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The falling cat as a port-controlled Hamiltonian system

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Abstract

The falling cat is modeled as jointed axial symmetric cylinders with arbitrary twist under the condition of the vanishing total angular momentum. There are three steps to the control of this model. The first step is a geometric setting on the configuration space of this model. The configuration space is made into an $SO(3)$ bundle and endowed with a connection together with a metric. The base space of this bundle is called the shape space. The second is to formulate the falling cat in the Hamiltonian formalism with the constraint of the vanishing total angular momentum. The last step is to set up the falling cat as a port-controlled Hamiltonian system, which is defined on the cotangent bundle of the shape space. A control is then designed according to the standard procedure. The whole motion of the falling cat is obtained after integrating the constraint equation of the vanishing total angular momentum.

1 Introduction

A model of the falling cat was set up in [1], and is called the Kane-Scher model, which was studied further in a geometric manner in [2] by Montgomery. In this article, the Kane-Scher model is extended a bit to include the freedom of twist. Geometric setting for the present model is in the line of [2], but the control inputs are designed in a manner different from that in [2]. A point to make here is that control inputs are torques. This article adopts the port-controlled Hamiltonian system (see [3]).

Three steps toward the control of the falling cat are as follows: (1) Geometry: The configuration space for the falling cat is made into an $SO(3)$ bundle. A key idea of somersault is the parallel translation associated with a connection on the $SO(3)$ bundle, which is linked with a first-order differential equation. (2) Mechanics: The equations of motion, second-order differential equations, arise from a Lagrangian with the constraint of the vanishing total angular momentum (*i.e*, parallel translation) along with torque inputs applied. (4) Control: The Lagrangian system with constraint and torques is translated into a port-controlled Hamiltonian system. The energy shaping technique provides a method for designing controls.

2 Geometric setting

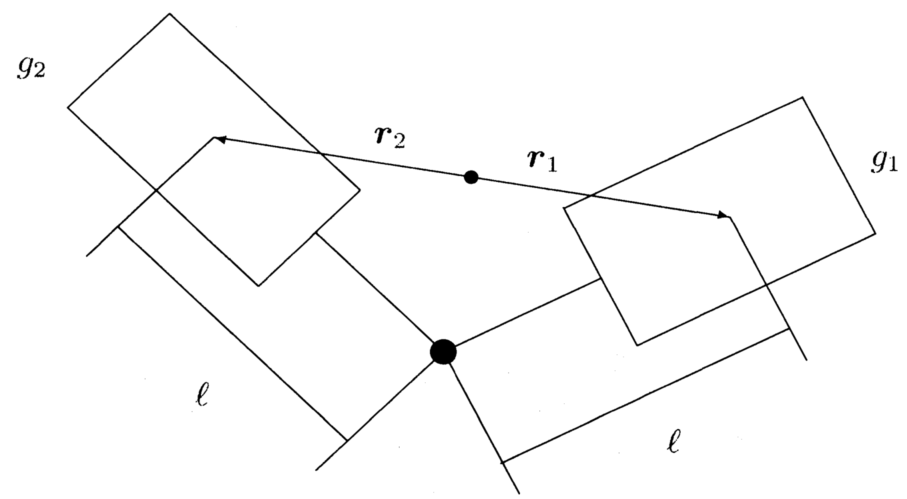

Figure 1: Jointed cylinders

Two identical axial symmetric cylinders are jointed together by a special type of joint that will give no constraints on the relative motion of the cylinders other than that they are jointed. The center-of-mass of the whole system is assumed to be fixed at the origin of $\mathbf{R}^3$. Let ℓ be the distance between the joint and the center-of-mass of each cylinder. Then the configuration space of this model is given by

$$X_0 = \{(\boldsymbol{r}_1, \boldsymbol{r}_2, g_1, g_2)\} \subset \mathbf{R}^3 \times \mathbf{R}^3 \times SO(3) \times SO(3), \tag{1}$$

where

$$\boldsymbol{r}_1 = \frac{\ell}{2}(g_1\boldsymbol{e}_3 - g_2\boldsymbol{e}_3) = -\boldsymbol{r}_2, \tag{2}$$

and where $\boldsymbol{e}_a$, $a = 1, 2, 3$, are the standard basis vectors of $\mathbf{R}^3$. It is obvious that $X_0 \cong SO(3) \times SO(3)$.

Since $SO(3)$ acts freely on X_0 to the left, X_0 is made into an $SO(3)$-bundle over $M := X_0/SO(3)$;

$$\pi: \; X_0 \cong SO(3) \times SO(3) \to M \cong SO(3), \quad \pi(g_1, g_2) = g_1^{-1}g_2. \tag{3}$$

Let $q = e^{-\theta_1\widehat{\boldsymbol{e}}_3}e^{\psi\widehat{\boldsymbol{e}}_1}e^{-\theta_2\widehat{\boldsymbol{e}}_3} \in U \subset SO(3) \cong M$, where $\widehat{\boldsymbol{e}}_a = R(\boldsymbol{e}_a)$ is the anti-symmetric matrix defined through $R(\boldsymbol{e}_a)\boldsymbol{x} = \boldsymbol{e}_a \times \boldsymbol{x}$ and where the open subset U is assigned by

$$0 < \psi < \pi, \quad 0 < \theta_1 < 2\pi, \quad 0 < \theta_2 < 2\pi.$$

Then, a local section $\sigma: U \subset M \to X_0$ is defined to be

$$\sigma(q) = (\boldsymbol{\sigma}_1(q), \boldsymbol{\sigma}_2(q), k_1(q), k_2(q)), \quad \begin{cases} \boldsymbol{\sigma}_1(q) = \boldsymbol{e}_2\ell\sin\dfrac{\psi}{2}, \\ \boldsymbol{\sigma}_2(q) = -\boldsymbol{e}_2\ell\sin\dfrac{\psi}{2}, \\ k_1(q) = e^{-\frac{\psi}{2}\widehat{\boldsymbol{e}}_1}e^{\theta_1\widehat{\boldsymbol{e}}_3}, \\ k_2(q) = e^{\frac{\psi}{2}\widehat{\boldsymbol{e}}_1}e^{-\theta_2\widehat{\boldsymbol{e}}_3}, \end{cases} \tag{4}$$

which is realized with the jointed cylinders shown in Fig.2.

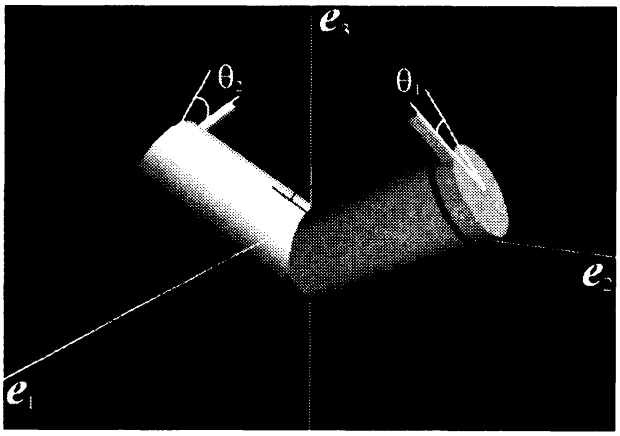

Figure 2: The realization of the local section $\sigma(q)$

It is to be noted that the angles $(\psi, \theta_1, \theta_2)$ determine the shape of the jointed cylinders. Since $\pi^{-1}(U) \cong U \times SO(3)$, $x \in \pi^{-1}(U) \subset X_0$ is put in the form

$$x = g\sigma(q) = (g\boldsymbol{\sigma}_1(q), g\boldsymbol{\sigma}_2(q), gk_1(q), gk_2(q)), \quad g \in SO(3). \tag{5}$$

In what follows, we introduce the inertia tensor, the total angular momentum, the connection form, and the metric which are defined on the configuration space $X_0 \cong SO(3) \times SO(3)$.

We take parameters for axial symmetric cylinders as

$$m := m_1 = m_2, \quad A_1 = A_2 = \mathrm{diag}(I_1, I_1, \alpha I_1), \quad \alpha := I_3/I_1, \quad m\ell^2 = \beta I_1. \tag{6}$$

Then, the inertia tensor of the whole system is defined at $\sigma(q)$ to be

$$A_{\sigma(q)}(\boldsymbol{v}) := 2m\boldsymbol{\sigma}(q) \times (\boldsymbol{v} \times \boldsymbol{\sigma}(q)) + \sum_{i=1}^{2} k_i(q) A_i k_i^{-1}(q)\boldsymbol{v}, \tag{7}$$

where $\boldsymbol{\sigma} := \boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_2$, and $\boldsymbol{v} \in \mathbf{R}^3$. Written out, the local expression of the inertia tensor is of the form

$$A_{\sigma(q)} = 2I_1 \begin{pmatrix} 1 + \beta \sin^2 \frac{\psi}{2} & 0 & 0 \\ 0 & \cos^2 \frac{\psi}{2} + \alpha \sin^2 \frac{\psi}{2} & 0 \\ 0 & 0 & \sin^2 \frac{\psi}{2} + \alpha \cos^2 \frac{\psi}{2} + \beta \sin^2 \frac{\psi}{2} \end{pmatrix}. \tag{8}$$

In general, the inertial tensor is expressed as

$$A_{g\sigma(q)} = gA_{\sigma(q)}g^{-1}. \tag{9}$$

The total angular momentum $\boldsymbol{\Lambda}_{\sigma(q)}$ at $\sigma(q)$ is given by and written out as

$$\begin{aligned} \boldsymbol{\Lambda}_{\sigma(q)} &= 2m\boldsymbol{\sigma}(q) \times d\boldsymbol{\sigma}(q) + \sum_{i=1}^{2} k_i(q) A_i R^{-1}(k_i(q)^{-1} dk_i(q)) \\ &= \alpha I_1 \Big(\sin \frac{\psi}{2}(d\theta_1 + d\theta_2)\boldsymbol{e}_2 + \cos \frac{\psi}{2}(d\theta_1 - d\theta_2)\boldsymbol{e}_3 \Big). \end{aligned} \tag{10}$$

The connection form is now defined at $\sigma(q)$ to be

$$\begin{aligned} \omega_{\sigma(q)} &= R(A_{\sigma(q)}^{-1} \boldsymbol{\Lambda}_{\sigma(q)}) \\ &= \frac{\alpha \sin \frac{\psi}{2}}{\cos^2 \frac{\psi}{2} + \alpha \sin^2 \frac{\psi}{2}} \frac{1}{2}(d\theta_1 + d\theta_2) R(\boldsymbol{e}_2) \\ &\quad + \frac{\alpha \cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2} + \alpha \cos^2 \frac{\psi}{2} + \beta \sin^2 \frac{\psi}{2}} \frac{1}{2}(d\theta_1 - d\theta_2) R(\boldsymbol{e}_3). \end{aligned} \tag{11}$$

In general, the connection form is expressed as

$$\omega_{g\sigma(q)} = \mathrm{Ad}_g(g^{-1}dg + \omega_{\sigma(q)}). \tag{12}$$

It is to be noted that $x(t) \in X_0$ is vibrational or the total angular momentum for $x(t)$ vanishes if and only if $\omega_{x(t)}(\dot{x}(t)) = 0$.

We now express $g \in SO(3)$ as

$$g = e^{\chi_2 \widehat{e}_2} e^{\chi_3 \widehat{e}_3} e^{\chi_1 \widehat{e}_1}, \quad \widehat{\boldsymbol{e}}_a := R(\boldsymbol{e}_a). \tag{13}$$

The variables χ_a, $a = 1, 2, 3$, are a kind of Euler angles, but not the same as the usual ones. The ranges of χ_a are

$$0 \le \chi_1 \le 2\pi, \quad 0 \le \chi_2 \le 2\pi, \quad -\pi/2 \le \chi_3 \le \pi/2. \tag{14}$$

In terms of the variables χ_a, the left invariant one-forms and vector fields on $SO(3)$ are respectively expressed as

$$\begin{cases} \Phi^1 &= d\chi_1 + \sin\chi_3 d\chi_2, \\ \Phi^2 &= \cos\chi_3 \cos\chi_1 d\chi_2 + \sin\chi_1 d\chi_3, \\ \Phi^3 &= -\cos\chi_3 \sin\chi_1 d\chi_2 + \cos\chi_1 d\chi_3, \end{cases} \tag{15}$$

$$\begin{cases} K_1 &= \dfrac{\partial}{\partial\chi_1}, \\ K_2 &= \dfrac{\cos\chi_1}{\cos\chi_3}\dfrac{\partial}{\partial\chi_2} + \sin\chi_1 \dfrac{\partial}{\partial\chi_3} - \tan\chi_3 \cos\chi_1 \dfrac{\partial}{\partial\chi_1}, \\ K_3 &= -\dfrac{\sin\chi_1}{\cos\chi_3}\dfrac{\partial}{\partial\chi_2} + \cos\chi_1 \dfrac{\partial}{\partial\chi_3} + \tan\chi_3 \sin\chi_1 \dfrac{\partial}{\partial\chi_1}. \end{cases} \tag{16}$$

Introducing new local coordinates by

$$\phi_1 = \frac{1}{2}(\theta_1 + \theta_2), \quad \phi_2 = \frac{1}{2}(\theta_1 - \theta_2), \quad (q^\alpha) = (\psi, \phi_1, \phi_2), \tag{17}$$

one can put the components of the connection form, $\omega_{g\sigma(q)} = \sum_a \omega^a \mathrm{Ad}_g R(\boldsymbol{e}_a)$, in the form

$$\begin{aligned} \omega^1 &= \Phi^1, \\ \omega^2 &= \Phi^2 + \Lambda_2^2(q) d\phi_1, \\ \omega^3 &= \Phi^3 + \Lambda_3^3(q) d\phi_2, \end{aligned} \tag{18}$$

where Φ_a are given in (15) and

$$\Lambda_2^2(q) = \frac{\alpha \sin\frac{\psi}{2}}{\cos^2\frac{\psi}{2} + \alpha\sin^2\frac{\psi}{2}}, \quad \Lambda_3^3(q) = \frac{\alpha\cos\frac{\psi}{2}}{\sin^2\frac{\psi}{2} + \alpha\cos^2\frac{\psi}{2} + \beta\sin^2\frac{\psi}{2}}. \tag{19}$$

The metric on X_0, which is associated with the kinetic energy, is defined and expressed as

$$\begin{aligned} ds^2 &= 2m d\boldsymbol{r} \cdot d\boldsymbol{r} + \sum_{i=1}^{2} \boldsymbol{\Theta}_i \cdot A_i \boldsymbol{\Theta}_i & (20) \\ &= \sum_{\alpha,\beta} a_{\alpha\beta} da^\alpha dq^\beta + \sum_{a,b} A_{ab}\omega^a\omega^b, & (21) \end{aligned}$$

where $\boldsymbol{r} = \boldsymbol{r}_1 = -\boldsymbol{r}_2$, $g_i^{-1}dg_i = R(\boldsymbol{\Theta}_i)$, $g_i = gk_i$, $(q^\alpha) = (\psi, \phi_1, \phi_2)$, and where

$$(a_{\alpha\beta}) = 2I_1 \mathrm{diag}\Big(\frac{1}{4}\big(1+\lambda\cos^2\frac{\psi}{2}\big), \frac{\nu\cos^2\frac{\psi}{2}}{\cos^2\frac{\psi}{2}+\nu\sin^2\frac{\psi}{2}}, \frac{(1+\lambda)\nu\sin^2\frac{\psi}{2}}{\sin^2\frac{\psi}{2}+\nu\cos^2\frac{\psi}{2}+\lambda\sin^2\frac{\psi}{2}}\Big), \tag{22a}$$

$$(A_{ab}) = 2I_1 \mathrm{diag}\Big(1+\lambda\sin^2\frac{\psi}{2}, \cos^2\frac{\psi}{2}+\nu\sin^2\frac{\psi}{2}, \sin^2\frac{\psi}{2}+\nu\cos^2\frac{\psi}{2}+\lambda\sin^2\frac{\psi}{2}\Big). \tag{22b}$$

Note that A_{ab} are components of $A_{\sigma(q)}$ given in (8). The first term of the right-hand side of (21) determines a metric on the shape space $M = X_0/SO(3)$,

$$d\sigma^2 = \sum_{\alpha,\beta} a_{\alpha\beta} dq^\alpha dq^\beta. \tag{23}$$

3 A mechanical model

The Lagrangian associated with the metric (21) is expressed as

$$L = \frac{1}{2}\sum_{\alpha,\beta} a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + \frac{1}{2}\sum_{a,b} A_{ab}\pi^a\pi^b, \tag{24}$$

where $\boldsymbol{\pi} = \sum \pi_a \boldsymbol{e}_a$ is defined through

$$R(\boldsymbol{\pi}) = g^{-1}\dot{g} + \sum_a\sum_\alpha \Lambda^a_\alpha(q)\dot{q}^\alpha R(\boldsymbol{e}_a). \tag{25}$$

We apply torques given by

$$\sum_\alpha v_\alpha(t) dq^\alpha. \tag{26}$$

Then, the equations of motion for the falling cat are expressed as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} - \sum_\beta \frac{\partial L}{\partial \boldsymbol{\pi}}\cdot \boldsymbol{\kappa}_{\alpha\beta}\dot{q}^\beta + \frac{\partial L}{\partial \boldsymbol{\pi}}\cdot(\boldsymbol{\pi}\times\boldsymbol{\lambda}_\alpha) = v_\alpha, \tag{27a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \boldsymbol{\pi}} - \frac{\partial L}{\partial \boldsymbol{\pi}}\times\boldsymbol{\pi} + \sum_\alpha \frac{\partial L}{\partial \boldsymbol{\pi}}\times\boldsymbol{\lambda}_\alpha\dot{q}^\alpha = 0, \tag{27b}$$

where $\boldsymbol{\kappa}_{\alpha\beta} = \sum_c \kappa^c_{\alpha\beta}\boldsymbol{e}_c$ is the curvature tensor defined to be

$$\kappa^c_{\alpha\beta} := \frac{\partial \Lambda^c_\beta}{\partial q^\alpha} - \frac{\partial \Lambda^c_\alpha}{\partial q^\beta} - \sum_{a,b}\varepsilon_{abc}\Lambda^a_\alpha\Lambda^b_\beta, \tag{28}$$

and where $\boldsymbol{\lambda}_\alpha = \sum_\alpha \Lambda^a_\alpha \boldsymbol{e}_a$. Since the condition of the vanishing total angular momentum is described as $\boldsymbol{\pi} = 0$, and since this condition is compatible with the above equation, the equations of motion with the constraint of the vanishing total angular momentum take the form

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{q}^\alpha} - \frac{\partial L_c}{\partial q^\alpha} = v_\alpha, \quad L_c := L|_{\boldsymbol{\pi}=0}, \tag{29}$$

which are viewed as reduced equations on $T(M)$.

After finding a solution curve $(\psi(t), \phi_1(t), \phi_2(t))$, $0 \le t \le T$, with suitably designed torques $v_a(t)$, we proceed to integrate the constraint equation,

$$\boldsymbol{\pi} = 0 \Leftrightarrow \frac{dg}{dt} = -g\Big(\Lambda_2^2(q)\frac{d\phi_1}{dt}R(\boldsymbol{e}_2) + \Lambda_3^3(q)\frac{d\phi_2}{dt}R(\boldsymbol{e}_3)\Big). \tag{30}$$

This equation linear in g will be solved to give $g(t) \in SO(3)$ with $g(0) = I$. The vibrational motion is then given by $x(t) = g(t)\sigma(q(t))$, a realization of a somersault.

The reduced Lagrangian system $(T(M), L_c)$ is now translated into a Hamiltonian system $(T^*(M), H_c)$ in a usual manner; the canonical one-form and the Hamiltonian on $T^*(M)$ are defined to be

$$\Theta_0 = \sum_\alpha P_\alpha dq^\alpha, \quad P_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}, \tag{31}$$

$$H_c = \frac{1}{2}\sum_{\alpha,\beta} a^{\alpha\beta}P_\alpha P_\beta, \quad (a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}, \tag{32}$$

respectively, and the Hamiltonian vector field X_{H_c} is determined as usual through $\iota(X_{H_c})d\Theta_0 = -dH_c$.

4 A port-controlled Hamiltonian system

Now the port-controlled Hamilton equations associated with (29) are given by

$$\frac{dp}{dt} = X_{H_c} + \sum_{\alpha=1}^{3} v_\alpha X_\alpha, \quad X_\alpha = \frac{\partial}{\partial P_\alpha}, \tag{33a}$$

$$y_\alpha = X_\alpha(H_c), \tag{33b}$$

where $p \in T^*(M)$, and where y_α are called system outputs.

A key idea of control in the port-controlled Hamiltonian system is energy shaping, according to which we introduce a new Hamiltonian of the form

$$\overline{H} = H_c + U(q), \quad U(q) \ge 0. \tag{34}$$

Then, new port-controlled Hamilton equations become

$$\frac{dp}{dt} = X_{\overline{H}} + \sum_{\alpha=1}^{3} \overline{v}_\alpha X_\alpha, \tag{35a}$$

$$\overline{y}_\alpha = X_\alpha(\overline{H}). \tag{35b}$$

These equations are equivalent to the old ones, if and only if

$$X_U + \sum (\overline{v}_\alpha - v_\alpha) X_\alpha = 0, \tag{36a}$$
$$\overline{y}_\alpha - y_\alpha = X_\alpha(U). \tag{36b}$$

If we choose the controls $\overline{v}_\alpha$ to be

$$\overline{v}_\alpha = -\overline{y}_\alpha, \tag{37}$$

the system becomes Lyapunov stable. In fact, the differentiation of $\overline{H}$ results in

$$\frac{d\overline{H}}{dt} = X_{\overline{H}}(\overline{H}) + \sum_{\alpha=1}^{3} \overline{v}_\alpha X_\alpha(\overline{H}) = -\sum_{\alpha=1}^{3} X_\alpha(\overline{H})^2 \leq 0. \tag{38}$$

Equations (36) and (37) are put together to determine the controls v_α as follows:

$$v_\alpha = -\frac{\partial U}{\partial q^\alpha} - \frac{\partial H_c}{\partial P_\alpha}. \tag{39}$$

With these controls, the state $(q(t), P(t)) \in T^*(M)$ tends to an equilibrium state $(q_\tau, 0)$, where $q_\tau \in M$ is a point at which U takes the minimum value.

5 An example

A potential function is designed to be

$$U = k_1 \sin^2(\frac{\phi_1 + \phi_2}{2}) + k_2 \sin^2(\frac{\phi_1 - \phi_2}{2}) + k_3 \sin^2(\psi - \frac{4\pi}{5}), \tag{40}$$

where k_α are constants. Note here that this potential function takes a minimum value at $q_\tau = e^{\frac{4\pi}{5}\widehat{e}_1}$, which is the target shape that the falling cat will take after a vibrational motion. We here choose the parameter values as follows:

$$I_1 = \frac{19}{12}, \quad \alpha = \frac{6}{19}, \quad \beta = \frac{48}{19}, \quad k_1 = k_2 = k_3 = 1. \tag{41}$$

Now we can solve the port-controlled Hamilton equations with the control well designed. If we find a solution $(q(t), P(t))$, we can integrate the constraint equation (30) along with the present $q(t)$.

If the initial shape $q_0 = q(0)$ is fixed, a family of solutions $q(t, K)$ will be found numerically, where K denotes initial momenta, $K = (P_\alpha(0))$. We choose a sufficiently large time T such that $q(T) \approx q_\tau$,

$$\phi_1(T) \approx -2\pi, \quad \phi_2(T) \approx 0, \quad \psi(T) \approx \frac{4\pi}{5}, \quad (\theta_1(T) = \theta_2(T) \approx -2\pi). \tag{42}$$

We wish to gain the rotation

$$g(0) = I \mapsto g(T) \approx e^{\pi \widehat{e}_2} \tag{43}$$

after completing a vibrational motion $g(t,K)\sigma(q(t,K))$, $0 \le t \le T$;

$$\chi_1(T,K) \approx 0, \quad \chi_2(T,K) \approx \pi, \quad \chi_3(T,K) \approx 0, \tag{44}$$

where $g(t,K)$, $0 \le t \le T$, is a family of solutions to the constraint equation. To this end, we have to solve numerically the equation

$$g(T,K) = e^{\pi \widehat{\mathbf{e}}_2} \tag{45}$$

for K by Newton's method.

The numerical solutions with K thus found are shown in the following graphs. The shape and the attitude are determined by $q(t) = e^{\chi_2(t)\widehat{\mathbf{e}}_2} e^{\chi_3(t)\widehat{\mathbf{e}}_3} e^{\chi_1(t)\widehat{\mathbf{e}}_1}$, and $g(t) = e^{-(\phi_1(t)+\phi_2(t))\widehat{\mathbf{e}}_3} e^{\psi(t)\widehat{\mathbf{e}}_1} e^{-(\phi_1(t)-\phi_2(t))\widehat{\mathbf{e}}_3}$, respectively.

References

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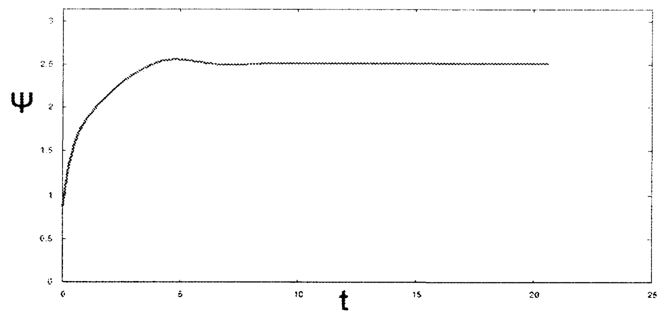

Figure 3: The graph of $\psi(t)$

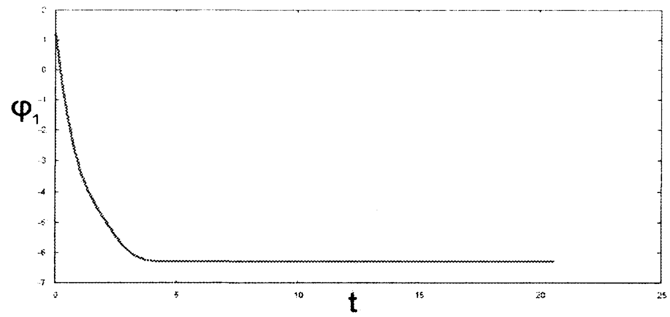

Figure 4: The graph of $\phi_1(t)$

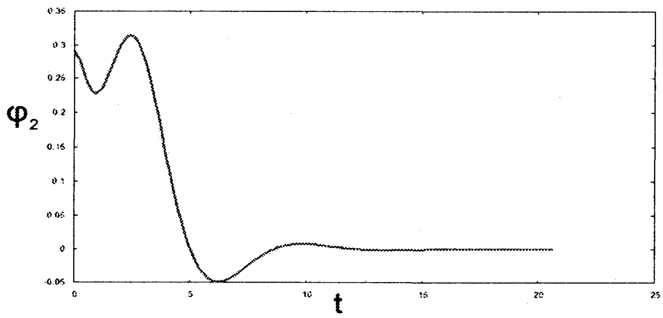

Figure 5: The graph of $\phi_2(t)$

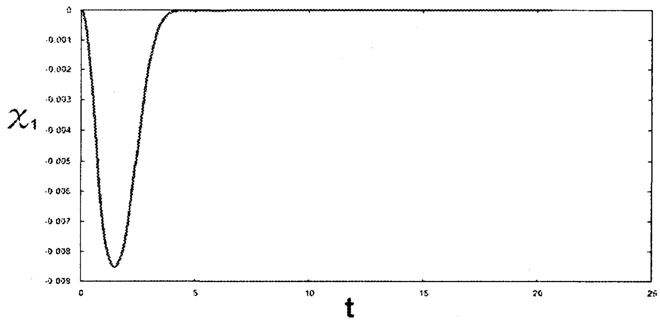

Figure 6: The graph of $\chi_1(t)$

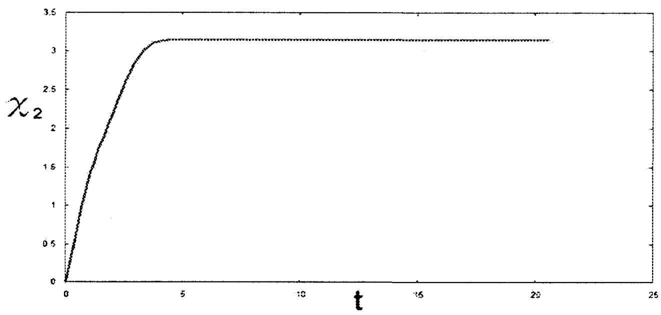

Figure 7: The graph of $\chi_2(t)$

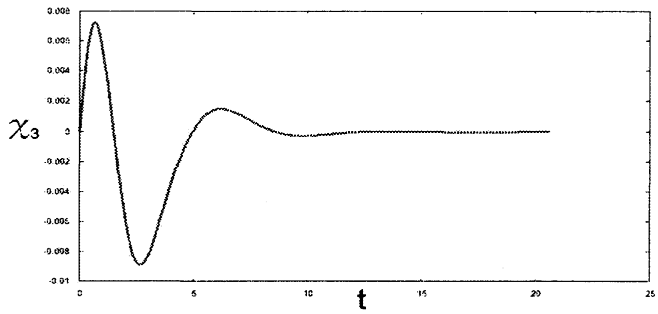

Figure 8: The graph of $\chi_3(t)$