, \tag{23}$$

let $X = x_2\frac{\partial}{\partial x_1} + \cdots + (a_1x_1 + \cdots + a_{n-1}x_{n-1})\frac{\partial}{\partial x_n}$ *be the associated vector field. Then if* $n \geq 4$, X *does not become a Nambu vector field.*

Proof. Suppose that $n \geq 4$ and that there exist $n-1$ Hamiltonians $H_1, \cdots H_{n-1}$ such that $X = X_{H_1\wedge\cdots\wedge H_{n-1}}$. By Lemma 3.1, $H_i, (1 \leq i \leq n-2)$ can be denoted by

$$H_i = c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{i,n-1}x_{n-1} + c_{in}x_n. \tag{24}$$

Since $dH_i/dt = 0$, we must have

$$0 = c_{i1}x_2 + \cdots + c_{i,n-1}x_n + c_{in}(a_1x_1 + \cdots + a_{n-1}x_{n-1}). \tag{25}$$

This is equivalent to

$$a_1c_{in} = 0, \;\; c_{i1} + a_2c_{in} = 0, \;\; \cdots, c_{i,n-2} + a_{n-1}c_{in} = 0, \;\; c_{i,n-1} = 0. \tag{26}$$

If $c_{in} = 0$, we would have $c_{i1} = c_{i2} = \cdots = c_{in} = 0$ and $H_i = 0$. Hence it must hold that $c_{in} \neq 0$, and that $a_1 = 0$. This means that H_i has the following form:

$$H_i = c_{i1}x_1 + \cdots c_{i,n-2}x_{n-2} + c_{in}x_n, \;\; (1 \leq i \leq n-2). \tag{27}$$

Since $i(X)\Omega$ contains the term $x_{n-2}dx_1 \wedge \cdots \wedge dx_{n-4} \wedge dx_{n-2} \wedge dx_{n-1} \wedge dx_n$, so does $i(X_{H_1} \wedge \cdots \wedge H_{n-1})\Omega$. Recalling that $H_1, \cdots, H_{n-2}$ are linear functions which do not contain the term x_{n-1} by (27) and that H_{n-1} is a quadratic function, we know that H_{n-1} must contain the term $x_{n-2}x_{n-1}$. On the other hand, the condition $dH_{n-1}/dt = 0$ implies that the coefficient of x_n^2 is 0 and hence also implies that the coefficient of x_{n-1}^2 is 0 in the expression of dH_{n-1}/dt. This means that the term $x_{n-2}x_{n-1}$ is not contained in the expression of H_{n-1}. This is the contradiction. □

Let us show another differential equation which becomes a Nambu system only for special cases.

Proposition 3.3. *Let* F *be a homogeneous polynomial of degree* k, $k \geq 2$, *which is defined on* $\mathbb{R}^3$. *Then the differential equation*

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_3} = \frac{dx_3}{F} \tag{28}$$

becomes a Nambu system if and only if $F = ax_1^{k-1}x_2$, $(a \in \mathbb{R})$. *In this case, the following* H_1 *and* H_2 *are the desired Hamiltonians:*

$$\begin{cases} H_1 &= x_3 - \frac{a}{k}x_1^k, \\ H_2 &= x_1x_3 - \frac{1}{2}x_2^2 - \frac{a}{k+1}x_1^{k+1}. \end{cases}$$

And the associated Nambu vector field is given by

$$X = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + ax_1^{k-1}x_2\frac{\partial}{\partial x_3} = X_{H_1\wedge H_2}. \tag{29}$$

Proof. We give here outline of proof. Let the associated vector field X be a Nambu vector field: $X = X_{H_1\wedge H_2}$. Then X satisfies the Liouville condition, we have

$\partial F/\partial x_3 = 0$. Put $F = a_1x_1^k + a_2x_1^{k-1}x_2 + a_3x_1^{k-2}x_2^2 + \cdots + a_kx_1x_2^{k-1} + a_{k+1}x_2^k$. Let $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ be the standard volume form on $\mathbb{R}^3$. Since it holds:

$$\begin{aligned} i(X)\Omega = i(X_{H_1\wedge H_2})\Omega &= dH_1 \wedge dH_2 \\ &= x_2dx_2 \wedge dx_3 + x_3dx_3 \wedge dx_1 + Fdx_1 \wedge dx_2, \end{aligned}$$

the coefficients of $dH_1 \wedge dH_2$ are linear with respect to x_3. Hence we can put $H_1 = f_0 + x_3f_1$, $H_2 = g_0 + x_3g_1 + x_3^2g_2$, where f_0, f_1, g_0, g_1 and g_2 are polynomial functions of x_1 and x_2. H_1 is time-independent, so we have

$$0 = \frac{dH_1}{dt} = x_3^2\frac{\partial f_1}{\partial x_2} + x_3(\frac{\partial f_0}{\partial x_2} + x_2\frac{\partial f_1}{\partial x_1}) + Ff_1 + \frac{\partial f_0}{\partial x_1}x_2. \tag{30}$$

By comparing the coefficients of x_3^2 and x_3, we obtain that $F = ax_1^{k-1}x_2$. (Here we put $a = a_2$). Substituting this F into the given differential equation, we can easily determine H_1 and H_2. □

Let us show that we can find a function A such that AX becomes a Nambu vector field for a *non* Nambu vector field X. The following theorem is essentially due to C.G.J.Jacobi (See for example [7]).

Theorem 3.4. *Let* $(\mathbb{R}^n, \eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n})$ *be the standard Nambu-Poisson manifold, and let*

$$\frac{dx_1}{f_1} = \frac{dx_2}{f_2} = \cdots = \frac{dx_n}{f_n} = dt \tag{31}$$

be the system of ordinary differential equation (ODE for short) on $(\mathbb{R}^n, \eta)$, *where* $f_i = f_i(x_1, \cdots, x_n)$, $(1 \le i \le n)$ *are given functions on* $\mathbb{R}^n$. *Suppose that the system (31) has* $n-1$ *time independent integrals* $H_1, \cdots, H_{n-1}$ *which are functionally independent one another. Then there exists a function* A *such that the following ODE:*

$$\frac{dx_1}{Af_1} = \frac{dx_2}{Af_2} = \cdots = \frac{dx_n}{Af_n} = \frac{dt}{A} \tag{32}$$

becomes a Nambu system. Put $Y = \sum_{j=1}^n Af_j\frac{\partial}{\partial x_j}$. *Then* Y *becomes a Nambu vector field and* Y *is expressed as* $Y = Y_{H_1\wedge\cdots\wedge H_{n-1}}$.

Proof. Since H_i is time-independent, we have

$$0 = \frac{dH_i}{dt} = \sum_{j=1}^n \frac{\partial H_i}{\partial x_j}\frac{dx_j}{dt} = \sum_{j=1}^n \frac{\partial H_i}{\partial x_j} \cdot f_j. \tag{33}$$

Put $a_{ij} = \partial H_i/\partial x_j$, and moreover put

$$\tilde{a}_j = \begin{pmatrix} a_{1j} \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1,j} \end{pmatrix}.$$

Since $H_1, \cdots, H_{n-1}$ are functionally independent, we can assume without loss of generality that rank $T = n-1$, where $T = (\tilde{a}_1, \cdots, \tilde{a}_{n-1})$.

Since (33) is equivalent to the following:

$$(\tilde{a}_1 \cdots \tilde{a}_{n-1}) \cdot \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{n-1} \end{pmatrix} = -f_n \tilde{a}_n,$$

we get the following relation:

$$f_j = (-1)^{n-j} \frac{f_n}{\det T} \cdot \det(\tilde{a}_1 \cdots \tilde{a}_{j-1} \tilde{a}_{j+1} \cdots \tilde{a}_n), \quad 1 \leq j \leq n-1. \tag{34}$$

Define a function A by $A = (-1)^{n-1} \frac{\tilde{A}}{f_n}$, where $\tilde{A} = \det T$. Then we have

$$\det(\tilde{a}_1 \cdots \tilde{a}_{j-1} \tilde{a}_{j+1} \cdots \tilde{a}_n) = f_j \cdot \frac{\det T}{f_n} \cdot \frac{1}{(-1)^{n-j}} = (-1)^{j-1} A f_j. \tag{35}$$

Using the relation (35), the following holds:

$$\begin{aligned} & dH_1 \wedge dH_2 \wedge \cdots \wedge dH_{n-1} \\ & = \det(\tilde{a}_2 \cdots \tilde{a}_n) dx_2 \wedge \cdots \wedge dx_{n-1} \\ & + \det(\tilde{a}_1 \tilde{a}_3 \cdots \tilde{a}_n) dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n \\ & + \cdots + \det(\tilde{a}_1 \cdots \tilde{a}_{n-1}) dx_1 \wedge \cdots \wedge dx_{n-1} \\ & = A f_1 \cdot dx_2 \wedge \cdots \wedge dx_n - A f_2 \cdot dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n \\ & + \cdots + (-1)^{n-1} A f_n \cdot dx_1 \wedge \cdots \wedge dx_{n-1}. \end{aligned}$$

Put $Y = \sum_{j=1}^n A f_j \frac{\partial}{\partial x_j}$. Let $\Omega = dx_1 \wedge \cdots \wedge dx_n$ be the standard volume form of $\mathbb{R}^n$. Then

$$i(Y)\Omega = dH_1 \wedge \cdots \wedge dH_{n-1} = i(Y_{H_1 \wedge \cdots \wedge H_{n-1}})\Omega. \tag{36}$$

Thus we get $Y = Y_{H_1 \wedge \cdots \wedge H_{n-1}}$. □

Corollary 3.5. *Let $X = \sum_{j=1}^n f_j \frac{\partial}{\partial x_j}$ be the associated vector field with the ODE system (31). Then X becomes a Nambu vector field i.e., $X = X_{H_1 \wedge \cdots \wedge H_{n-1}}$ with respect to the new Nambu-Poisson structure $\tilde{\eta} = \frac{1}{A} \cdot \eta$. In particular, X is a Nambu vector field if and only if we can find $(n-1)$ Hamiltonians $H_1, \cdots, H_{n-1}$ such that $A = 1$.*

Remark 3.1. In the given system of ODE (31) of Theorem 3.4, assume that each $f_i = f_i(x_1, \cdots, x_n)$ is a function of C^1-class. Then it is well-known that (31) has n general solutions with n arbitrary constants $C_1, \cdots, C_n$:

$$\begin{cases} x_1 = & \phi_1(t, C_1, \cdots, C_n), \\ & \cdots \\ x_n = & \phi_n(t, C_1, \cdots, C_n). \end{cases} \tag{37}$$

By eliminating a variable t from the above relations (37), $n-1$ functions $H_1, \cdots, H_{n-1}$ are obtained, which are time-independent and functionally independent one another.

4. Examples

1. Let us consider a 6-dimensional ODE system:

$$\frac{dx_1}{x_4} = \frac{dx_2}{x_5} = \frac{dx_3}{x_6} = \frac{dx_4}{0} = \frac{dx_5}{0} = \frac{dx_6}{0} = dt. \tag{38}$$

This is an ODE system of *motion of free particles.* The associated vector field

$$X = x_4\frac{\partial}{\partial x_1} + x_5\frac{\partial}{\partial x_2} + x_6\frac{\partial}{\partial x_3}$$

satisfies the Liouville condition, but X is not a Nambu vector field.

Five integrals of (38) are easily obtained:

$$H_1 = x_1x_5 - x_2x_4, \quad H_2 = x_2x_6 - x_3x_5, \quad H_3 = x_4, \quad H_4 = x_5, \quad H_5 = x_6. \tag{39}$$

Using the above five integrals, we have

$$\begin{pmatrix} -x_4 & 0 & -x_2 & x_1 & 0 \\ x_6 & -x_5 & 0 & -x_3 & x_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_5 \\ x_6 \\ 0 \\ 0 \\ 0 \end{pmatrix} = -x_4 \cdot \begin{pmatrix} x_5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Put

$$T = \begin{pmatrix} -x_4 & 0 & -x_2 & x_1 & 0 \\ x_6 & -x_5 & 0 & -x_3 & x_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\tilde{A} = \det T = x_4x_5$, we have $A = f_1/\tilde{A} = x_5$. Hence by Theorem 3.4, $Y = AX$ becomes a Nambu vector field on $(\mathbb{R}^6, \eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_6})$, and $Y = Y_{H_1\wedge\cdots\wedge H_5}$. Or equivalently, X becomes a Nambu vector field on $(\mathbb{R}^6, \frac{1}{x_5}\eta)$.

2. S.Codriansky *et al.*[1] studied the following 3-dimensional ODE system:

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_3} = \frac{dx_3}{a_1x_1 + a_2x_2} = dt. \tag{40}$$

The associated vector field is

$$X = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + (a_1x_1 + a_2x_2)\frac{\partial}{\partial x_3},$$

and they found that X becomes a Nambu vector field if and only if $a_1 = 0$.

Here we consider the case: $a_1 = 1$ and $a_2 = 0$. So the given system is not a Nambu system, and is given by

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_3} = \frac{dx_3}{x_1} = dt. \tag{41}$$

Then the solutions of (41) are given by

$$\begin{cases} x_1 &= c_1e^t + c_2e^{\omega t} + c_3e^{\omega^2 t}, \\ x_2 &= c_1e^t + c_2\omega e^{\omega t} + c_3\omega^2e^{\omega^2 t}, \\ x_3 &= c_1e^t + c_2\omega^2e^{\omega t} + c_3\omega e^{\omega^2 t}, \end{cases}$$

where ω is a cube root of 1 which is not 1.

Using x_1, x_2 and x_3, we get the following two integrals:

$$\begin{cases} H_1 &= \frac{1}{\omega-1} \cdot \frac{(x_1+x_2+x_3)^{\omega}}{\omega x_1+x_2+\omega^2 x_3^{\omega^2}}, \\ H_2 &= \frac{1}{\omega-1} \cdot \frac{(x_1+x_2+x_3)^{\omega}}{\omega x_1+\omega^2 x_2+x_3}. \end{cases}$$

Since

$$T = \begin{pmatrix} \partial H_1/\partial x_1 & \partial H_1/\partial x_2 \\ \partial H_2/\partial x_1 & \partial H_2/\partial x_2 \end{pmatrix}$$

we have

$$\tilde{A} = \det T = \frac{1-\omega^2}{\omega} \cdot \frac{x_1}{(x_1^3+x_2^3+x_3^3-3x_1x_2x_3)^2}.$$

Thus

$$A = \frac{\tilde{A}}{x_1} = \frac{1-\omega^2}{\omega \cdot (x_1^3+x_2^3+x_3^3-3x_1x_2x_3)^2}.$$

Then by Theorem 3.4, $Y = AX$ becomes a Nambu vector field: $Y = Y_{H_1 \wedge H_2}$ on a manifold $(\mathbb{R}^3, \eta = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3})$. Equivalently X becomes a Nambu vector field: $X = X_{H_1 \wedge H_2}$ on a manifold $(\mathbb{R}^3, \frac{1}{A}\eta)$.

Remark 4.1. The differential system (41) is not a Nambu system, and the associated vector field $X = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}$ is not a Nambu vector field with $\mathrm{div}(X) = 0$. But X is a Hamiltonian vector field in our sense:

$$X = X_{G_1 \wedge G_2 + H_1 \wedge H_2 + K_1 \wedge K_2} \in \mathcal{H}, \tag{42}$$

where $G_1 = \frac{1}{2}x_3^2$, $G_2 = x_1$, $H_1 = \frac{1}{2}x_2^2$, $H_2 = x_3$, $K_1 = \frac{a_1}{2}x_1^2 + a_2x_1x_2$, $K_2 = x_2$. This fact is guaranteed by the following proposition. (See [6].)

Proposition 4.1. *Let (M, η) be an m-dimensional Nambu-Poisson manifold with non-vanishing η of order m. Then $\mathcal{L}/\mathcal{H}$ is isomorphic to $H_{dR}^{m-1}(M)$.*

3. Let us consider the 2D isotropic harmonic oscillator. It is defined by

$$\frac{dx_1}{-x_3} = \frac{dx_2}{-x_4} = \frac{dx_3}{x_1} = \frac{dx_4}{x_2} = dt. \tag{43}$$

It is easy to find 3 Hamiltonians:

$$\begin{cases} H_1 &= x_1x_4 - x_2x_3, \\ H_2 &= \frac{1}{2}(x_1x_2 + x_3x_4), \\ H_3 &= \frac{1}{2}(x_1^2 + x_3^2 - x_2^2 - x_4^2) \end{cases}$$

The associated vector field $X = -x_3\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4}$ is not a Nambu vector field. The matrix expression corresponding to (33) in Theorem 3.4 is:

$$\begin{pmatrix} x_4 & -x_3 & -x_2 \\ \frac{1}{2}x_2 & \frac{1}{2}x_1 & \frac{1}{2}x_4 \\ x_1 & -x_2 & x_3 \end{pmatrix} \cdot \begin{pmatrix} -x_3 \\ -x_4 \\ x_1 \end{pmatrix} = -x_2 \cdot \begin{pmatrix} x_1 \\ \frac{1}{2}x_3 \\ -x_4 \end{pmatrix}.$$

Hence we have $A = -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$. Thus we obtain the Nambu vector field $Y = AX = Y_{H_1 \wedge H_2 \wedge H_3}$ on $(\mathbb{R}^4, \eta = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$. Or equivalently we have $X = X_{H_1 \wedge H_2 \wedge H_3}$ on $(\mathbb{R}^4, \frac{1}{A}\eta)$.

4. The differential equation

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_1} = \frac{dx_3}{x_3} = dt$$

is not a Nambu system. In fact, the associated vector field $X = x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$ does not satisfy the Liouville condition. We can easily find two Hamiltonians: $H_1 = \frac{1}{2}(x_1^2 - x_2^2)$, and $H_2 = \frac{x_1+x_2}{x_3}$. Following the necessary procedures of Theorem 3.4, we have the last multiplier $A = \frac{x_1+x_2}{x_3^2}$. Then $Y = AX = Y_{H_1\wedge H_2}$ is a Nambu vector field on $(\mathbb{R}^3, \eta = dx_1 \wedge dx_2 \wedge dx_3)$, and div $(Y) = 0$.

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