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Higher-order Schrödinger operators with singular potentials

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Abstract. The selfadjointness of $\Delta^2+\kappa|x|^{-4}(\kappa\in\mathbb{R})$ in $L^2(\mathbb{R}^N)$ and the m-accretivity of $\Delta^2+\kappa|x|^{-4}$ $(\kappa\in\mathbb{C})$ in $L^2(\mathbb{R}^N)$ are established as applications of perturbation theorems for nonnegative selfadjoint operators. The key lies in two new inequalities derived by using two real or complex parameters.

1. Introduction and results

Let $N\in\mathbb{N}$. Then this paper is concerned with the selfadjointness of $\Delta^2+\kappa|x|^{-4}$ (when $\kappa\in\mathbb{R}$), and the m-accretivity of $\Delta^2+\kappa|x|^{-4}$ (when $\kappa\in\mathbb{C}$) in the (complex) Hilbert space $L^2(\mathbb{R}^N)$. Here Δ^2 and $|x|^{-4}$ are nonnegative selfadjoint operators in $L^2(\mathbb{R}^N)$, with domains $D(\Delta^2):=H^4(\mathbb{R}^N)$ and $D(|x|^{-4}):=\{u\in L^2(\mathbb{R}^N);\ |x|^{-4}u\in L^2(\mathbb{R}^N)\}$, respectively.

First we consider the selfadjointness of $\Delta^2+\kappa|x|^{-4}$ $(\kappa\in\mathbb{R})$. On the one hand, it is worth noticing that the relation between simpler operators $-\Delta$ and $|x|^{-2}$ is already known as a model case. In [8] it has been proved that $-\Delta+t|x|^{-2}$ is m-accretive in $L^p(\mathbb{R}^N)$ for $t>a_0(p)$ and $-\Delta+a_0(p)|x|^{-2}$ is essentially m-accretive in $L^p(\mathbb{R}^N)$ $(1<p<\infty)$, where $a_0(p)$ is defined as

$$a_0(p):=\begin{cases} p^{-2}(p-1)(2p-N)N, & 2(1-N^{-1})\le p<\infty,\\ -p^{-2}(p-1)(N-2)^2, & 1<p<2(1-N^{-1}).\end{cases}$$

In particular, if $p=2$, then $a_0(2)=4^{-1}(4-N)N$ and m-accretivity is replaced with nonnegative selfadjointness. A proof of the selfadjointness in [7] is based on the inequality

$$\operatorname{Re}(-\Delta u,(|x|^2+n^{-1})^{-1}u)\ge -a_0(2)\|(|x|^2+n^{-1})^{-1}u\|^2,\quad u\in H^2(\mathbb{R}^N),$$

where $(|x|^2+n^{-1})^{-1}=|x|^{-2}(1+n^{-1}|x|^{-2})^{-1}$ is the Yosida approximation of $|x|^{-2}$ $(n\in\mathbb{N})$. On the other hand, there seems to be few works about the selfadjointness of higher order elliptic operators. In [6] Nguyen discussed the selfadjointness of general even order elliptic operators under several assumptions. However, his result cannot be applied to determine the critical bound of κ for the selfadjointness of $\Delta^2+\kappa|x|^{-4}$.

The first purpose of this paper is to establish the following

Theorem 1.1. *Put $A:=\Delta^2$ and $B:=|x|^{-4}$. Let $\kappa_0(N)$ be defined as*

$$\kappa_0(N):=\begin{cases} k_1:=112-3(N-2)^2, & N\le 8,\\ k_2:=-(N/16)(N-8)(N^2-16), & N\ge 9.\end{cases}\tag{1.1}$$

Then the following (i) *and* (ii) *hold.*

(i) *If $N \le 8$, then B is $(A+\kappa B)$-bounded for $\kappa > \kappa_0(N)$ as*

$$\|Bu\| \le (\kappa - \kappa_0(N))^{-1}\|(A+\kappa B)u\|, \quad u \in D(A+\kappa B) := D(A) \cap D(B),$$

and $A+\kappa B$ is nonnegative selfadjoint for $\kappa > \kappa_0(N)$. Moreover, $A+\kappa_0(N)B$ is nonnegative and essentially selfadjoint.

(ii) *If $N \ge 9$, then B is A-bounded as*

$$\|Bu\| \le \frac{16}{N(N-8)(N^2-16)}\|Au\|, \quad u \in D(A) \subset D(B), \tag{1.2}$$

and $A+\kappa B$ is nonnegative selfadjoint for $\kappa > \kappa_0(N)$. Moreover, $A+\kappa_0(N)B$ is nonnegative and essentially selfadjoint in $L^2(\mathbb{R}^N)$.

Next we shall find $\Omega \subset \mathbb{C}$ such that $\{\Delta^2 + \kappa|x|^{-4}; \kappa \in \Omega\}$ is a holomorphic family of type (A) in the sense of Kato [4, Section VII.2]. We review it in a simple case.

Definition 1. *Let X be a reflexive complex Banach space. Let Ω be a domain in $\mathbb{C}$ and $\{T(\kappa);\ \kappa \in \Omega\}$ a family of linear operators in X. Then $\{T(\kappa);\ \kappa \in \Omega\}$ is said to be a holomorphic family of type (A) if*

(i) *$T(\kappa)$ is closed in X and $D(T(\kappa)) = D$ independent of κ;*

(ii) *$\kappa \mapsto T(\kappa)u$ is holomorphic in Ω for every $u \in D$.*

Kato [5] proved that $\{-\Delta + \kappa|x|^{-2};\ \kappa \in \Omega_1\}$ forms a holomorphic family of type (A) in $L^2(\mathbb{R}^N)$, where

$$\Omega_1 := \{\xi + i\eta \in \mathbb{C};\ \eta^2 > 4(\beta - \xi)\}, \quad \beta := (N-2)^2/4.$$

Borisov-Okazawa [1] proved that $\{d/dx + \kappa|x|^{-1};\ \kappa \in \Omega_2\}$ forms a holomorphic family of type (A) in $L^p(0,\infty)$ $(1 < p < \infty)$, where

$$\Omega_2 := \left\{\kappa \in \mathbb{C}\ ;\ \mathrm{Re}\,\kappa > -\frac{1}{p'}\right\}, \quad p^{-1} + p'^{-1} = 1.$$

In both cases it is essential to find $\Sigma_j := \Omega_j{}^c$, the complement of Ω_j $(j = 1,2)$. Concerning forth order elliptic operators, there seems to be no preceding work on holomorphic family of type (A). So we clarify the region where $\Delta^2 + \kappa|x|^{-4}$ forms a holomorphic family of type (A) and where $\Delta^2 + \kappa|x|^{-4}$ is m-accretive in $L^2(\mathbb{R}^N)$ (the definition of (regular) m-accretivity will be given in Section 3). Our second result here is stated as follows.

Theorem 1.2. *Let A and B be the same as in* Theorem 1.1. *Let Σ be a closed convex subset of $\mathbb{C}$ (see* Figure 1) *such that*

$$\Sigma := \left\{\xi + i\eta \in \mathbb{C};\ \xi \le k_1,\ \eta^2 \le 64\Big(\sqrt{k_1 - \xi} + \Big(10 + N - \frac{N^2}{4}\Big)\Big)\Big(\sqrt{k_1-\xi} + 8\Big)^2\right\},$$

where the constant k_1 is defined in (1.1); replace Σ with

$$\Sigma = \left\{\xi + i\eta \in \mathbb{C};\ \xi \le k_2,\ \eta^2 \le \frac{64(k_2 - \xi)(\sqrt{k_1-\xi}+8)^2}{\sqrt{k_1 - \xi} + (N^2/4 - N - 10)}\right\},$$

if $N \geq 9$ [the constant k_2 is also defined in (1.1)]. *Then the following* (**i**) *and* (**ii**) *hold.*
(**i**) *B is $(A+\kappa B)$-bounded for $\kappa \in \Sigma^c$, with*

$$\|Bu\| \leq \operatorname{dist}(\kappa, \Sigma)^{-1} \|(A+\kappa B)u\|, \quad u \in D(A) \cap D(B),$$

and $\{A+\kappa B;\ \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A) *in $L^2(\mathbb{R}^N)$.*
(**ii**) *$A+\kappa B$ is m-accretive on $D(A)\cap D(B)$ for $\kappa \in \Sigma^c$ with $\operatorname{Re}\kappa \geq -\alpha_0$, $A+\kappa B$ is regularly m-accretive on $D(A)\cap D(B)$ for $\kappa \in \Sigma^c$ with $\operatorname{Re}\kappa > -\alpha_0$ and $A+\kappa B$ is essentially m-accretive in $L^2(\mathbb{R}^N)$ for $\kappa \in \partial\Sigma$ with $\operatorname{Re}\kappa \geq -\alpha_0$, where α_0 is defined as*

$$\alpha_0 := \begin{cases} 0, & N \leq 4, \\ \dfrac{N^2}{16}(N-4)^2, & N \geq 5. \end{cases} \tag{1.3}$$

In particular, if $\kappa \in \mathbb{R}$, then m-accretivity can be replaced with nonnegative selfadjointness.

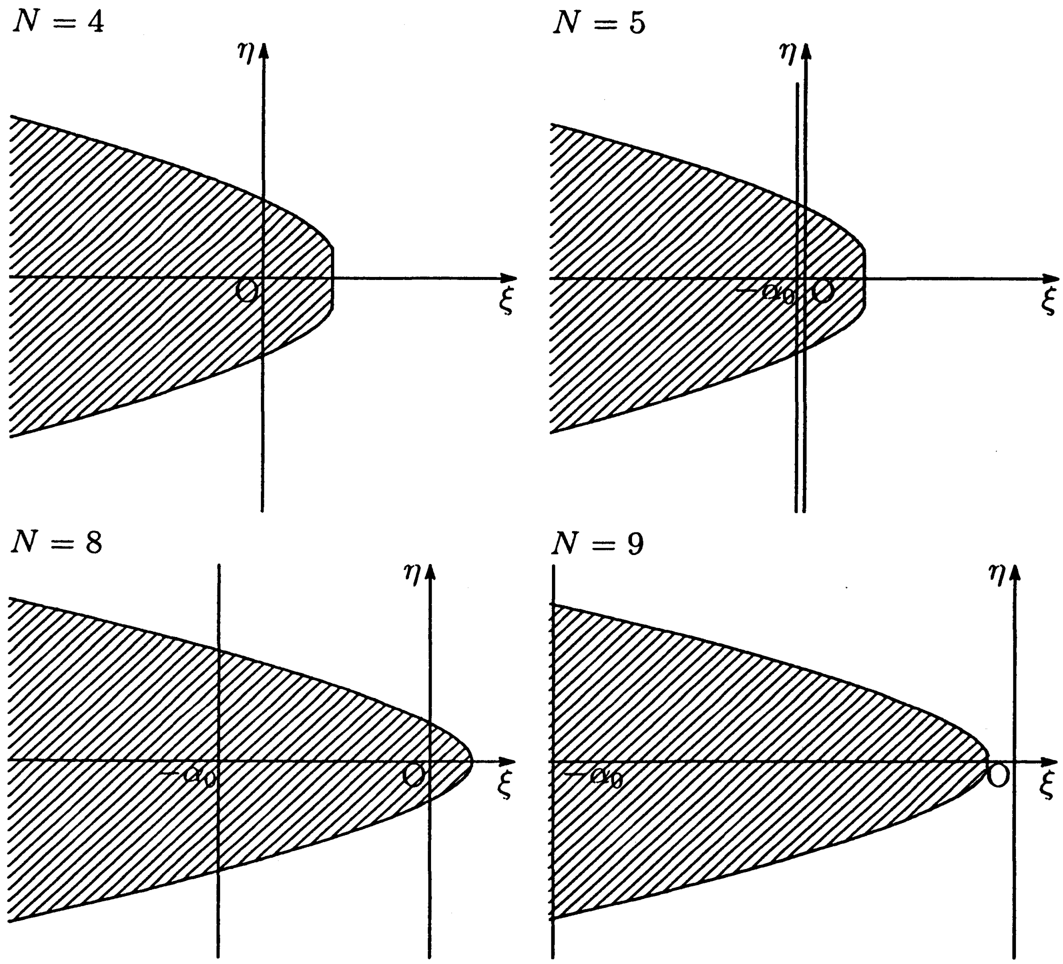

Figure 1: The images of Σ for $N = 4, 5, 8, 9$ and the value of $-\alpha_0$

The constant α_0 in (1.3) appears in the Rellich inequality

$$\frac{N(N-4)}{4} \| |x|^{-2} u \| \leq \|\Delta u\|, \quad u \in H^2(\mathbb{R}^N). \tag{1.4}$$

In [3] Davies-Hinz have shown Hardy or Rellich type inequalities between $(-\Delta)^m$ and $|x|^{-2m}$ ($m \in \mathbb{N}$), and it helps us to construct the theory of the selfadjointness.

In Section 2 we review abstract theorems based on [8]. In Section 3 we prepare abstract theorems based on Kato [5] (however, the assumption and conclusions are slightly changed). In Section 4 we derive some new inequalities by using two real parameters and prove Theorem 1.1 by applying abstract theorems prepared in Section 2. In Section 5 we generalize inequalities obtained in Section 4 by using two complex parameters and prove Theorem 1.2 by applying abstract theorems prepared in Section 3.

2. Perturbation theory toward Theorem 1.1

This section is a short review of the perturbation theory developed in [7] and [8] for m-accretive operators in a Banach space. The following two theorems are the special cases of those in [8].

Theorem 2.1 ([8, Theorem 1.6]). *Let A and B be nonnegative selfadjoint operators in a Hilbert space H. Let $B_\varepsilon := B(1+\varepsilon B)^{-1}$ be the Yosida approximation of B. Assume that there exists some $k_0 \geq 0$ such that*

$$\mathrm{Re}(Au, B_\varepsilon u) \geq -k_0\|B_\varepsilon u\|^2, \quad u \in D(A). \tag{2.1}$$

Then B is $(A+kB)$-bounded for $k > k_0$ as

$$\|Bu\| \leq (k-k_0)^{-1}\|(A+kB)u\|, \quad u \in D(A+kB), \tag{2.2}$$

and hence $A+kB$ is closed in H for $k > k_0$. Moreover, $A+kB$ is nonnegative selfadjoint on $D(A) \cap D(B)$ for $k > k_0 \geq 0$ and $A+k_0B$ is nonnegative and essentially selfadjoint in H.

Theorem 2.2 ([8, Theorem 1.7]). *Let A, B and B_ε be the same as those in* Theorem 2.1. *Assume that there exists some $m_1 > 0$ such that*

$$\mathrm{Re}(Au, B_\varepsilon u) \geq m_1\|B_\varepsilon u\|^2, \quad u \in D(A). \tag{2.3}$$

Then B is A-bounded as

$$\|Bu\| \leq m_1^{-1}\|Au\|, \quad u \in D(A) \subset D(B), \tag{2.4}$$

and $A+kB$ is closed in H for $k > -m_1$. Assume further that there exists some $m_2 \geq \sqrt{m_1}$ such that $m_2^2(B_\varepsilon u, u) \leq (Au, u)$, $u \in D(A)$, or equivalently

$$m_2\|B^{1/2}(1+\varepsilon B)^{-1/2}v\| \leq \|A^{1/2}v\|, \quad v \in D(A^{1/2}) \tag{2.5}$$

Then $A+kB$ is nonnegative selfadjoint in H for $k > -k_1$, and $A-k_1B$ is nonnegative and essentially selfadjoint in H.

3. Perturbation theory toward Theorem 1.2

First we review some definitions required to state Theorems 3.1 and 3.5. Let A be a linear operator with domain $D(A)$ and range $R(A)$ in a (complex) Hilbert space H. Then A is said to be *accretive* if $\mathrm{Re}\,(Au, u) \geq 0$ for every $u \in D(A)$. An accretive operator A is said to be *m-accretive* if $R(A+1) = H$. An m-accretive operator A is said to be *regularly m-accretive* if A satisfies for some $\omega \in [0, \pi/2)$ that

$$|\mathrm{Im}\,(Au, u)| \leq (\tan\omega)\mathrm{Re}\,(Au, u), \quad u \in D(A).$$

Let A be m-accretive in H. Then $R(A+\lambda) = H$ holds, with

$$\|(A+\lambda)^{-1}\| \leq (\mathrm{Re}\,\lambda)^{-1} \quad \forall\,\lambda \in \mathbb{C} \text{ with } \mathrm{Re}\,\lambda > 0.$$

Therefore we can define the Yosida approximation $\{A_\varepsilon; \varepsilon > 0\}$ of A:

$$A_\varepsilon := A(1+\varepsilon A)^{-1}$$

A nonnegative selfadjoint operator is a typical example of m-accretive operator, while a symmetric m-accretive operator is nonnegative and selfadjoint (see Brézis [**2,** Proposition VII.6] or Kato [**4,** Problem V.3.32]).

Next we consider the m-accretivity of $A+\kappa B$ ($\kappa \in \mathbb{C}$) where A and B are nonnegative selfadjoint operators. Since m-accretive operators are closed and densely defined, we will first find the set of $\kappa \in \mathbb{C}$ where $A+\kappa B$ is closed (and densely defined). Hence we can connect the two notions of m-accretivity and holomorphic family of closed operators.

Theorem 3.1. *Let A and B be nonnegative selfadjoint operators in H. Let $\Sigma \subset \mathbb{C}$, and $\gamma : \mathbb{R} \to \mathbb{R}$. Assume that Σ and γ satisfy* $(\gamma 1)$–$(\gamma 4)$ *and* $(\gamma 5)_0$:

(γ1) *γ is continuous and $-\gamma$ is convex,*
(γ2) *$\gamma(\eta) = \gamma(-\eta)$ for $\eta \in \mathbb{R}$,*
(γ3) *$\Sigma = \{\xi + i\eta \in \mathbb{C}\ ;\ \xi \leq \gamma(\eta)\}$,*
(γ4) *$-(Au, B_\varepsilon u) \in \Sigma$ for $u \in D(A)$, $\|B_\varepsilon u\| = 1$ for any $\varepsilon > 0$,*
(γ5)$_0$ *$0 \leq \gamma(0) \Leftrightarrow 0 \in \Sigma$.*

Then the following (**i**) *and* (**ii**) *hold.*

(**i**) *B is $(A+\kappa B)$-bounded for $\kappa \in \Sigma^c$, with*

$$\|Bu\| \leq \mathrm{dist}(\kappa, \Sigma)^{-1}\|(A+\kappa B)u\|, \quad u \in D(A)\cap D(B), \tag{3.1}$$

and $\{A+\kappa B;\ \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A).

(**ii**) *$A+\kappa B$ is m-accretive in H for $\kappa \in \Sigma^c$ with* $\mathrm{Re}\,\kappa \geq 0$, *$A+\kappa B$ is regularly m-accretive in H for $\kappa \in \Sigma^c$ with* $\mathrm{Re}\,\kappa > 0$ *and $A+\kappa B$ is essentially m-accretive in H for $\kappa \in \partial\Sigma$ with* $\mathrm{Re}\,\kappa \geq 0$.

The proof of Theorem 3.1 is divided into several lemmas.

Lemma 3.2. *The assertion* (**i**) *of* Theorem 3.1 *holds.*

Proof. Let $\kappa \in \Sigma^c$ and $\varepsilon > 0$. To prove (3.1) we shall show that

$$\|B_\varepsilon u\| \le \mathrm{dist}\,(\kappa, \Sigma)^{-1}\|(A + \kappa B_\varepsilon)u\|, \quad u \in D(A). \tag{3.2}$$

Here we may assume that $B_\varepsilon u = B(1 + \varepsilon B)^{-1}u \neq 0$. Setting $v := \|B_\varepsilon u\|^{-1}u$, we see that $v \in D(A)$ and $\|B_\varepsilon v\| = 1$. it then follows from (γ4) that $-(Av, B_\varepsilon v) \in \Sigma$. Since Σ is closed and convex by (γ1), we have

$$0 < \mathrm{dist}\,(\kappa, \Sigma) \le |\kappa + (Av, B_\varepsilon v)| = \|B_\varepsilon u\|^{-2}\,|((A + \kappa B_\varepsilon)u, B_\varepsilon u)|,$$

and hence $\|B_\varepsilon u\|^2 \le \mathrm{dist}\,(\kappa, \Sigma)^{-1}|((A+\kappa B_\varepsilon)u, B_\varepsilon u)|$. Now the Cauchy-Schwarz inequality applies to give (3.2). Letting $\varepsilon \downarrow 0$ in (3.2) with $u \in D(A) \cap D(B)$ yields (3.1). The closedness of $A+\kappa B$ is a consequence of (3.1). This completes the proof of (i) in Theorem 3.1 □

Lemma 3.3. *$A + \kappa B$ is m-accretive in H for $\kappa \in \Sigma^c$ with* $\mathrm{Re}\,\kappa \ge 0$. *In particular, if* $\mathrm{Re}\,\kappa > 0$, *then $A + \kappa B$ is regularly m-accretive in H, with*

$$|\mathrm{Im}\,((A + \kappa B)u, u)| \le (\tan|\arg\kappa|)\mathrm{Re}\,((A + \kappa B)u, u), \quad u \in D(A) \cap D(B). \tag{3.3}$$

Proof. Since the sum of accretive operators is also accretive, it suffices to show that

$$R(A + \kappa B + \lambda) = H, \quad \lambda > 0 \tag{3.4}$$

for $\kappa \in \Sigma^c$ with $\mathrm{Re}\,\kappa \ge 0$. Since $A + \kappa B_\varepsilon$ is also m-accretive (see **[10**, Corollary 3.3.3]), for $f \in H$ and $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in D(A)$ of approximate equation

$$Au_\varepsilon + \kappa B_\varepsilon u_\varepsilon + \lambda u_\varepsilon = f, \tag{3.5}$$

satisfying $\|u_\varepsilon\| \le \lambda^{-1}\|f\|$ and hence $\|Au_\varepsilon + \kappa B_\varepsilon u_\varepsilon\| = \|f - \lambda u_\varepsilon\| \le 2\|f\|$. Therefore we see from (3.2) that

$$\|B_\varepsilon u_\varepsilon\| \le 2\,\mathrm{dist}\,(\kappa, \Sigma)^{-1}\|f\|,$$

and hence $\|Au_\varepsilon\| \le 2(1+|\kappa|\mathrm{dist}\,(\kappa, \Sigma)^{-1})\|f\|$. Thus $\|u_\varepsilon\|$, $\|Au_\varepsilon\|$ and $\|B_\varepsilon u_\varepsilon\|$ are bounded as ε tends to zero. This implies that there exist convergent subsequences $\{u_{\varepsilon_n}\}$, $\{Au_{\varepsilon_n}\}$ and $\{B_{\varepsilon_n}u_{\varepsilon_n}\} = \{B(1 + \varepsilon_n B)^{-1}u_{\varepsilon_n}\}$ for some null sequence $\{\varepsilon_n\}$. Since A and B are (weakly) closed, there exists $u :=\text{w-}\lim_{n\to\infty} u_{\varepsilon_n} \in D(A) \cap D(B)$ such that

$$Au_{\varepsilon_n} \to Au \quad \text{and} \quad B_{\varepsilon_n}u_{\varepsilon_n} \to Bu \ (n \to \infty) \text{ weakly;}$$

note that $u_\varepsilon - (1 + \varepsilon B)^{-1}u_\varepsilon = \varepsilon B_\varepsilon u_\varepsilon$. Letting $n \to \infty$ in (3.5) with $\varepsilon = \varepsilon_n$ in the weak topology of H, we obtain (3.4). The regular m-accretivity of $A + \kappa B$ for $\kappa \in \Sigma^c$ with $\mathrm{Re}\,\kappa > 0$ follows to consider the numerical range of $A + \kappa B$;

$$\begin{aligned}((A + \kappa B)u, u) &= \|A^{1/2}u\|^2 + \kappa\|B^{1/2}u\|^2 \\ &\in \{a + \kappa b \in \mathbb{C};\ a \ge 0, b \ge 0\} \\ &\subset \{z \in \mathbb{C}; |\arg z| \le |\arg\kappa|\}, \quad u \in D(A) \cap D(B).\end{aligned}$$

This proves (3.3). □

Lemma 3.4. *The closure of $A + \kappa B$ is m-accretive in H for $\kappa \in \partial\Sigma$ with* $\operatorname{Re}\kappa \geq 0$.

Proof. Let $\kappa \in \partial\Sigma$ with $\operatorname{Re}\kappa \geq 0$. First we note that $A + \kappa B$ is closable and its closure is also accretive (cf. **[10,** Theorem 1.4.5]). Now (γ**1**) means that there exists some (not unique in general) unit outward normal vector ν of $\partial\Sigma$ at κ. This implies that $\kappa + t\nu \in \Sigma^c$ $(t > 0)$, with the properties:

$$\operatorname{Re}(\kappa + t\nu) \geq 0, \quad \operatorname{dist}(\kappa + t\nu, \Sigma) = t, \quad t > 0.$$

This implies that $A + \kappa B$ $(\kappa \in \partial\Sigma)$ is approximated by $A + (\kappa + \nu/n)B$ $(\kappa + \nu/n \in \Sigma^c)$ with $n \in \mathbb{N}$. Since $\operatorname{Re}\kappa + \nu/n \geq 0$, we see that $A + (\kappa + (\nu/n))B$ is m-accretive (see Lemma 3.3), that is, $f \in H$ there exists a unique solution $u_n \in D(A) \cap D(B)$ of

$$(A + \kappa B)u_n + (\nu/n)Bu_n + \lambda u_n = Au_n + (\kappa + (\nu/n))Bu_n + \lambda u_n = f, \tag{3.6}$$

satisfying

$$\|u_n\| \leq \lambda^{-1}\|f\|. \tag{3.7}$$

Now we can prove that $\|(\nu/n)Bu_n\| = n^{-1}\|Bu_n\| \leq 2\|f\|$. In fact, it follows from (3.1) that

$$\begin{aligned}\|Bu_n\| &\leq \operatorname{dist}(\kappa + n^{-1}\nu, \Sigma)^{-1}\|(A + (\kappa + \nu/n)B)u_n\| = n\|f - \lambda u_n\| \\ &\leq 2n\|f\|.\end{aligned}$$

This yields together with (3.6) that

$$\|(A + \kappa B)u_n\| \leq 4\|f\| \quad \forall\, n \in \mathbb{N} \tag{3.8}$$

To finish the proof we show that $(\nu/n)Bu_n$ converges to zero weakly in H. It follows from (3.7) that for every $v \in D(B)$,

$$|((\nu/n)Bu_n, v)| = n^{-1}|(u_n, Bv)| \leq n^{-1}\lambda^{-1}\|f\| \cdot \|Bv\| \to 0, \ n \to \infty.$$

Since $D(B)$ is dense in H and $n^{-1}\|Bu_n\|$ is bounded, we can conclude that $n^{-1}Bu_n \to 0$ weakly as $n \to \infty$. Now let $\{u_{n_k}\}$ be a convergent subsequence of $\{u_n\}$ and put $u :=$w-$\lim_{k\to\infty} u_{n_k}$. Then we have

$$\begin{aligned}(A + \kappa B)u_{n_k} &= f - \lambda u_{n_k} - (\nu/n)Bu_{n_k} \\ &\to f - \lambda u \ (k \to \infty) \text{ weakly.}\end{aligned}$$

It follows from the (weak) closedness that $u \in D((A + \kappa B)^{\sim})$ and

$$(A + \kappa B)^{\sim}u + \lambda u = f$$

This completes the proof of essential m-accretivity of $A+\kappa B$ for $\kappa \in \partial\Sigma$ with $\operatorname{Re}\kappa \geq 0$. □

We can improve Theorem 3.1 in the case where $B^{1/2}$ is $A^{1/2}$-bounded.

Theorem 3.5. *Let H, A, B, B_ε, Σ and γ be the same as those in* Theorem 3.1 *with* $(\gamma 1)$–$(\gamma 4)$. *Let $\alpha_0 > 0$. Assume that $B_\varepsilon^{1/2}$ is $A^{1/2}$-bounded, with*

$$(3.9) \qquad \alpha_0 \|B_\varepsilon^{1/2} u\|^2 \le \|A^{1/2} u\|^2, \quad u \in D(A^{1/2}).$$

Assume further that Σ and γ satisfy $(\gamma 5)_{\alpha_0}$ instead of $(\gamma 5)_0$:

$(\gamma 5)_{\alpha_0}$ $-\alpha_0 \le \gamma(0)$.

Then the following (i) *and* (ii) *hold.*

(i) *B is $(A + \kappa B)$-bounded for $\kappa \in \Sigma^c$, with*

$$(3.10) \qquad \|Bu\| \le \mathrm{dist}(\kappa, \Sigma)^{-1} \|(A + \kappa B)u\|, \quad u \in D(A) \cap D(B),$$

and $\{A + \kappa B;\ \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A). *In particular, if $\gamma(0) < 0$, then B is A-bounded with*

$$(3.11) \qquad \|Bu\| \le \mathrm{dist}(0, \Sigma)^{-1} \|Au\|, \quad u \in D(A) \subset D(B).$$

(ii) *$A + \kappa B$ is m-accretive in H for $\kappa \in \Sigma^c$ with $\mathrm{Re}\,\kappa \ge -\alpha_0$ and $A + \kappa B$ is essentially m-accretive in H for $\kappa \in \partial\Sigma$ with $\mathrm{Re}\,\kappa \ge -\alpha_0$. Moreover, $A + \kappa B$ is regularly m-accretive in H for $\kappa \in \Sigma^c$ with $\mathrm{Re}\,\kappa > -\alpha_0$, with*

$$(3.12) \quad |\mathrm{Im}\,((A + \kappa B)u, u)| \le (\tan|\arg(\kappa + \alpha_0)|) \mathrm{Re}\,((A + \kappa B)u, u), \quad u \in D(A) \cap D(B).$$

Proof. (i) The closedness of $A + \kappa B$ for $\kappa \in \Sigma^c$ is a consequence of Theorem 3.1. Noting that $\gamma(0) < 0$ implies $0 \in \Sigma^c$, we see from $(\gamma 4)$ that if $\gamma(0) < 0$, then

$$(3.13) \qquad \|B_\varepsilon u\| \le \mathrm{dist}(0, \Sigma)^{-1} \|Au\|, \quad \varepsilon > 0,\ u \in D(A).$$

Letting $\varepsilon \downarrow 0$ in (3.13) for $u \in D(A)$, we obtain (3.11).
(ii) Let $f \in H$, $\lambda > 0$ and $\kappa \in \Sigma^c$ with $\mathrm{Re}\,\kappa \ge -\alpha_0$. Then we consider the equation

$$(3.14) \qquad Au_\varepsilon + \kappa B_\varepsilon u_\varepsilon + \lambda u_\varepsilon = f.$$

In order to prove $R(A + \kappa B + \lambda) = H$ we only have to show that $\|u_\varepsilon\|$, $\|Au_\varepsilon\|$ and $\|B_\varepsilon u_\varepsilon\|$ are bounded as ε tends to zero. (3.9) implies that $A + \kappa B_\varepsilon$ is accretive:

$$\begin{aligned} \mathrm{Re}\,((A + \kappa B_\varepsilon)u, u) &= \|A^{1/2}u\|^2 + (\mathrm{Re}\,\kappa)\|B_\varepsilon^{1/2}u\|^2 \\ &\ge (\alpha_0 + \mathrm{Re}\,\kappa)\|B_\varepsilon^{1/2}u\|^2 \\ &\ge 0. \end{aligned}$$

The accretivity of $A + \kappa B_\varepsilon$ yields that $\|u_\varepsilon\| \le \lambda^{-1}\|f\|$. $(\gamma 1)$–$(\gamma 4)$ yield that there exists $c > 0$ such that $\|Au_\varepsilon\| \le c\|f\|$ and $\|B_\varepsilon u_\varepsilon\| \le c\|f\|$. As in the proof of Theorem 3.1, we obtain $R(A + \kappa B + \lambda) = H$. In particular, if $\mathrm{Re}\,\kappa > -\alpha_0$, then the numerical range of $A + \kappa B$, together with (3.9), proves the regular m-accretivity of $A + \kappa B$ with (3.12). □

4. Proof of Theorem 1.1

In order to prove Theorem 1.1 we need some inequalities in the real or complex Hilbert space $L^2(\mathbb{R}^N)$. We review the following lemma proposed by Ozawa-Sasaki [9].

Lemma 4.1. [9, Theorem 1.1] *Let* $1 \le p < \infty$. *If* $v \in L^p(\mathbb{R}^N)$ *and* $x \cdot \nabla v \in L^p(\mathbb{R}^N)$, *then*

$$\frac{N}{p}\|v\| \le \|x \cdot \nabla v\|. \tag{4.1}$$

Here we give a simple proof of (4.1) when $p = 2$.

Proof. Let $v \in L^2(\mathbb{R}^N)$ and $x \cdot \nabla v \in L^2(\mathbb{R}^N)$. Integration by parts gives

$$\mathrm{Re}(v, x \cdot \nabla v) = -\frac{N}{2}\|v\|^2. \tag{4.2}$$

Then the Cauchy-Schwarz inequality applies to give (4.1). □

Using two real parameters, we can obtain the following lemma which plays an important role to derive some inequalities.

Lemma 4.2. *If* $v \in L^2(\mathbb{R}^N)$ *and* $|x|^2\Delta v \in L^2(\mathbb{R}^N)$, *then* $|x|\,|\nabla v| \in L^2(\mathbb{R}^N)$ *and*

$$0 \le \big\||x|\nabla v\big\|^4 + 4\|x \cdot \nabla v\|^2\|v\|^2 - 2N\big\||x|\nabla v\big\|^2\|v\|^2 \le \big\||x|^2\Delta v\big\|^2\|v\|^2. \tag{4.3}$$

Proof. Let $v \in L^2(\mathbb{R}^N)$ with $|x|^2\Delta v \in L^2(\mathbb{R}^N)$ and $c_1, c_2 \in \mathbb{R}$. We start with the trivial inequality

$$\begin{aligned} 0 &\le \big\||x|^2\Delta v + c_1 x \cdot \nabla v + c_2 v\big\|^2 \\ &= \big\||x|^2\Delta v\big\|^2 + c_1^2\|x \cdot \nabla v\|^2 + c_2^2\|v\|^2 \\ &+ 2c_1\mathrm{Re}(x \cdot \nabla v, |x|^2\Delta v) + 2c_2\mathrm{Re}(|x|^2\Delta v, v) + 2c_1c_2\mathrm{Re}(v, x \cdot \nabla v). \end{aligned} \tag{4.4}$$

Integration by parts gives

$$\begin{aligned} \mathrm{Re}(x \cdot \nabla v, |x|^2\Delta v) &= \sum_{j,k=1}^{N} \mathrm{Re} \int_{\mathbb{R}^N} |x|^2 x_j \frac{\partial v}{\partial x_j}\overline{\frac{\partial^2 v}{\partial x_k^2}}\, dx \\ &= -\sum_{j,k=1}^{N} \mathrm{Re} \int_{\mathbb{R}^N} \left(2x_j x_k \frac{\partial v}{\partial x_j} + |x|^2\delta_{jk}\frac{\partial v}{\partial x_j} + |x|^2 x_j \frac{\partial^2 v}{\partial x_j \partial x_k}\right)\overline{\frac{\partial v}{\partial x_k}}\, dx \\ &= -2\|x \cdot \nabla v\|^2 - \big\||x|\nabla v\big\|^2 - \frac{1}{2}\sum_{j,k=1}^{N}\int_{\mathbb{R}^N} |x|^2 x_j \frac{\partial}{\partial x_j}\left|\frac{\partial v}{\partial x_k}\right|^2 dx \\ &= -2\|x \cdot \nabla v\|^2 + \frac{N}{2}\big\||x|\nabla v\big\|^2, \end{aligned} \tag{4.5}$$

$$(4.6)\qquad (|x|^2\Delta v, v) = \sum_{k=1}^{N}\int_{\mathbb{R}^N} |x|^2\overline{v}\frac{\partial^2 v}{\partial x_k^2}\,dx$$
$$= -\sum_{k=1}^{N}\int_{\mathbb{R}^N}\left(|x|^2\overline{\frac{\partial v}{\partial x_k}} + 2x_k\overline{v}\right)\frac{\partial v}{\partial x_k}\,dx$$
$$= -\big\||x|\nabla v\big\|^2 - 2(x\cdot\nabla v, v).$$

In view of (4.2) and (4.6) we have

$$(4.7)\qquad \mathrm{Re}(|x|^2\Delta v, v) = -\big\||x|\,\nabla v\big\|^2 + N\|v\|^2.$$

Putting (4.2), (4.5) and (4.6) in (4.4), we have

$$(4.8)\qquad 0 \le \big\||x|^2\Delta v\big\|^2 + (c_1^2 - 4c_1)\|x\cdot\nabla v\|^2 + (Nc_1 - 2c_2)\big\||x|\,\nabla v\big\|^2 + c_2(c_2 + 2N - Nc_1)\|v\|^2.$$

Minimizing the right-hand side of (4.8), i.e., setting $c_1 = 2$, $c_2 = \big\||x|\,\nabla v\big\|^2/\|v\|^2$ for $v \ne 0$, we can obtain the second inequality of (4.3). The first inequality of (4.3) can be shown by completing the square as

$$\left(\big\||x|\,\nabla v\big\|^2 - N\|v\|^2\right)^2 + 4\|v\|^2\left(\|x\cdot\nabla v\|^2 - \frac{N^2}{4}\|v\|^2\right).$$

In fact, the nonnegativity of the second term is a consequence of (4.1). □

Lemma 4.3. *Let $\varepsilon > 0$. Then*

$$(4.9)\qquad \mathrm{Re}(\Delta^2 u, (|x|^4+\varepsilon)^{-1}u) \ge -\kappa_0(N)\|(|x|^4+\varepsilon)^{-1}u\|^2,\ u \in H^4(\mathbb{R}^N),$$
$$(4.10)\qquad \|\Delta u\|^2 \ge \alpha_0(N)\|\,(|x|^2+\varepsilon)^{-1}u\|^2,\ u\in H^2(\mathbb{R}^N),\ N\ge 5.$$

Here $\kappa_0(N)$ and $\alpha_0(N)$ are defined as

$$\kappa_0(N) := \begin{cases} 112 - 3(N-2)^2, & N \le 8,\\ -\dfrac{N}{16}(N-8)(N^2-16), & N\ge 9,\end{cases}$$
$$\alpha_0(N) := \frac{N^2}{16}(N-4)^2,\ N \ge 5.$$

The approximate Rellich inequality (4.10) is already shown in [7, Theorem 6.8] in 1982. Here we can give another proof of (4.10).

Proof. First we shall prove (4.9). Put IP $:= (\Delta^2 u, (|x|^4+\varepsilon)^{-1}u)$ and $v := (|x|^4+\varepsilon)^{-1}u$ for $u \in H^4(\mathbb{R}^N)$. Then IP is written as

$$(4.11)\qquad \mathrm{IP} = (\Delta^2((|x|^4+\varepsilon)v), v)$$
$$= (\Delta((|x|^4+\varepsilon)v), \Delta v)$$
$$= (|x|^4\Delta v + 8|x|^2 x\cdot\nabla v + 4(N+2)|x|^2 v, \Delta v) + \varepsilon\|\Delta v\|^2$$
$$= (|x|^2\Delta v + 8x\cdot\nabla v + 4(N+2)v, |x|^2\Delta v) + \varepsilon\|\Delta v\|^2.$$

From (4.5) and (4.6) we have

$$\text{Re}\,\text{IP} \geq \big\||x|^2\Delta v\big\|^2 - 16\|x\cdot\nabla v\|^2 - 8\big\||x|\,\nabla v\big\|^2 + 4N(N+2)\|v\|^2. \tag{4.12}$$

Applying Lemma 4.2 to the first term of the right-hand side of (4.12) multiplied by $\|v\|^2$, we have

$$\begin{aligned}\|v\|^2\text{Re}\,\text{IP} \geq{}& \big\||x|\nabla v\big\|^4 - 12\|x\cdot\nabla v\|^2\|v\|^2\\ &- 2(N+4)\big\||x|\nabla v\big\|^2\|v\|^2 + 4N(N+2)\|v\|^4.\end{aligned}$$

Since $\|x\cdot\nabla v\|^2 \leq \big\||x|\,\nabla v\big\|^2$, it follows that

$$\begin{aligned}\|v\|^2\text{Re}\,\text{IP} &\geq \big\||x|\,\nabla v\big\|^4 - 2(N+10)\big\||x|\nabla v\big\|^2\|v\|^2 + 4N(N+2)\|v\|^4\\ &= \Big[\big\||x|\,\nabla v\big\|^2 - (N+10)\|v\|^2\Big]^2 - [112 - 3(N-2)^2]\|v\|^4.\end{aligned} \tag{4.13}$$

Hence we obtain $\text{Re}\,\text{IP} \geq -[112-3(N-2)^2]\|v\|^2$. In particular, if $N \geq 9$, then we see from Lemma 4.1 that

$$\begin{aligned}\big\||x|\,\nabla v\big\|^2 - (N+10)\|v\|^2 &\geq \|x\cdot\nabla v\|^2 - (N+10)\|v\|^2\\ &\geq (N^2/4 - N - 10)\|v\|^2\\ &\geq 0.\end{aligned}$$

Applying this inequality to (4.13) implies

$$\begin{aligned}\|v\|^2\text{Re}\,\text{IP} &\geq \Big[\Big(\frac{N^2}{4} - N - 10\Big)\|v\|^2\Big]^2 - [112 - 3(N-2)^2]\|v\|^4\\ &= -\Big[-\frac{N}{16}(N-8)(N^2-16)\Big]\|v\|^4.\end{aligned}$$

Therefore we obtain $\text{Re}\,\text{IP} \geq -\kappa_0(N)\|v\|^2$ which is nothing but (4.9).

Next we give a simplified proof of (4.10). Let $v := (|x|^2+\varepsilon)^{-1}u$ for $u \in H^2(\mathbb{R}^N)$. Then it follows from (4.2) that

$$\begin{aligned}\text{Re}(-\Delta u, (|x|^2+\varepsilon)^{-1}u) &= \text{Re}(-\Delta(|x|^2 v + \varepsilon v), v)\\ &= \text{Re}(\nabla(|x|^2 v + \varepsilon v), \nabla v)\\ &= \text{Re}(|x|^2\nabla v + 2xv + \varepsilon\nabla v, \nabla v)\\ &= \big\||x|\nabla v\big\|^2 - N\|v\|^2 + \varepsilon\|\nabla v\|^2.\end{aligned}$$

Hence Lemma 4.1 implies

$$\begin{aligned}\text{Re}(-\Delta u, (|x|^2+\varepsilon)^{-1}u) &\geq \|x\cdot\nabla v\|^2 - N\|v\|^2\\ &\geq \frac{N}{4}(N-4)\|v\|^2.\end{aligned}$$

Therefore the Schwarz inequality applies to give (4.10). □

Proof of Theorem 1.1. Let $H := L^2(\mathbb{R}^N)$, $A := \Delta^2$ with $D(A) := H^4(\mathbb{R}^N)$ and $B := |x|^{-4}$ with $D(B) := \{u \in H;\ |x|^{-4}u \in H\}$. Then we see that $B_\varepsilon = |x|^{-4}(1+\varepsilon|x|^{-4})^{-1} = (|x|^4 + \varepsilon)^{-1}$ for $\varepsilon > 0$. Therefore Lemma 4.3 allows us to apply Theorem 2.1 with $k_0 = \kappa_0(N)$ if $N \leq 8$ and Theorem 2.2 with $k_1 = -\kappa_0(N)$ and $k_2 = \alpha_0(N)$ if $N \geq 9$. □

5. Proof of Theorem 1.2

In this section we generalize the inequalities obtained in Section 4. To see this we propose the generalized discriminant of bi-form in Hilbert spaces.

Lemma 5.1. *Let X be a complex Hilbert space with inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$. Let $\varphi \in X$, $c \in \mathbb{R}$ and let M be a selfadjoint operator in X. Assume that for every $\zeta \in D(M)$,*

$$(M\zeta,\zeta)_X + 2\mathrm{Re}\,(\varphi,\zeta)_X + c \geq 0. \tag{5.1}$$

Then M is nonnegative and

$$\sup_{\varepsilon>0}((M+\varepsilon)^{-1}\varphi,\varphi)_X \leq c. \tag{5.2}$$

In particular, if M is positive, then

$$(M^{-1}\varphi,\varphi)_X \leq c. \tag{5.3}$$

Proof. First we shall show that M is nonnegative. Considering $\zeta/\|\zeta\|_X$ instead of ζ, it suffices to show that $(M\zeta,\zeta)_X \geq 0$ for $\zeta \in D(M)$ with $\|\zeta\|_X = 1$. Let $t \in \mathbb{R}$ with $t \neq 0$. Then it follows from (5.1) with ζ replaced with $t\zeta$ that

$$\begin{aligned} 0 &\leq t^2(M\zeta,\zeta)_X + 2t\mathrm{Re}\,(\varphi,\zeta)_X + c \\ &\leq t^2(M\zeta,\zeta)_X + 2|t|\,\|\varphi\|_X + c. \end{aligned}$$

This is equivalent to

$$-2|t|^{-1}\|\varphi\|_X - ct^{-2} \leq (M\zeta,\zeta)_X.$$

Letting $|t| \to \infty$ yields that $(M\zeta,\zeta)_X \geq 0$. Next we shall prove (5.2). Let $M_\varepsilon := M+\varepsilon$. Since M is nonnegative selfadjoint in X, we see that M_ε^{-1} is well-defined as a bounded symmetric operator with $\|M_\varepsilon^{-1}\zeta\|_X \leq \varepsilon^{-1}\|\zeta\|_X$. Then (5.1) implies that

$$\begin{aligned} 0 &\leq (M_\varepsilon\zeta,\zeta)_X + 2\mathrm{Re}\,(\varphi,\zeta)_X + c \\ &= (M_\varepsilon(\zeta + M_\varepsilon^{-1}\varphi),\zeta + M_\varepsilon^{-1}\varphi)_X - (M_\varepsilon^{-1}\varphi,\varphi)_X + c. \end{aligned}$$

Taking $\zeta = -M_\varepsilon^{-1}\varphi$, we see that $(M_\varepsilon^{-1}\varphi,\varphi)_X \leq c$ for $\varepsilon > 0$. Therefore we obtain (5.2). In particular, if M is positive, then we can take $\varepsilon = 0$. □

Using two complex parameters, we can obtain the following lemma which is a strict version of Lemma 4.1

Lemma 5.2. *If $v \in L^2(\mathbb{R}^N)$ and $x\cdot\nabla v \in L^2(\mathbb{R}^N)$, then*

$$|\mathrm{Im}\,(v, x\cdot\nabla v)|^2 \leq \|v\|^2\Big(\|x\cdot\nabla v\|^2 - \frac{N^2}{4}\|v\|^2\Big). \tag{5.4}$$

Proof. Let $v \in L^2(\mathbb{R}^N)$ with $x\cdot\nabla v \in L^2(\mathbb{R}^N)$. From the Schwarz inequality we have

$$|(v, x\cdot\nabla v)|^2 \leq \|v\|^2\|x\cdot\nabla v\|^2. \tag{5.5}$$

Combining (4.2) with (5.5), we obtain (5.4). □

If $X := \mathbb{C}^2$, then Lemma 5.1 is regarded as a two-complex-parameter technique to derive a new inequality.

Corollary 5.3. *Let M be a Hermite matrix on $\mathbb{C}^2$:*

$$M = \begin{pmatrix} b & \gamma \\ \overline{\gamma} & a \end{pmatrix},$$

where $a,\ b \in \mathbb{R}$ and $\gamma \in \mathbb{C}$. Assume that there are $\varphi := {}^t(\overline{\alpha}, \beta) \in \mathbb{C}^2$ and $c \in \mathbb{R}$, satisfying (5.1)*. Then it follows from* (5.2) *that*

$$a|\alpha|^2 + b|\beta|^2 - 2\mathrm{Re}\,(\alpha\beta\gamma) \le c(ab - |\gamma|^2).$$

Setting $\alpha := \alpha_1 + i\alpha_2$, $\beta := \beta_1 + i\beta_2$, $\gamma := \gamma_1 + i\gamma_2$, one has

$$\begin{aligned} &a\alpha_2^2 + b\beta_2^2 + c\gamma_2^2 + 2(\alpha_1\beta_2\gamma_2 + \alpha_2\beta_1\gamma_2 + \alpha_2\beta_2\gamma_1) \\ &\le abc + 2\alpha_1\beta_1\gamma_1 - (a\alpha_1^2 + b\beta_1^2 + c\gamma_1^2). \end{aligned} \tag{5.6}$$

The following lemma together with Lemma 5.2 give a strict version of Lemma 4.2.

Lemma 5.4. *If $v \in L^2(\mathbb{R}^N)$ and $|x|^2\Delta v \in L^2(\mathbb{R}^N)$, then $|x|\,|\nabla v| \in L^2(\mathbb{R}^N)$ and*

$$\begin{aligned} &\Big[\|v\|^2\mathrm{Im}\,(x\cdot\nabla v, |x|^2\Delta v) - \big\||x|\nabla v\big\|^2\mathrm{Im}\,(v, x\cdot\nabla v)\Big]^2 \\ &\le \Big[\|v\|^2\|x\cdot\nabla v\|^2 - \frac{N^2}{4}\|v\|^4 - |\mathrm{Im}\,(v, x\cdot\nabla v)|^2\Big] \\ &\times \Big[\big\||x|^2\Delta v\big\|^2\|v\|^2 + 2N\big\||x|\nabla v\big\|^2\|v\|^2 - \big\||x|\nabla v\big\|^4 - 4\|x\cdot\nabla v\|^2\|v\|^2\Big]. \end{aligned} \tag{5.7}$$

Proof. Let $v \in L^2(\mathbb{R}^N)$ with $|x|^2\Delta v \in L^2(\mathbb{R}^N)$. Then for $\zeta = {}^t(\zeta_1, \zeta_2) \in \mathbb{C}^2$ we have an inequality of the form (5.1):

$$\begin{aligned} 0 &\le \big\||x|^2\Delta v + \zeta_1(x\cdot\nabla)v + \zeta_2 v\big\|^2 \\ &= (M\zeta, \zeta)_{\mathbb{C}^2} + 2\,\mathrm{Re}\,(\varphi, \zeta)_{\mathbb{C}^2} + c, \end{aligned}$$

where $\varphi = {}^t(\overline{\alpha}, \beta) := (\overline{((x\cdot\nabla)v, |x|^2\Delta v)}, (|x|^2\Delta v, v))$, $\;c := \big\||x|^2\Delta v\big\|^2$ and

$$M = \begin{pmatrix} b & \gamma \\ \overline{\gamma} & a \end{pmatrix} := \begin{pmatrix} \|(x\cdot\nabla)v\|^2 & (v, (x\cdot\nabla)v) \\ \overline{(v, (x\cdot\nabla)v)} & \|v\|^2 \end{pmatrix}.$$

Thus we obtain (5.6) as a consequence of Corollary 5.3. Now it is easy to see from (4.2), (4.5) and (4.6) that

$$\alpha_1 = \mathrm{Re}\,\alpha = \frac{N}{2}\widetilde{b} - 2b, \tag{5.8}$$

$$\beta_1 = \mathrm{Re}\,\beta = Na - \widetilde{b}, \tag{5.9}$$

$$\gamma_1 = \mathrm{Re}\,\gamma = -\frac{N}{2}a, \tag{5.10}$$

where $\widetilde{b} := \| |x| \nabla v \|^2$. It follows from (5.8)–(5.10) that the right-hand side of (5.6) equals

$$(b - (N^2/4)a)(ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab).$$

Multiplying (5.6) by a and using the equality $\beta_2 = 2\gamma_2$, we have

$$\begin{aligned} a^2\alpha_2^2 + 2a(\beta_1 + 2\gamma_1)\alpha_2\gamma_2 + a(4\alpha_1 + 4b + c)\gamma_2^2& \\ \le a(b - (N^2/4)a)(ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab).& \end{aligned} \tag{5.11}$$

We see from (5.8)–(5.10) that the left-hand side of (5.11) equals

$$(a\alpha_2 - \widetilde{b}\gamma_2)^2 + (ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab)\gamma_2^2,$$

which implies that

$$(a\alpha_2 - \widetilde{b}\gamma_2)^2 \le (ab - (N^2/4)a^2 - \gamma_2^2)(ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab).$$

This proves (5.7). □

Lemma 5.5. *Let $u \in H^4(\mathbb{R}^N)$ and $\varepsilon > 0$. Let k_1 and k_2 be constants defined as*

$$\begin{aligned} k_1 &:= 112 - 3(N-2)^2, \\ k_2 &:= -\frac{N}{16}(N-8)(N^2-16), \ N \ge 9. \end{aligned}$$

Put $\mathrm{IP} := (\Delta^2 u, (|x|^4 + \varepsilon)^{-1}u)$ *and* $a := \|(|x|^4 + \varepsilon)^{-1}u\|^2$. *Then*

$$(\mathrm{Im}\,\mathrm{IP})^2 \le 64\sqrt{a}\Big(\sqrt{\mathrm{Re}\,\mathrm{IP} + k_1 a} + \Big(10 + N - \frac{N^2}{4}\Big)\sqrt{a}\Big)\Big(\sqrt{\mathrm{Re}\,\mathrm{IP} + k_1 a} + 8\sqrt{a}\Big)^2. \tag{5.12}$$

If $N \ge 9$, then it is equivalent to

$$(\mathrm{Im}\,\mathrm{IP})^2 \le \frac{64\sqrt{a}(\mathrm{Re}\,\mathrm{IP} + k_2 a)\Big(\sqrt{\mathrm{Re}\,\mathrm{IP} + k_1 a} + 8\sqrt{a}\Big)^2}{\sqrt{\mathrm{Re}\,\mathrm{IP} + k_1 a} + \Big(\frac{N^2}{4} - N - 10\Big)\sqrt{a}}. \tag{5.13}$$

Proof. Let $u \in H^4(\mathbb{R}^N)$ and $\varepsilon > 0$. Put $v := (|x|^4 + \varepsilon)^{-1}u$. Using the same notations as in the proof of Lemma 5.4, we see that (5.7) is written as

$$L := \frac{(a\alpha_2 - \widetilde{b}\gamma_2)^2}{ab - (N^2/4)a^2 - \gamma_2^2} \le ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab =: R. \tag{5.14}$$

Here we note (4.11) that

$$\mathrm{IP} = \| |x|^2 \Delta v \|^2 + 8(\,(x \cdot \nabla)v, |x|^2\Delta v) + 4(N+2)(v, |x|^2\Delta v) + \varepsilon\|\Delta v\|^2.$$

Since $\beta_2 = 2\gamma_2$, it follows that

$$c = \| |x|^2 \Delta v \|^2 \le \mathrm{Re}\,\mathrm{IP} + 16b + 8\widetilde{b} - 4N(N+2)a, \tag{5.15}$$

$$\alpha_2 = \mathrm{Im}\,((x \cdot \nabla)v, |x|^2\Delta v) = \frac{1}{8}\mathrm{Im}\,\mathrm{IP} + (N+2)\gamma_2. \tag{5.16}$$

Applying (5.16) to L yields

$$L = \frac{\left(\frac{a}{8}\mathrm{Im}\,\mathrm{IP} + ((N+2)a - \widetilde{b})\gamma_2\right)^2}{a(b-(N^2/4)a) - \gamma_2^2} = \frac{(c_1\gamma_2 + c_2)^2}{c_0 - \gamma_2^2},$$

where

$$c_0 := a(b-(N^2/4)a) \geq \gamma_2^2, \tag{5.17}$$

$$c_1 := (N+2)a - \widetilde{b}, \tag{5.18}$$

$$c_2 := \frac{a}{8}\mathrm{Im}\,\mathrm{IP}; \tag{5.19}$$

note that the inequality in (5.17) is nothing but (5.4). Since the quadratic equation $L(c_0 - t^2) = (c_1 t + c_2)^2$ has a real root $t = \gamma_2$, the discriminant is nonnegative:

$$L(c_0 L + c_0 c_1^2 - c_2^2) \geq 0. \tag{5.20}$$

It is clear that $L \geq 0$. If $L > 0$, then (5.20) yields

$$L \geq (c_2^2/c_0) - c_1^2. \tag{5.21}$$

If $L = 0$, then $\gamma_2 = -c_2/c_1$ and hence (5.17) yields that $0 \geq (c_2^2/c_0) - c_1^2$. This means that (5.21) holds for $L \geq 0$. Hence it follows from (5.17)–(5.19) and (5.21) that

$$L \geq \frac{a|\mathrm{Im}\,\mathrm{IP}|^2}{64(b-(N^2/4)a)} - (\widetilde{b} - (N+2)a)^2. \tag{5.22}$$

On the other hand, since $b \leq \widetilde{b}$, (5.14) and (5.15) yields

$$\begin{aligned} R &\leq a\mathrm{Re}\,\mathrm{IP} + 12ab + 2(N+4)a\widetilde{b} - \widetilde{b}^2 - 4N(N+2)a^2 \\ &\leq a(k_1 a + \mathrm{Re}\,\mathrm{IP}) - (\widetilde{b} - (N+10)a)^2, \end{aligned} \tag{5.23}$$

where $k_1 := (N+10)^2 - 4N(N+2) = 112 - 3(N-2)^2$. Since $L \leq R$, it follows from (5.22) and (5.23) that

$$\frac{a|\mathrm{Im}\,\mathrm{IP}|^2}{64(b - N^2 a/4)} - (\widetilde{b} - (N+2)a)^2 \leq a(k_1 a + \mathrm{Re}\,\mathrm{IP}) - (\widetilde{b} - (N+10)a)^2. \tag{5.24}$$

Therefore we obtain

$$\frac{|\mathrm{Im}\,\mathrm{IP}|^2}{64(b-(N^2/4)a)} - 16(\widetilde{b} - (N+6)a) \leq k_1 a + \mathrm{Re}\,\mathrm{IP} =: K. \tag{5.25}$$

Now we see from (5.23) that

$$(\widetilde{b} - (N+10)a)^2 \leq R + (\widetilde{b} - (N+10)a)^2 \leq aK$$

and hence

$$b \leq \widetilde{b} \leq \sqrt{aK} + (N+10)a. \tag{5.26}$$

Applying (5.26) to (5.25), we obtain

$$\frac{|\mathrm{Im\,IP}|^2}{64\sqrt{a}\big[\sqrt{K}-((N^2/4)-N-10)\sqrt{a}\big]} \le K + 16(\sqrt{aK}+4a) = (\sqrt{K}+8\sqrt{a})^2.$$

This proves (5.12). Next note that $N^2/4 - N - 10 \ge 0$ for $N \ge 9$. To obtain (5.13), we have only to use the equality

$$\sqrt{K}-((N^2/4)-N-10)\sqrt{a} = \frac{k_2 a + \mathrm{Re\,IP}}{\sqrt{K}+((N^2/4)-N-10)\sqrt{a}}$$

where $k_2 = -N(N-8)(N^2-16)/16$. □

Proof of Theorem 1.2. Let $H := L^2(\mathbb{R}^N)$, $A := \Delta^2$ with $D(A) := H^4(\mathbb{R}^N)$ and $B := |x|^{-4}$ with $D(B) := \{u \in H;\ |x|^{-4}u \in H\}$. For $u \in D(A)$ and $\varepsilon > 0$ take $v := B_\varepsilon u = (|x|^4+\varepsilon)^{-1}u$ with $\sqrt{a} := \|v\| = 1$. Then set

$$\xi + i\eta := -\mathrm{IP} = -(Au, B_\varepsilon u).$$

If $N \le 8$, then $\xi \le k_1 := 112 - 3(N-2)^2$. In fact, we see from (4.9) that

$$-\xi = \mathrm{Re\,IP} \ge -[112-3(N-2)^2] \ \text{ for } v \in H \ \text{ with } \|v\| = 1.$$

Thus (5.12) (with $\mathrm{Re\,IP} = -\xi, \mathrm{Im\,IP} = -\eta, a = 1$) allows us to apply Theorem 3.1 with

$$\Sigma := \{\xi + i\eta \in \mathbb{C};\ \xi \le k_1, \eta^2 \le \varphi_N(\xi)\},$$
$$\gamma(\eta) + i\eta \in \partial\Sigma\,(\Longrightarrow \gamma(0) = k_1 > 0),$$

where

$$\varphi_N(\xi) := 64\big[\sqrt{k_1-\xi} + (10+N-(N^2/4))\big](\sqrt{k_1-\xi}+8)^2, \quad \xi \le k_1.$$

In more detail γ is given by

$$\gamma(\eta) := \begin{cases} k_1, & |\eta| \le \eta_N, \\ \varphi_N^{-1}(\eta^2) \iff \eta^2 = \varphi_N(\gamma(\eta)), & |\eta| \ge \eta_N, \end{cases}$$

where $\eta_N := \sqrt{\varphi_N(k_1)} = \sqrt{\min \varphi_N} = 64\sqrt{10+N-(N^2/4)}$. In particular, if $N \ge 5$, then the Rellich inequality (4.10)

$$(N/4)(N-4)\|(|x|^2+\varepsilon)^{-1}u\| \le \|\Delta u\|, \quad u \in H^2(\mathbb{R}^N)$$

applies to give (3.9) with $\alpha_0 := (N^2/16)(N-4)^2$. In fact, it follows for every $u \in D(A) \cap D(B)$ that $u \in D(A^{1/2}) \subset D(B^{1/2})$ and

$$\alpha_0((|x|^4+\varepsilon)^{-1}u, u) \le \alpha_0(|x|^{-4}u, u) = \alpha_0\big\||x|^{-2}u\big\|^2 \le \|\Delta u\|^2 = (\Delta^2 u, u).$$

Thus we can apply Theorem 3.5 with those Σ, γ and α_0.

If $N \geq 9$, then we have $\xi \leq k_2 := -(N/16)(N-8)(N^2-16)$. In fact, it follows from (4.9) that

$$-\xi = \mathrm{Re}\,\mathrm{IP} \geq (N/16)(N-8)(N^2-16) \quad \text{for } v \in H \text{ with } \|v\| = 1.$$

Thus (5.13) allows us to apply Theorem 3.5 with $\alpha_0 := (N^2/16)(N-4)^2$ and

$$\Sigma := \{\xi + i\eta \in \mathbb{C}\ ;\ \xi \leq k_2, \eta^2 \leq \varphi_N(\xi)\},$$
$$\gamma(\eta) + i\eta \in \partial\Sigma\,(\Longrightarrow -\alpha_0 < \gamma(0) = k_2 < 0).$$

where

$$\varphi_N(\xi) := \frac{64(k_2-\xi)(\sqrt{k_1-\xi}+8)^2}{\sqrt{k_1-\xi}+((N^2/4)-N-10)}, \quad \xi \leq k_2.$$

γ is given by $\gamma(\eta) := \varphi_N^{-1}(\eta^2)$. This completes the proof of Theorem 1.2. □

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