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Global existence and blow up for harmonic map heat flows into ellipsiod

Dedicated to Professor Nobuyuki Kenmochi
on the occasion of his retirement from Chiba University

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1 Introduction

Let $\mathcal{M}$ be a d_M-dimensional Riemannian manifold (with or without boundary) and let $\mathcal{N}$ be another compact d_N-dimensional Riemannian without boundary. We will assume that $\mathcal{N}$ is isometrically embedded in $\mathbb{R}^k$ $(k>d_N)$. Let u be a map from $\mathcal{M}$ to $\mathcal{N}$ which belongs to $H^1(\mathcal{M};\mathbb{R}^k)$. The energy of u is defined by

$$\mathbf{E}(u)=\frac{1}{2}\int_M |du(x)|^2 d\mu_g.$$

If $\partial\mathcal{M}\neq\varnothing$, we assume that $u\big|_{\partial\mathcal{M}}=\gamma$ for some given $\gamma\in H^{1/2}(\partial\mathcal{M};\mathbb{R}^k)$ with $\gamma(x)\in\mathcal{N}$.

The map u is (weakly) harmonic if it is a critical point of $\mathbf{E}$. The Euler-Lagrange equation satisfied by the (weakly) harmonic maps is

$$\tau(u)=\operatorname{trace}\nabla\, du=0,\quad u\big|_{\partial\mathcal{M}}=\gamma$$

where τ is called a tension field.

Let $(x_1,\cdots,x_m)$, $(y_1,\cdots,y_n)$ be a local coordinates of $\mathcal{M}$ and $\mathcal{N}$, respectively. The metrics in local coordinates are written as

$$ds_M^2=\sum_{k,j=1}^{d_M} g_{ij}dx^k dx^j \quad\text{and}\quad ds_N^2=\sum_{\alpha,\beta=1}^{d_N} h_{\alpha\beta}dy^\alpha dy^\beta$$

respectively. Then, the tension field $\tau(u)$ can be expressed as follows: for $u^\alpha=y_\alpha\circ u$, $\tau(u)(x)\in T_{u(x)}\mathcal{N}$, where $\tau(u)(x)=\sum_{\gamma=1}^{n}\tau(u)^\gamma\frac{\partial}{\partial y_\gamma}$ with

$$\begin{aligned}\tau(u)^\gamma(x)&=\sum_{i,j=1}^{m} g^{ij}\{\frac{\partial^2 u^\gamma}{\partial x_i\partial x_j}-\sum_{k=1}^{m}\Gamma_{ij}^k(x)\frac{\partial u^\gamma}{\partial x_k}+\sum_{\alpha,\beta=1}^{n}{}^N\Gamma_{\alpha,\beta}^\gamma(u(x))\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}\}\\&=-\Delta_M u^\gamma+\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n} g^{ij\,N}\Gamma_{\alpha,\beta}^\gamma(u(x))\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}\end{aligned}$$

where Γ_{ij}^k and ${}^N\Gamma_{\alpha\beta}^\gamma$ are Christoffel symbols of Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ respectively, (g^{ij}) is the reciprocal matrix of $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and Δ_M is the Laplace Beltrami operator of $(\mathcal{M}, g)$.

The heat flow associated to the above Euler-Lagrange equation is

$$\frac{\partial u}{\partial t} - \tau(u) = 0.$$

As is well known, the global existence of smooth solutions of the initial boundary value problem for a heat flow of harmonic maps from a Riemannian manifold $\mathcal{M}$ into a Riemannian manifold $\mathcal{N}$ depends on the geometry of $\mathcal{N}$

If the sectorial curvature of $\mathcal{N}$ is non-positive, then there exists a unique global smooth solution of the problem for $C^{2+\alpha}$ data. Moreover, the solution subconverges to a harmonic map as the time goes to infinity.(see e.g. Ellis-Sampson [8], R.S. Hamilton [12]).

In general, the structures of solutions of the harmonic map heat flows are very complicated, and the study is still quite incomplete.

There are extensive works when the target space $\mathcal{N}$ is a sphere $\mathbb{S}^{d_N-1}$. In this situation, assuming that $\mathcal{M}$ is an open domain in $\mathbb{R}^{d_M}$, the harmonic map heat flow equation is written as

$$\frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 u.$$

Then, the local existence of smooth solutions and global existence of weak solutions with bounded energy can be established. Moreover, the partial regularity results hold for $d_M = 3, 4$. More precisely, flows of bounded energy are regular in the interior of domain except for a closed set of Hausdorff dimension at most $d_M - 3$ (e.g. Y. Chen [5], M. Feldman [16], M. Struwe [20]). There are also uniqueness and non-uniqueness results on the weak solutions of bounded energy (e.g., J-M. Coron [6], P. Topping [21], A. Freire [10]). The finite time blow-up results have been investigated by many authors (e.g., Chang-Ding [4], Coron-Ghidaglia [7]). Here we say that the solution $u(x,t)$ blows up at $t = T$ if

$$\limsup_{t \to T^-} \|\nabla u(\cdot, t)\|_\infty = \infty.$$

We now consider the case when the target manifold $\mathcal{N}$ is a d_N-dimensional ellipsoid, say,

$$\mathcal{N} = \left\{ (u_1, u_2, \cdots, u_{d_N+1}) \in \mathbb{R}^{d_N+1} \ : \ \sum_{j=1}^{d_N+1} \frac{u_j^2}{a_j^2} = 1 \right\}, \quad a_j > 0$$

and $\mathcal{M}$ is an open domain Ω in $\mathbb{R}^d$ or a flat torus T^d.

Definition 1 $u : \Omega \to \mathcal{N}$ *is said to be weakly harmonic if* u *is a critical point of the energy*

$$\mathbf{E}(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx$$

under the following constraints:

$$u \in H^1 \cap L^\infty(\Omega;\mathcal{N}), \quad u\big|_{\partial\Omega} = \gamma \in H^{1/2}(\partial\Omega;\mathcal{N})$$

where

$$H^1 \cap L^\infty(\Omega;\mathcal{N}) = \{u \in H^1 \cap L^\infty(\Omega;\mathbb{R}^{d_N+1}) \,:\, u(x) \in \mathcal{N} \quad a.e.\ in\ \Omega\}$$

and

$$H^{1/2}(\partial\Omega;\mathcal{N}) = \{u \in H^{1/2}(\partial\Omega;\mathbb{R}^{d_N+1}) \,:\, u(x) \in \mathcal{N} \quad a.e.\ on\ \partial\Omega\}.$$

We have

Lemma 1 *u is weakly harmonic if and only if u satisfies the constraints and for any $\phi \in C_0^\infty(\Omega, \mathbb{R}^{d_M+1})$*

$$\sum_{j=1}^{d} \int_\Omega \{\langle \partial_{x_j} u, \partial_{x_j}\phi\rangle - |\partial_{x_j} u|^2 \langle A^2 u, \phi\rangle \\ -\langle u, \partial_{x_j} u\rangle(\langle A^2 \partial_{x_j} u, \phi\rangle - \langle A^2 u, \partial_{x_j}\phi\rangle)\}\, dx = 0 \qquad (1.1)$$

where A is a linear mapping $A : \mathbb{R}^{d_N+1} \longrightarrow \mathbb{R}^{d_N+1}$ defined by

$$A : (u_1, \cdots, u_{d_N*1}) \mapsto \left(\frac{u_1}{a_1}, \cdots, \frac{u_{d_N+1}}{a_{d_N+1}}\right).$$

The proof is accomplished by the straghtforward calculation of the left hand side of the following:

$$\frac{d\mathbf{E}(u_s)}{ds}\bigg|_{s=0} = 0, \quad u\big|_{\partial\Omega} = \gamma$$

where $u_s = \dfrac{u + s\phi}{|A(u + s\phi)|} \in \mathcal{N}$ with $\phi \in C_0^\infty(\Omega;\mathbb{R}^{d_N+1})$.

Lemma 2 *If u is smooth and weakly harmonic, then u satisfies*

$$\Delta u + \lambda A^2 u = 0 \qquad (1.2)$$

where

$$\lambda = \left\{ \frac{\displaystyle\sum_{j=1}^{d_N+1} \frac{|\nabla u_j|^2}{a_j^2}}{\displaystyle\sum_{j=1}^{d_N+1} \frac{u_j^2}{a_j^4}} \right\}.$$

Proof If u is smooth, then (1.1) is rewritten as

$$\int_\Omega \langle \Delta u - \langle u, \Delta u \rangle A^2 u, \phi \rangle dx = 0, \quad \forall \phi \in C_0^\infty(\Omega, \mathbb{R}^{d_M+1}).$$

Hence, we have

$$\Delta u - \langle u, \Delta u \rangle A^2 u = 0. \tag{1.3}$$

Since $|Au|^2 = 1$, we have

$$\partial_{x_j} |Au|^2 = 2\langle A^2 u, \partial_{x_j} u \rangle = 0, \tag{1.4}$$

$$\Delta |Au|^2 = |A\nabla u|^2 + \langle A^2 u, \Delta u \rangle = 0. \tag{1.5}$$

From (1.3) we have

$$\langle \Delta u, A^2 u \rangle - \langle u, \Delta u \rangle |A^2 u|^2 = 0.$$

Then, we make use of (1.5) to obtain

$$\langle u, \Delta u \rangle = -\frac{|A\nabla u|^2}{|A^2 u|^2}.$$

Hence, (1.3) is rewritten as

$$\Delta u + \frac{|A\nabla u|^2}{|A^2 u|^2} A^2 u = 0$$

which gives (1.2). ■

Several authors ([15], [1], [13], [14]) investigated special case of the ellipsoid $\mathcal{N}' \subset \mathbb{R}^{d+1}$ with $a_j = 1, (j = 1, 2, \cdots, d)$. One of their main results is concerned with stability of the equator map. Here we say that the map $U^* = x/|x|$ from the unit ball B^d of $\mathbb{R}^d$ into the equator of $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ or of the ellipsoid $\mathcal{N}'$ is a critical point for the energy **E**. U^* is called the equator map. Jäger and Kaul [15] (1979) showed that the equator map U^* into sphere is an absolute minimum if $d \geq 7$, but is is unstable if $3 \leq d \leq 6$. Baldes [1] (1984) considered the equator maps into ellipsoid $\mathcal{N}'$ and showed U^* is stable if $a^2 \geq 1$ and $n \geq 7$, and unstable $a^2 < 4(d-1)/(d-2)^2$. Helein [13] (1988) showed that if $a^2 < 4(d-1)/(d-2)^2$, there is a smooth minimizing map and if $a^2 > 4(d-1)/(d-2)^2$, U^* is a unique minimizer. Here we remind that

Definition 2 *Let $u \in H^1(\Omega; \mathcal{N})$ be a weakly harmonic map. u is called (weakly) stable if for* $u_s = \dfrac{u + s\phi}{|A(u + s\phi)|}$

$$\frac{d^2}{ds^2}\mathbf{E}(u_s)\Big|_{s=0} \geq 0 \quad \forall \phi \in C_0^\infty(\Omega, \mathbb{R}^{d_N+1}) \quad \textit{with } u \cdot A^2\phi = 0.$$

It might be an interesting open problem to generalize the equator maps to gneneral ellipsoid and to ingestigate thier properties.

In this paper our aim is to investigate the initial boundary value problem to the heat flow of harmonic maps into general ellipsoid $\mathcal{N}$ and to generalize results previously obtained for the maps into sphere or for the maps into the special ellipsoid mentioned above. We summ up our results mostly with rough sketches of proofs. Details will be published elsewhere.

2 Global existence of weak solutions

Let u_0 be a map from $\overline{\Omega}$ into $\mathcal{N}$. We consider the initial-boundary value problem : find $u : \overline{\Omega} \times [0, +\infty) \to \mathcal{N}$ such that

$$\frac{\partial u}{\partial t} = \Delta u + \lambda A^2 u, \quad (x, t) \in \Omega \times [0, +\infty) \tag{2.1}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega \tag{2.2}$$

$$u(x, t) = u_0(x), \quad (x, t) \in \partial\Omega \times [0, +\infty) \tag{2.3}$$

Definition of global weak solutions is as follows:

Definition 3 *Let $u_0 \in H^1 \cap L^\infty(\Omega; \mathcal{N})$ and $\gamma \in H^{1/2}(\partial\Omega : \mathcal{N})$. Then, u is a weak solution of* (2.1) – (2.3) *if u satisfies*

$$u \in L^\infty(0, \infty; H^1 \cap L^\infty(\Omega; \mathcal{N})),$$
$$\partial_t u \in L^2((0, \infty) \times \Omega; \mathbb{R}^{d_N+1}),$$
$$u\big|_{\partial\Omega} = \gamma,$$

and

$$\int_0^\infty \int_\Omega \{\langle \partial_t u, \phi\rangle + \langle \nabla u, \nabla \phi\rangle - \lambda \langle A^2 u, \phi\rangle\}\, dxdt = 0,$$
$$\forall \phi \in C_0^\infty((0, \infty) \times \Omega; \mathbb{R}^{d_N+1}).$$

Denote $\mathcal{N}_+$ the open upper hemisphere, i.e., $\mathcal{N}_+ = \{u \in \mathcal{N} : u_{i_0} > 0\}$ where the subscript $i_0 \in \{1, 2, \cdots, d_N + 1\}$ is chosen scuh that

$$a_{i_0} = \min_{i=1,\cdots,d_N+1} a_i.$$

We first consider the global existence of weak solutions.

Theorem 3 *Let $u_0 \in H^1 \cap L^\infty(\Omega; \mathcal{N})$. Then there exists a global weak solution of (1.1)-(1.3).*

If Ω is a bounded domain with smooth boundary $\partial\Omega$ such that

$$(x, n_x) \geq 0, \quad \forall x \in \partial\Omega$$

where n_x denotes the unit outer normal vector at $x \in \partial\Omega$, and $u_0\big|_{\partial\Omega} = \gamma_0$being a constant, the mapping $u(t)$ subconverges strongly to a constant in $H^1(\Omega)$ as $t \to \infty$

Moreover, if $u_0 \in \mathcal{N}_+$, then the solution is uniquely determined by the data.

The proof of Theorem 3 is accomplished by the following Ginzburg-Landau approximation: for $\alpha > 1$ and any $k > 0$

$$\frac{\partial u_{(k)}}{\partial t} = \Delta u_{(k)} + k|1 - |Au_{(k)}|^2|^{2\alpha-2}(1 - |Au_{(k)}|^2)u_{(k)}, \tag{2.4}$$

$$u = u_0 \quad \text{on } \partial\Omega \times [0, +\infty) \bigcup (\Omega \times \{t = 0\}). \tag{2.5}$$

By using the standard theory of semilinear parabolic systems (see [17]), it can be proved that for every integer $k \geq 1$, the problem (2.4)-(2.5) has a global solution $u_{(k)}$ which is smooth in $\Omega \times (0, \infty)$. In order to prove the convergence of $\{u_{(k)}\}$, we need a priori estimates of them.

We have

Lemma 4 *The following equality holds:*

$$\int_0^t \int_\Omega |\partial_t A u_{(k)}|^2 dx dt + \frac{1}{2}\int_\Omega |\nabla A u_{(k)}|^2 dx + \frac{k}{4\alpha}\int_\Omega |1 - |Au_{(k)}|^2|^{2\alpha} dx = \frac{1}{2}\int_\Omega |\nabla A u_0|^2 dx.$$

Proof. Multiplication of the both sides of (2.4) with $A^2 \partial_t u_{(k)}$ and integration by parts in x, t on $\Omega \times [0, t]$ yield the result. ■

Lemma 5 *Let φ is a continuous function on $[0, \infty)$ with bounded derivative. We have*

$$\begin{aligned} &\int_\Omega \Phi(|Au_{(k)}(x,t)|^2 - 1)^2 dx \\ &+ \int_0^t \int_\Omega \varphi(|Au_{(k)}|^2 - 1)|\nabla A u_{(k)}|^2 dx dt + 2\int_0^t \int_\Omega \varphi'(|Au_{(k)}|^2 - 1)|\langle \nabla A u_{(k)}, Au_{(k)}\rangle|^2 dx dt \\ &+ k\int_0^t \int_\Omega \varphi(|Au_{(k)}|^2 - 1)||Au_{(k)}|^2 - 1|^{2\alpha - 2}(|Au_{(k)}|^2 - 1)||Au_{(k)}|^2 dx dt = 0. \end{aligned}$$

where Φ is the primitive of 2φ.

Proof. We multiply the both sides of (2.4) with $\varphi(|Au_{(k)}|^2 - 1)A^2 u_{(k)}$ and integrate by parts in x, t on $\Omega \times [0, t]$ to obtain the result. ■

Lemma 6 (Maximum principle) For any $k \geq 1$

$$|Au_{(k)}| \leq 1, \quad \forall (x,t) \in \Omega \times [0, \infty).$$

Proof. In Lemma 5 taking φ as

$$\varphi(s) = \begin{cases} 0, & \text{for } x \leq 0 \\ s, & \text{for } x > 0 \end{cases}$$

and denoting

$$[f]_+ = \begin{cases} 0, & \text{for } f \leq 0 \\ f, & \text{for } f > 0 \end{cases},$$

we have

$$\int_\Omega [|Au_{(k)}(x,t)|^2 - 1]_+^2 dx \leq 0, \quad \forall t \geq 0$$

from which it follows that

$$|Au_{(k)}| \leq 1, \quad \forall(x,t) \in \Omega \times [0,\infty).$$

■

We continue the proof of Theorem 3.

From Lemma 4 and Lemma 6 , we see that

$$\{u_{(k)}\} \text{ is a bounded set in } L^\infty(0,\infty; H^{1,2}(\Omega) \cap L^\infty(\Omega))$$
$$\{\partial u_{(k)}\} \text{ is a bounded set in } L^2(0,\infty; L^2(\Omega))$$

Hence wee that

$$\{u_{(k)}\} \text{ is a bounded set in } H^{1,2}([0,\infty) \times \Omega).$$

It is a standard manner that we make use of the above mentioned boundedness to extract a weakly convergent subsequence of $\{u_{(k)}\}_{k\in\mathbb{N}}$ in $H^{1,2}([0,\infty) \times \Omega)$. We can show the limit function u a weak solution to (2.1)-(2.3) by a suitable modification of Evans' argument (see [9]).

Moreover, taking $\varphi(x) = x$ in Lemma 5 we obtain

$$\{k|1 - |Au_{(k)}|^2|^{2\alpha-2}(1 - |Au_{(k)}|^2)u_{(k)}\} \text{ is a bounded set in } L^1(\Omega \times [0,\infty])$$

Then, in view of (2.4) we see that

$$\{\triangle u_{(k)}\} \text{ is a bounded set in } L^1(\Omega \times [0,\infty])$$

from which it follows that $\{\triangle u_{(k)}\}$ and $\{k|1-|Au_{(k)}|^2|^{2\alpha-2}(1-|Au_{(k)}|^2)u_{(k)}\}$ subconverges to $\triangle u$ and $\lambda A^2 u$ in measure, respectively. ■

We also have

Lemma 7 *If Ω is a bounded domain in $\mathbb{R}^{d_M}$ with smooth boundary $\partial\Omega$ and*

$$(x, n_x) \geq 0, \quad \forall x \in \partial\Omega$$

where n_x denotes the unit outer normal vector at $x \in \partial\Omega$, $u_0\big|_{\partial\Omega} = \gamma$, γ being a constant, then there exists a $\delta > 0$ such that

$$\int_\Omega |\nabla u(x,t)|^2(|x|^2+1)dx \leq e^{-\delta t}\int_\Omega |\nabla u_0(x)|^2(|x|^2+1)dx. \tag{2.6}$$

Thus we have the convergence assertion as $t \to \infty$.

Uniqueness assertion is obtained essentially by the same maximum principle as in the proof of the regularity theorem mentioned below.

2. Regularity

For regularity of solutions we have

Theorem 8 *Let $0 < a \leq 1$ and $\partial\Omega = \varnothing$. Assume that $u_0 \in W^{2,\infty}(\Omega)$ and the image of u_0 belongs to a compact subset of $\mathcal{N}_+$. Then, u is smooth and*

$$\|\nabla u\|_\infty \leq C\|\nabla u_0\|_\infty,$$
$$\left\|\frac{\partial u}{\partial t}\right\|_\infty + \|\Delta u\|_\infty \leq C(\|\Delta u_0\|_\infty + \|\nabla u_0\|_\infty^2).$$

We here consider the regular local solution u constructed by the standard local existence theorems (e.g. Hamilton [12], Ladyzenskaya -Uralceva [19], Fuwa-T [18]) and to establish the maximum principle to the derivatives of solutions.

First we note that the assumption that $u_0 = (u_{01}, \cdots, u_{0d_N+1}) \in \mathbb{R}^{d_N+1}$ lies a compact subset in $\mathcal{N}_+$ implies there exists a positive constant b such that $u_{0i_0} \geq b$. Then, by the maximum principle yields that

$$u_{i_0}(x,t) \geq b, \quad \forall(x,t) \in \overline{\Omega} \times [0,\infty)$$

We utilize the following maximum principle for a parabolic operator P defined by

$$P(f) = \mathrm{div}(e^{-\Phi}\mathrm{grad}f) - e^{-\Phi}\frac{\partial f}{\partial t}$$

where Φ is a smooth function on $\overline{\Omega} \times [0,T]$ for some $T > 0$.

Lemma 9 *(Maximun Principle) Assume that f is smooth on $\overline{\Omega} \times [0,T]$ and satisfies $P(f) \geq 0$ on $\Omega \times (0,T)$ for some $T > 0$. Then,*

$$\max_Q f \leq \max_\Gamma f$$

where $Q = \overline{\Omega} \times [0,T)$ and $\Gamma = (\partial\Omega) \times (0,T] \cup \Omega \times \{t = 0\}$.

For the proof we refer to Friedman [17].

Put $f = \psi e^\Phi$ where ψ and Φ are smooth functions on $\Omega \times [0,T]$. Then

$$P(f) = \nabla\Phi\nabla\psi + \psi(\Delta\Phi - \partial_t\Phi) + \Delta\psi - \partial_t\psi.$$

We take $\Phi = -2\log u_{i_0}$ and

$$\psi = \frac{1}{2}|\nabla u|^2$$

or

$$\psi = \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2.$$

Simple calculation shows that

$$\Delta\Phi - \partial_t\Phi = 2\frac{\lambda}{a_{i_0}^2} + \frac{1}{2}|\nabla\Phi|^2.$$

For $\psi = \frac{1}{2}|\nabla u|^2$ we have

$$\Delta\psi - \partial_t\psi = |D^2u|^2 - \lambda|A\nabla u|^2$$

where D^2u denotes the Hessian of u.

Then, by the Cauchy-Schwartz inequality we have

$$\begin{aligned} P(f) &= \nabla\Phi \cdot \frac{1}{2}\nabla|\nabla u|^2 + \frac{1}{2}\left(2\frac{\lambda}{a_{i_0}^2} + \frac{1}{2}|\nabla\Phi|^2\right)|\nabla u|^2 \\ &\quad - \lambda|A\nabla u|^2 + |D^2u|^2 \\ &\geq -|\nabla\Phi||\nabla u||D^2u| + \frac{1}{4}|\nabla u|^2|\nabla\Phi|^2 + |D^2u|^2 \\ &\quad + \lambda\left(\frac{1}{a_{i_0}^2}|\nabla u|^2 - \sum_{i=1}^{d_N+1}\frac{1}{a_i^2}\nabla u_i|^2\right) \\ &\geq 0. \end{aligned}$$

A similar calculation holds for $\psi = \dfrac{1}{2}\left|\dfrac{\partial u}{\partial t}\right|^2$.

Thus we obtain

$$\|\nabla u\|_{L^\infty(\Omega\times[0,\infty))} \leq \|\nabla u_0\|_{L^\infty(\Omega)}$$

and

$$\|\partial_t u\|_{L^\infty(\Omega\times[0,\infty))} \leq \|\Delta u_0\|_{L^\infty(\Omega)} + \|\nabla u_0\|^2_{L^\infty(\Omega)}.$$

Then, we have

$$\|\Delta u\|_{L^\infty(Q)} \leq \|\partial_t u\|_{L^\infty(Q)} + C\|\nabla u\|^2_{L^\infty(Q)} \leq C.$$

As to the case $\partial\Omega \neq \varnothing$, a similar result holds by making use of different strategy to obtain higher spacial regularity of u besides the maximum principle to $\dfrac{\partial u}{\partial t}$. ■

Finally we remark that for the proof of uniqueness of (weak) solutions we take

$$\begin{aligned} \Phi &= -\log u_{i_0}^1 - \log u_{i_0}^2 \\ \psi &= -\frac{1}{2}|u^1 - u^2|^2 \end{aligned}$$

where u^1 and u^2 are two solutions with the same initial and boundary data.

3. Blow-up of solutions

Most results of the finite time blow-up are shown for (axially) symmetric solutions for the harmonic map heat flow into sphere. We extend the notion of axially symmetric solutions to the ellipsoid. It is straightforward if we consider an ellipsoid of the form

$$\mathcal{N}' = \{(u_1, \cdots, u_{n+1}) \in \mathbb{R}^{n+1} : u_1^2 + \cdots + u_n^2 + \frac{1}{a^2}u_{n+1}^2 = 1\}.$$

General cases are left for further studies.

we first consider the 2-dimensional ellipsoid

$$\mathcal{N}' = \{(u_1, u_2, u_2) \in \mathbb{R}^3 \, : \, u_1^2 + u_2^2 + \frac{1}{a^2}u_3^2 = 1\}.$$

Let

$$u = (\cos\psi\cos\chi, \sin\psi\sin\chi, a\cos\psi).$$

Then, the equation (1.1) becomes

$$\frac{\partial\psi}{\partial t} = \Delta\psi + \frac{(a^2-1)(\sin\psi\cos\psi}{(a^2-1)\sin^2\psi+1}|\nabla\psi|^2 - \frac{\sin\psi\cos\psi}{(a^2-1)\sin^2\psi+1}|\nabla\chi|^2$$
$$\frac{\partial\chi}{\partial t} = \Delta\chi + 2\mathrm{cot}\psi\nabla\psi\cdot\nabla\chi.$$

Let $\mathcal{M} = B^2 = \{(x,y) \in \mathbb{R}^2 \, : \, x^2+y^2 \leq 1\}$. Introducing the polar coordinates on the plane, i.e. $x = r\cos\theta, y = r\sin\theta$. As is the case of sphere, we say that the solution u is axially symmetric if

$$\chi = m\theta, \quad (m \in \mathbb{N}), \quad \psi = \psi(r,t).$$

For axially symmetric solutions we have $|\nabla\chi|^2 = m^2/r^2$, $\nabla\chi\cdot\nabla\psi = 0$ and $\Delta\chi = 0$. Hence, ψ satisfies

$$\psi_t = \psi_{rr} + \frac{1}{r}\psi_r + \frac{(a^2-1)(\sin\psi\cos\psi}{(a^2-1)\sin^2\psi+1}|\psi_r|^2 - \frac{m^2\sin\psi\cos\psi}{r^2((a^2-1)\sin^2\psi+1)}.$$

When $m=1$, we can also extend the notion of axially symmetric solutions to the case of n-dimensional ellipsoidal target space $\mathcal{N}'$,

Let $\Omega = B^n$ or $\mathbb{R}^n$ Let $u \, : \, \Omega \longrightarrow \mathcal{N}'$

$$u = \left(\frac{x}{r}\sin\psi(r,t), a\cos\psi(r,t)\right), \quad r = |x|. \tag{2.7}$$

Then, we have

$$\psi_t = \psi_{rr} + \frac{n-1}{r}\psi_r + \frac{(a^2-1)(\sin\psi\cos\psi}{(a^2-1)\sin^2\psi+1}|\psi_r|^2 - \frac{(n-1)\sin\psi\cos\psi}{r^2((a^2-1)\sin^2\psi+1)}.$$

We say that u of the form (2.7) is an axially symmetric solution of (2.1). The energy $\mathbf{E}$ is of the form

$$\mathbf{E}(\psi) = \frac{1}{2}\int_0^\infty \left((a^2\sin^2\psi + \cos^2\psi)|\psi_r|^2 + \frac{n-1}{r^2}\sin^2\psi\right) r^{n-1}dr \tag{2.8}$$

Finite energy yields that $\sin\psi(0,t) = 0$, say, $\psi(0,t) = k\pi, \quad k \in \mathbb{Z}$.

Our result is

Theorem 10 *There exist regular axially symmetric initial and boundary data for which the solution to the harmonic heat flow (1.1) blows up in finite time.*

For simplicity we consider $\Omega = \mathbb{R}^n$ and use a variant of the method of Coron and Ghidaglia (1989) [7].

Let $w : \mathbb{R}^n \to \mathcal{N}'$ satisfy

$$-\Delta w + \frac{1}{2}(x \cdot \nabla)w = \lambda A^2 w, \quad \lambda = \frac{|A\nabla w|^2}{|A^2 w|^2}$$

For $\tau > 0$, $u(x,t) = W(x/(\tau - t)^{1/2})$ is a solution of (2.1)-(2.3).

Set

$$A(g) = -g_{rr} - \frac{n-1}{r}g_r + \frac{r}{2}g_r - \frac{(a^2-1)(\sin g \cos g}{(a^2-1)\sin^2 g + 1}|g_r|^2 + \frac{(n-1)\sin g \cos g}{r^2((a^2-1)\sin^2 g + 1)}.$$

If $A(g) \le 0$, $H(r,t) = g(x/(\tau - -t)^{1/2})$ satisfies

$$\psi_t - \psi_{rr} - \frac{n-1}{r}\psi_r - \frac{(a^2-1)(\sin\psi\cos\psi}{(a^2-1)\sin^2\psi + 1}|\psi_r|^2 + \frac{(n-1)\sin\psi\cos\psi}{r^2((a^2-1)\sin^2\psi + 1)} \le 0.$$

In order to prove the blow-up of solutions, it is crucial to construct a function g such that $A(g) \le 0$. Candidates of g are

$$\phi^{\sharp}(r,\mu) = 2\arctan\frac{r}{\mu}, \quad \phi^{\flat}(r,\mu) = 2\arctan\frac{\mu}{r}, \quad \mu \in \mathbb{R}$$

which satisfy

$$\phi_{rr} + \frac{1}{r}\phi_r - \frac{\sin\phi\cos\phi}{r^2} = 0.$$

Here we choose $\phi^{\flat}$. Then,

$$\lim_{r\to 0}\phi^{\flat}(r,\mu) = \pm\pi, \quad \lim_{r\to\infty}\phi^{\flat}(r,\mu) = 0.$$

Long and tedious calculation shows that $A(\phi^{\flat}) \le$ for sufficiently large $\mu > 0$ for any $a > 0$.

More precise investigations will be done near future.

This work has been done with my student H. Miyoshi.

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