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Transition layers for a bistable reaction-diffusion equation in heterogeneous media

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1 Introduction

We will consider the following reaction-diffusion problem:

$$\begin{cases} u_t = \varepsilon^2 (d(x)^2 u_x)_x + h(x)^2 f(u), & 0 < x < 1,\ t > 0, \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases} \tag{1.1}$$

Here ε denotes a positive parameter and f is given by $f(u) = u(1-u)(u-1/2)$. Moreover d and h are C^2-functions with the following properties:

(Φ1) $d(x) > 0$ and $h(x) > 0$ in $[0,1]$.

(Φ2) Define $\varphi(x) := d(x)h(x)$ and $\Sigma := \{x \in [0,1]\,;\, \varphi_x(x) = 0\}$. Then Σ is a non-empty finite set and $\varphi_{xx}(x) \neq 0$ at any $x \in \Sigma$.

(Φ3) $d_x(0) = d_x(1) = h_x(0) = h_x(1) = 0$.

In this paper, we will mainly discuss the stationary problem of (1.1) as follows:

$$\begin{cases} \varepsilon^2 (d(x)^2 u_x)_x + h(x)^2 f(u) = 0, & 0 < x < 1, \\ u_x(0) = u_x(1) = 0. \end{cases} \tag{1.2}$$

This problem describes phase transition phenomena in various fields such as physics, chemistry and mathematical biology. It should be noted that (1.2) possesses spatial inhomogeneity both in its diffusion and reaction terms. Hence we may consider that this problem describes a certain phenomenon in heterogeneous media. One of the most usual examples of such media is a porus medium. We remark that their description of reaction-diffusion processes should be made in terms of the porosity of the respective material. We will discuss the relation between our problem and the porosity of porus media in Appendix.

It is well known that (1.1) describes a phase transition phenomenon and that this kind of problem admits a solution with **transition layers** when ε is sufficiently small. Here a transition layer means a part of a solution where its value is drastically changing when x varies in a very small interval.

The appearance of such solutions has close connections in the bistability of our problem. As an energy functional of (1.1), one can take

$$E(u) := \int_0^1 \left[\frac{\varepsilon^2}{2} d(x)^2 (u_x(x))^2 + h(x)^2 W(u(x))\right] dx$$

with

$$W(u) := -\int_0^u f(s)ds. \tag{1.3}$$

Here W is called a **bistable potential** because W attains its local minima both at $u = 0$ and 1. It is well known that every solution of (1.1) converges a solution of (1.2) as $t \to \infty$ and that $E(u(\cdot, t))$ is monotone decreasing with respect to t. Hence the minimizer of E will be a stable steady-states. For proofs of these facts, see Matano [2].

In this paper, we concentrate ourselves to investigate all patterns of solutions of (1.2) with transition layers. For transition layers, we can observe that several transition layers appear in a vicinity of a certain point as a cluster. This is called a **multi-layer**, while a transition layer which is away from other transition layers is called a **single-layer**. We will also discuss multi-layers.

We now present some related results. When both d and h are constant functions, Chafee and Infant [1] proved that, for any $n \in \mathbb{N}$, if ε is sufficiently small, then (1.2) admits a solution with n transition layers placed evenly spaced apart, and that every non-constant solution is unstable. See Miyata and Yanagida [3], they have discussed the case that only h is a constant function. They proved that there exists a stable solution with single-layers and each of them is located in a neighborhood of a local minimum point of d. In the case that only d is a constant function, Nakashima [4] has proved that there exists a solution with single-layers and multi-layers. She has shown that each of them must be in a vicinity of a critical point of h and that any multi-layer appears only in a neighborhood of a local maximum point of h. Moreover, she discussed the stability of such solutions. She proved that if every transition layer is a single-layer and each of them appears in a neighborhood of a local minimum point of h, then the solution is stable, and that any solution with a multi-layer is unstable by using Morse index. We also refer to the work of Urano, Nakashima and Yamada [6], who have studied patterns of solutions with transition layers including multi-layers for $\varepsilon^2 u_{xx} + u(1-u)(u - a(x)) = 0$ where a denotes a C^2-function lying between 0 and 1.

We now introduce the notion of **n-mode solutions** as follows:

Definition 1.1. Let u^ε be a solution of (1.2). Then u^ε is called an ***n*-mode solution** of (1.2), if $u^\varepsilon - 1/2$ has exactly n zero points in $(0,1)$.

Since our target is a solution with oscillatory profiles, the notion of n-mode solutions is very convenient. In what follows we will denote the set of all n-mode solutions of (1.2) by $S_{n,\varepsilon}$. Furthermore, for $u^\varepsilon \in S_{n,\varepsilon}$, set

$$\Xi := \{x \in (0,1)\,;\, u^\varepsilon(x) = 1/2\}.$$

For any n-mode solution u^ε, one will find out in Section 3 that the interval $[0,1]$ is divided into the following form when ε is small:

$$[0,1] = \{x\,;\ u^\varepsilon(x) \text{ is very close to either 0 or 1}\} \cup \{x\,;\ u^\varepsilon(x) \text{ forms a transition layer}\}.$$

Therefore, it is essential to consider n-mode solutions in the study of solutions with transition layers. Then we obtain the following theorems (Theorems 1.1 and 1.2) concerning the pattern of solutions with transition layers:

Theorem 1.1 (Location of transition layers for solutions of (1.2)). *For $u^\varepsilon \in S_{n,\varepsilon}$, if u^ε has a transition layer, then it appears only in an $O(\varepsilon|\log\varepsilon|)$-neighborhood of a point in Σ. Moreover, the following assertions hold true:*
(i) If u^ε has a multi-layer, then it appears only in a neighborhood of a local maximum point of φ.
(ii) If u^ε has a transition layer in a neighborhood of a local minimum point of φ, then it must be a single-layer.
(iii) If $\varphi_{xx}(0) > 0$ (resp. $\varphi_{xx}(1) > 0$), then u^ε has no transition layer in a neighborhood of 0 (resp. 1).

Theorem 1.2. *Let $\sigma \in \Sigma$ satisfy $\varphi_{xx}(\sigma) < 0$ and δ be a small positive number. For $u^\varepsilon \in S_{n,\varepsilon}$ and $m \in \mathbb{N}$ with $2 \le m \le n$, set $\{\xi_k\}_{k=1}^m = \Xi \cap (\sigma-\delta, \sigma+\delta)$. Moreover, let $\{\zeta_k\}_{k=0}^m$ be a unique set of critical points of u^ε satisfying $\zeta_0 < \xi_1 < \zeta_1 < \xi_2 < \cdots < \xi_m < \zeta_m$, where $\zeta_0 := \sup\{y\,;\, u^\varepsilon_y(y) = 0 \text{ and } y < \xi_1\}$ and $\zeta_m := \inf\{y\,;\, u^\varepsilon_y(y) = 0 \text{ and } y > \xi_m\}$. If $u^\varepsilon_y(\xi_1) < 0$ (resp. $u^\varepsilon_y(\xi_1) > 0$), then it holds that*

$$\begin{cases} u^\varepsilon(\zeta_{2j-2}) > u^\varepsilon(\zeta_{2j}) & \text{if } \zeta_0 \le \zeta_{2j-2} < \zeta_{2j} < \sigma, \\ u^\varepsilon(\zeta_{2j-1}) < u^\varepsilon(\zeta_{2j+1}) & \text{if } \zeta_1 \le \zeta_{2j-1} < \zeta_{2j+1} < \sigma, \end{cases}$$

and

$$\begin{cases} u^\varepsilon(\zeta_{2j-2}) < u^\varepsilon(\zeta_{2j}) & \text{if } \sigma \le \zeta_{2j-2} < \zeta_{2j} \le \zeta_m, \\ u^\varepsilon(\zeta_{2j-1}) > u^\varepsilon(\zeta_{2j+1}) & \text{if } \sigma \le \zeta_{2j-1} < \zeta_{2j+1} \le \zeta_m. \end{cases}$$

(*resp.*

$$\begin{cases} u^\varepsilon(\zeta_{2j-2}) < u^\varepsilon(\zeta_{2j}) & \text{if } \zeta_0 \le \zeta_{2j-2} < \zeta_{2j} < \sigma, \\ u^\varepsilon(\zeta_{2j-1}) > u^\varepsilon(\zeta_{2j+1}) & \text{if } \zeta_1 \le \zeta_{2j-1} < \zeta_{2j+1} < \sigma, \end{cases}$$

and

$$\begin{cases} u^\varepsilon(\zeta_{2j-2}) > u^\varepsilon(\zeta_{2j}) & \text{if } \sigma \le \zeta_{2j-2} < \zeta_{2j} \le \zeta_m, \\ u^\varepsilon(\zeta_{2j-1}) < u^\varepsilon(\zeta_{2j+1}) & \text{if } \sigma \le \zeta_{2j-1} < \zeta_{2j+1} \le \zeta_m.) \end{cases}$$

The following figure denotes a typical profile of an n-mode solution (Figure 1):

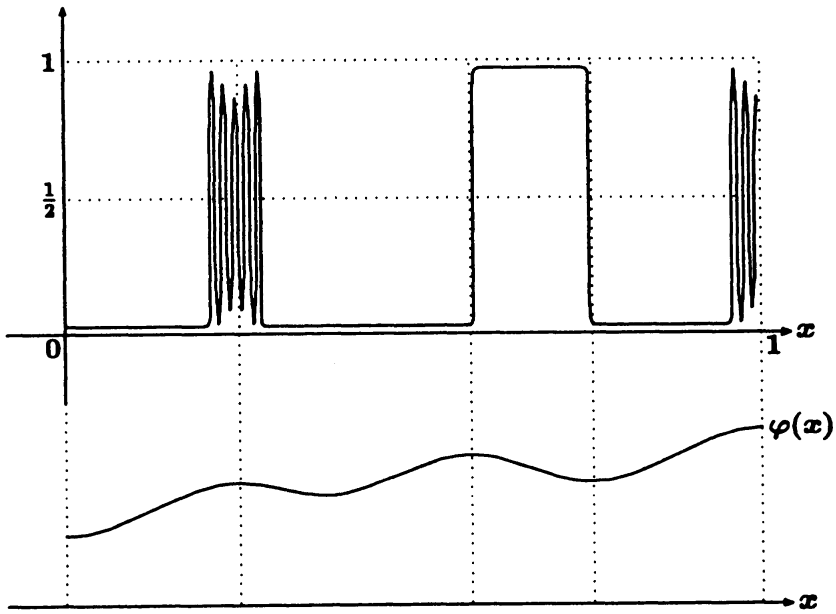

Figure 1: Typical profile of an n-mode solution

2 Change of variables

In the study of (1.2), it is suitable to make the change of variables

$$x \mapsto y = \int_0^x \frac{h(s)}{d(s)} ds. \tag{2.1}$$

Then (1.2) is transformed into

$$\begin{cases} \varepsilon^2 u_{yy} + \varepsilon^2 \gamma(y) u_y + f(u) = 0 & \text{in } (0, L), \\ u_y(0) = u_y(L) = 0, \end{cases} \tag{2.2}$$

where

$$\gamma(y) := \frac{\varphi_x(x)}{h(x)^2} \quad \text{and} \quad L := \int_0^1 \frac{h(s)}{d(s)} ds.$$

From (Φ1)-(Φ3), γ satisfies the following conditions:

(Γ1) $\gamma(0) = \gamma(L) = 0$.

(Γ2) Let $\tilde{\Sigma} := \{y \in [0, L] ; \gamma(y) = 0\}$. Then $\tilde{\Sigma}$ is a non-empty finite set and $\gamma_y(y) \neq 0$ at any $y \in \tilde{\Sigma}$.

Moreover, we will introduce the following notation:

$$\tilde{\Sigma}^+ := \{y \in \tilde{\Sigma} ; \gamma_y(y) > 0\} \qquad \text{and} \qquad \tilde{\Sigma}^- := \{y \in \tilde{\Sigma} ; \gamma_y(y) < 0\}.$$

Since $d(x) > 0$ and $h(x) > 0$ in $[0, 1]$, every solution of (1.2) has a one-to-one correspondence to that of (2.2). In particular, every n-mode solution of (1.2) corresponds to an n-mode solution of (2.2), which will be defined as in Definition 2.1:

Definition 2.1. Let $\tilde{u}^\varepsilon$ be a solution of (2.2). Then $\tilde{u}^\varepsilon$ is called an ***n*-mode solution** of (2.2), if $\tilde{u}^\varepsilon - 1/2$ has exactly n zero points in $(0, L)$.

We will study n-mode solutions of (2.2) as subsutitute for that of (1.2). Hereafter we will denote the set of all n-mode solutions of (2.2) by $\tilde{S}_{n,\varepsilon}$ and, for $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$, we put

$$\tilde{\Xi} := \{y \in [0, L] ; \tilde{u}^\varepsilon(y) = 1/2\}.$$

Since $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$ satisfies $\tilde{u}^\varepsilon_y(0) = \tilde{u}^\varepsilon_y(L) = 0$, one can extend $\tilde{u}^\varepsilon$ to a function over $\mathbb{R}$ by the standard reflection. Therefore, if necessary, we may regard $\tilde{u}^\varepsilon$ as a function in $\mathbb{R}$ satisfying (2.2) for all $y \in \mathbb{R}$.

By the change of variables (2.1), we see that each $y_0 \in \tilde{\Sigma}$ corresponds to an $x_0 \in \Sigma$. If $y_0 \in \tilde{\Sigma}^-$, then it follows from

$$\gamma_y(y) = \frac{d(x)\{\varphi_{xx}(x)h(x) - 2\varphi_x(x)h_x(x)\}}{h(x)^4}$$

that x_0 satisfies $\varphi_{xx}(x_0) < 0$. In other words, any element of $\tilde{\Sigma}^-$ corresponds to a local maximum point of φ, while any element of $\tilde{\Sigma}^+$ corresponds to a local minimum point of φ. Moreover, for $y_1, y_2 \in [0, L]$ with $y_1 \leq y_2$, it holds that

$$M_*(x_2 - x_1) \leq y_2 - y_1 = \int_{x_1}^{x_2} \frac{h(s)}{d(s)} ds \leq M^*(x_2 - x_1).$$

Here x_1 and x_2 are corresponding points in $[0, 1]$ to y_1 and y_2, respectively, and M^* and M_* are positive constants defined by

$$M_* := \min_{x \in [0,1]} \frac{h(x)}{d(x)} \quad \text{and} \quad M^* := \max_{x \in [0,1]} \frac{h(x)}{d(x)}.$$

Therefore, Theorem 1.1 is equivalent to the following theorem:

Theorem 2.1 (Location of transition layers for solutions of (2.2)). *For $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$, if $\tilde{u}^\varepsilon$ has a transition layer, then it appears only in an $O(\varepsilon|\log \varepsilon|)$-neighborhood of a point in $\tilde{\Sigma}$. Moreover, the following assertions hold true:*
(i) If $\tilde{u}^\varepsilon$ has a multi-layer, then it appears only in a neighborhood of a point in $\tilde{\Sigma}^-$.
(ii) If $\tilde{u}^\varepsilon$ has a transition layer in a neighborhood of a point in $\tilde{\Sigma}^+$, then it must be a single-layer.
(iii) If $\gamma_y(0) > 0$ (resp. $\gamma_y(L) > 0$), then $\tilde{u}^\varepsilon$ has no transition layer in a neighborhood of 0 *(resp.* 1*).*

Now our goal is to give a proof of this theorem, which will be found in Section 4.

3 Some properties of n-mode solutions

In this section, we will collect some properties of n-mode solutions of (2.2).

Lemma 3.1. *For $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$, it holds that $0 < \tilde{u}^\varepsilon(y) < 1$ in $[0, L]$.*

Lemma 3.2. *For $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$, let $\tilde{\Xi} = \{\tilde{\xi}_k\}_{k=1}^n$ with $\tilde{\xi}_1 < \tilde{\xi}_2 < \cdots < \tilde{\xi}_n$. Then there exist a unique set of critical points $\{\tilde{\zeta}_k\}_{k=0}^n$ of $\tilde{u}^\varepsilon$ such that*

$$0 = \tilde{\zeta}_0 < \tilde{\xi}_1 < \tilde{\zeta}_1 < \tilde{\xi}_2 < \cdots < \tilde{\xi}_n < \tilde{\zeta}_n = L.$$

Moreover $\tilde{u}^\varepsilon$ takes either its local maximum or minimum at $\tilde{\zeta}_k$ for each $k = 0, 1, \ldots, n$.

The following two lemmas (Lemmas 3.3 and 3.4) play a fundamental role in our analysis:

Lemma 3.3. *For $n \in \mathbb{N}$, it holds that*

$$\lim_{\varepsilon\to 0}\sup_{\tilde{u}^\varepsilon\in\tilde{S}_{n,\varepsilon}}\max_{y\in[0,L]}\left|\tilde{u}^\varepsilon(y)(1-\tilde{u}^\varepsilon(y))\left[\frac{1}{2}\varepsilon^2(\tilde{u}^\varepsilon_y(y))^2 - W(\tilde{u}^\varepsilon(y))\right]\right| = 0$$

where W is a function defined in (1.3).

Lemma 3.4. *For $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$, let $\tilde{\xi}^\varepsilon$ be a point in $\tilde{\Xi}$ and define $U^\varepsilon(z) = \tilde{u}^\varepsilon(\tilde{\xi}^\varepsilon + \varepsilon z)$. Then it holds that*

$$\lim_{\varepsilon\to 0} U^\varepsilon = U \quad \text{in } C^2_{loc}(\mathbb{R}),$$

where $U \in C^2(\mathbb{R})$ is the unique solution of

$$\begin{cases} U_{zz} + f(U) = 0 & \text{in } \mathbb{R}, \\ U(-\infty) = 0,\ U(\infty) = 1 & (\text{resp. } U(-\infty) = 1,\ U(\infty) = 0) \\ U(0) = 1/2 & \text{in } \mathbb{R}, \end{cases}$$

if $U_z(0) > 0$ (resp. $U_z(0) < 0$).

We will have some comments on Lemmas 3.3 and 3.4. For $\tilde{u}^\varepsilon \in S^{n,\varepsilon}$ and any small positive constant δ, which is independent of ε, define $Y^\delta := \{y \in [0,L]\,;\, \delta < \tilde{u}^\varepsilon(y) < 1-\delta\}$. By virtue of Lemma 3.3, for any $\eta > 0$, if ε is sufficiently small, then it holds that

$$\left|\frac{1}{2}\varepsilon^2(\tilde{u}^\varepsilon_y(y))^2 - W(\tilde{u}^\varepsilon(y))\right| < \eta \qquad \text{in } Y^\delta. \tag{3.1}$$

Note that $C\delta^2 < W(\tilde{u}^\varepsilon(y)) \le 1/64$ in Y^δ with some positive constant C. Hence it follows from (3.1) that there exist some positive constants C_1 and C_2 satisfying

$$\frac{C_1}{\varepsilon} \le |\tilde{u}^\varepsilon_y(y)| \le \frac{C_2}{\varepsilon} \qquad \text{in } Y^\delta. \tag{3.2}$$

We should note that $\tilde{\Xi} \subset Y^\delta$. Therefore, Lemma 3.4 enables us to see that (3.2) is also valid for all y belonging to an $O(\varepsilon)$-neighborhood of a point of $\tilde{\Xi}$; i.e., we have

$$\frac{C_1}{\varepsilon} \le |\tilde{u}^\varepsilon_y(y)| \le \frac{C_2}{\varepsilon} \qquad \text{in } \bigcup_{\tilde{\xi}\in\tilde{\Xi}}(\tilde{\xi}-\varepsilon, \tilde{\xi}+\varepsilon),$$

provided that ε is sufficiently small.

Summarizing these arguments, roughly speaking, we can conclude that the graph of an n-mode solution $\tilde{u}^\varepsilon$ of (2.2) is classified into the following two parts when ε is sufficiently small:

(i) $\tilde{u}^\varepsilon(y)$ is very close to either 0 or 1.

(ii) $\tilde{u}^\varepsilon(y)$ forms a transition layer connecting 0 and 1.

Moreover, one can also see that any transition layer appears only in a vicinity of a point in $\tilde{\Xi}$. Hence, in our analysis, it is essential to determin the location of the elements belonging to $\tilde{\Xi}$.

Now we will state some asymptotic results of $\tilde{u}^\varepsilon(y)$ and $1-\tilde{u}^\varepsilon(y)$ as ε goes to 0 in a certain interval which contains a local minimum or maximum point of $\tilde{u}^\varepsilon$.

Theorem 3.5 (Asymptotic profiles of n-mode solutions). *For $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$, let $\tilde{\xi}_1$ and $\tilde{\xi}_2$ be successive points in $\tilde{\Xi}$ satisfying $\tilde{u}^\varepsilon(y) > 1/2$ (resp. $\tilde{u}^\varepsilon(y) < 1/2$) in $(\tilde{\xi}_1, \tilde{\xi}_2)$. Moreover let $\tilde{\zeta}$ be a unique critical point in $(\tilde{\xi}_1, \tilde{\xi}_2)$ where $\tilde{u}^\varepsilon$ takes its local maximum (resp. minimum). Then there exist some positive constants C and r such that*

$$1-\tilde{u}^\varepsilon(y) < C\exp\left(-\frac{rl(y)}{\varepsilon}\right) \quad in\ [\tilde{\xi}_1, \tilde{\xi}_2].$$

$$\left(\textit{resp.} \qquad \tilde{u}^\varepsilon(y) < C\exp\left(-\frac{rl(y)}{\varepsilon}\right) \quad in\ [\tilde{\xi}_1, \tilde{\xi}_2]. \right)$$

Here l denotes a function defined by

$$l(y) := \begin{cases} y-\tilde{\xi}_1 & in\ [\tilde{\xi}_1, \tilde{\zeta}] \\ \tilde{\xi}_2 - y & in\ [\tilde{\zeta}, \tilde{\xi}_2]. \end{cases}$$

Finally, we will give some more estimates.

Lemma 3.6. *For $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$ and $\tilde{\xi} \in \tilde{\Xi}$, define $\tilde{\zeta}_0 := \sup\{y\,;y < \tilde{\xi}$ and $\tilde{u}^\varepsilon_y(y) = 0\}$ and $\tilde{\zeta}_1 := \inf\{y\,;y > \tilde{\xi}$ and $\tilde{u}^\varepsilon_y(y) = 0\}$. Moreover, let $\tilde{\sigma}_1, \tilde{\sigma}_2 \in \tilde{\Sigma}$ satisfy $\gamma(y) > 0$ (resp. $\gamma(y) < 0$) in $(\tilde{\sigma}_1, \tilde{\sigma}_2)$. If $\tilde{\xi} \in (\tilde{\sigma}_1 + \delta, \tilde{\sigma}_2 - \delta)$ with some positive constant δ, which is independent of ε, then there exists a positive constant K satisfying*

$$\begin{cases} 1-\tilde{u}^\varepsilon(\tilde{\zeta}_1) > K\sqrt{\varepsilon} & if\ \tilde{u}^\varepsilon_y(\tilde{\xi}) > 0, \\ \tilde{u}^\varepsilon(\tilde{\zeta}_1) > K\sqrt{\varepsilon} & if\ \tilde{u}^\varepsilon_y(\tilde{\xi}) < 0. \end{cases}$$

$$\left(\textit{resp.} \qquad \begin{cases} \tilde{u}^\varepsilon(\tilde{\zeta}_0) > K\sqrt{\varepsilon} & if\ \tilde{u}^\varepsilon_y(\tilde{\xi}) > 0, \\ 1-\tilde{u}^\varepsilon(\tilde{\zeta}_0) > K\sqrt{\varepsilon} & if\ \tilde{u}^\varepsilon_y(\tilde{\xi}) < 0. \end{cases} \right)$$

Furthermore, it holds that

$$\tilde{\zeta}_1 - \tilde{\xi} = O(\varepsilon|\log\varepsilon|) \qquad (\textit{resp. } \tilde{\xi} - \tilde{\zeta}_0 = O(\varepsilon|\log\varepsilon|)).$$

Proofs of all theorems and lemmas in this section will be found in Urano [5].

4 Location of transition layers and their multiplicity

We will discuss the location of transition layers. It follows from Lemmas 3.3, 3.4 and the argument behind them that any n-mode solution forms a transition layer in a vicinity of a point in $\tilde{\Xi}$. Thus our goal is now to know the location of the points belonging to $\tilde{\Xi}$.

In this section, we will prove Theorem 2.1. However, it requires lengthly arguments to show all assertions in Theorem 2.1; so that we only give a proof of the following lemma:

Lemma 4.1. *Let $\tilde{u}^\varepsilon \in \tilde{S}_{n,\varepsilon}$ and take any small positive constant δ independently of ε. If $\tilde{u}^\varepsilon$ has a transition layer, then it appears only in a δ-neighborhood of a point in $\tilde{\Sigma}$ when ε is sufficiently small.*

Proof. For $l \in \mathbb{N}$, set $\tilde{\Sigma} = \{\tilde{\sigma}_k\}_{k=0}^{l}$ with $0 = \tilde{\sigma}_0 < \tilde{\sigma}_1 < \cdots < \tilde{\sigma}_l = L$. We will show this lemma by using a contradiction method. For this purpose, suppose

$$\tilde{\Xi} \cap \bigcup_{k=0}^{l-1} (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta) \neq \emptyset.$$

In this case, we can choose a point $\tilde{\xi}_1 \in \tilde{\Xi}$ and a number $k \in \{0, 1, \ldots, l-1\}$ satisfying

$$\tilde{\xi}_1 \in (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta).$$

Moreover, we put $\tilde{\zeta}_1 := \inf\{y\,; y > \tilde{\xi}_1 \text{ and } \tilde{u}^\varepsilon_y(y) = 0\}$. We only consider the case that $\tilde{u}^\varepsilon_y(\tilde{\xi}_1) > 0$ and $\gamma(y) > 0$ in $(\tilde{\sigma}_k, \tilde{\sigma}_{k+1})$.

By virtue of Lemma 3.6, there exists a positive constant K_1 such that

$$1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1) > K_1\sqrt{\varepsilon} \quad \text{and} \quad \tilde{\zeta}_1 - \tilde{\xi}_1 = O(\varepsilon|\log\varepsilon|). \tag{4.1}$$

This implies $\tilde{\zeta}_1 < L$ when ε is sufficiently small. Therefore, (4.1) together with Lemma 3.2 and Theorem 3.5 enables us to see that there exists a point $\tilde{\xi}_2 := \inf\{\tilde{\xi} \in \tilde{\Xi}\,; \tilde{\xi} > \tilde{\xi}_1\}$ satisfying

$$K_1\sqrt{\varepsilon} < 1 - \tilde{u}^\varepsilon(\tilde{\zeta}_1) < C_1 \exp\left(-\frac{r_1(\tilde{\xi}_2 - \tilde{\zeta}_1)}{\varepsilon}\right)$$

with some positive constants C_1 and r_1. Hence we get

$$\tilde{\xi}_2 - \tilde{\zeta}_1 = O(\varepsilon|\log\varepsilon|) \tag{4.2}$$

Furthermore, it follows from (4.1) and (4.2) that

$$\tilde{\xi}_2 - \tilde{\xi}_1 = O(\varepsilon|\log\varepsilon|).$$

Here, we should note that $\tilde{\xi}_2 \in (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta)$ when ε is sufficiently small. This means that there exists another point of $\tilde{\Xi} \cap (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta)$ except for $\tilde{\xi}_1$.

Putting $\tilde{\zeta}_2 := \inf\{y\,;\, y > \tilde{\xi}_2 \text{ and } \tilde{u}^\varepsilon_y(y) = 0\}$ and using Lemma 3.6 again, one can obtain that there is a positive constant K_2 satisfying

$$K_2\sqrt{\varepsilon} < \tilde{u}^\varepsilon(\tilde{\zeta}_2) \quad \text{and} \quad \tilde{\zeta}_2 - \tilde{\xi}_2 = O(\varepsilon|\log\varepsilon|)$$

because $\tilde{u}^\varepsilon_y(\tilde{\xi}_2) < 0$. This implies $\tilde{\zeta}_2 < L$ when ε is sufficiently small; so that Lemma 3.2 and Theorem 3.5 assure the existence of a point $\tilde{\xi}_3 := \inf\{\tilde{\xi} \in \tilde{\Xi}\,;\, \tilde{\xi} > \tilde{\xi}_2\}$ satisfying

$$K_2\sqrt{\varepsilon} < \tilde{u}^\varepsilon(\tilde{\zeta}_2) < C_2 \exp\left(-\frac{r_2(\tilde{\xi}_3 - \tilde{\zeta}_2)}{\varepsilon}\right)$$

with some positive constants C_2 and r_2 Therefore, it holds that

$$\tilde{\xi}_3 - \tilde{\xi}_2 = O(\varepsilon|\log\varepsilon|)\,;$$

i.e., we have shown the existence of another element of $\tilde{\Xi} \cap (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta)$ except for $\tilde{\xi}_1$ and $\tilde{\xi}_2$.

Repeating these procedure, one can see that the number of points in $\tilde{\Xi} \cap (\tilde{\sigma}_k + \delta, \tilde{\sigma}_{k+1} - \delta)$ increases in each process. However, this is impossible because $\tilde{u}^\varepsilon$ belongs to $\tilde{S}_{n,\varepsilon}$.

Thus, the proof is complete. □

For proofs of other assertions in Theorem 2.1, see Urano [5].

Appendix Reaction and diffusion in porus media

We will consider reaction and diffusion in porus media. We will describe the role of d and h in terms of their porosity. Here, the porosity is defined as follows:

Definition A.1. Let Ω be a porus medium with volume $|\Omega|$ and let Ω_p be the space occupied by the porus in Ω. Then the porosity $\phi = \phi(x)$ of the medium Ω is defined by $\phi = |\Omega_p|/|\Omega|$.

By use of the porosity ϕ, it holds that

$$(\phi(x)u)_t + (-\varepsilon^2\phi(x)u_x)_x = \phi(x)\kappa(x)f(u),$$

where κ denotes the number of grams of a spew. Therefore, we can obtain that

$$u_t = \frac{1}{\phi(x)}(\varepsilon^2\phi(x)u_x)_x + \kappa(x)f(u).$$

Moreover, for the stationary problem, this implies that

$$\varepsilon^2(\phi(x)u_x)_x + \phi(x)\kappa(x)f(u) = 0.$$

Hence, we can choose the functions d and h as

$$d(x)^2 = \phi(x) \quad \text{and} \quad h(x)^2 = \phi(x)\kappa(x),$$

respectively.

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