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LOCAL SEMIPOSITIVITY OF RELATIVE CANONICAL BUNDLES

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Abstract

This is an announcement of the recent development on the local semipositivity of relative canonical bundles for projective families. And we prove the boundedness of the effectively parmetrized families of canonically polarized varieties or minimal algebraic varieties with semiample canonical divisors. I hope this will give an approach for the Kobayashi hyperbolicity of the moduli space of canonically polarized varieties and Viehweg's conjecture (cf. Conjecture 1.1 below).

Contents

1 Introduction

Let $f; X \longrightarrow Y$ be a proper surjective projective morphism with connected fibers over a smooth projective variety Y. Then it is well known that the direct image $f_*\mathcal{O}_X(mK_{X/Y})$ is semipositive in an appropriate sense (cf. Theorems 2.1 and 2.6 below). This implies that the canonical bundle K_X of X is more positive than the canonical bundle K_Y of the base space, if the Kodaira dimension of a general fiber is nonnegative.

The first aim of this paper is to present the quantitative version of Theorems 2.1 and 2.6 in the case that a general fiber is of general type, i.e., we give an explicit pointwise lower bound of the positivity how much K_X is more positive than K_Y (cf. Theorem 4.5).

Next we shall consider the boundedness of the smooth effectively prametrized family of canonically polarized varieties over the fixed base space of log general type. In this direction, a lot of results have already been known ([Kov1, Kov2, Kov3, Kov4, Kov5]). In particular [Kov4], S. Kovács proved the Shafarevich type theorem for families of canonocally polarized varieties. On the other hand the following conjecture is well known.

Conjecture 1.1 *(Viehweg's conjecture) Let $f : X \longrightarrow Y$ be an effectively parametrized smooth projective family of canonically polarized varieties over a smooth quasi projective variety Y. Then Y is of log general type.* □

There have been several partial affirmative answer to Conjecture 1.1.
Conjecture 1.1 asserts that the (log)canonical bundle of the base space Y is positive. In this direction, E. Viehweg and K. Zuo proved that for an effectively parmetrized smooth canonically polarized varieties $f : X \longrightarrow Y$, Y is Brody hyperbolic ([V-Z]), i.e., there does not exist a nonconstant holomorphic map $\phi : \mathbb{C} \longrightarrow Y$. Their proof uses the Higgs bundles and Ahlfors type Schwarz lemma. But the Higgs bundle depends on the Brody curve (which is supposed not to exist), hence their proof does not lead the Kobayashi hyperbolicity of the base space.

Conjecture 1.2 *Let $f : X \longrightarrow Y$ be an effectively parametrized projective family of canonically polarized varieties over a smooth quasi projective variety Y. Then Y is Kobayashi hyperbolic.* □

The purpose of this paper is to present some estimates of local positivity of the relative canonical bundles in terms of the Kähler-Einstein currents or canonical measures and give some applications.

The proof seems to be essentially local, hence it is expected to give an affirmative answer to Conjecture 1.2.

2 Review of the global semipositivity results

In this section, we shall review the global semipositivity resuls on the direct image of a relative pluricanonical systems due to Y. Kawamata and E. Viehweg. The feature of these semipositivity is that the semipositivity is on the direct images of tensor powers and not the relative canonical bundle itself. In the next section, we shall consider the local version of these global semipositivity. The feature of the local semipositivity is essentially on the relative canonical bundle itself and not on the direct images of the tensor powers.

2.1 Kawamata's semipositivity theorem

The first result on the semipositivity of the relative pluricanonical system is the following theorem due to Y. Kawamata in 1982.

Theorem 2.1 *([Ka2]) Let $f : X \longrightarrow Y$ be an algebraic fiber space. Suppose that $\dim Y = 1$. Then for every positive integer m, $f_*\mathcal{O}_X(mK_{X/Y})$ is a semipositive vector bundle on Y, in the sense that every quotient $\mathcal{Q}$ of $f_*\mathcal{O}_X(mK_{X/Y})$, $\deg \mathcal{Q} \geqq 0$ holds.* □

The proof of Theorem 2.1 depends on the variation of Hodge structure due to P.A. Griffiths and W. Schmidt (cf. [G, Sch]). We note that before Theorem 2.1, T. Fujita proved the case of $m = 1$ in [F1] by using the curvature computation of the Hodge metrics of P.A. Griffiths ([G]). In this special case, Fujita gave a singular hermitian metric on the vector bundle $f_*\mathcal{O}_X(K_{X/Y})$ with semipositive curvature in the sense of Griffiths. In contrast to Fujita's result, for $m \geqq 2$ Theorem 2.1 does not give a (singular) hermitian metric on $f_*\mathcal{O}_X(mK_{X/Y})$ with semipositive curvature, because the proof relies on the semipositivity of the curvature of the Finslar metric on $f_*\mathcal{O}_X(mK_{X/Y})$ defined by

$$\| \sigma \| := \left(\int_{X/Y} |\sigma|^{\frac{2}{m}} \right)^{\frac{m}{2}} \tag{2.1}$$

which is a singular hermitian metric on the tautological line bundle on $\mathbb{P}((f_*\mathcal{O}_X(mK_{X/Y}))^*)$.

2.2 Viehweg's semipositivity theorem

In 1995 E. Viehweg extended Theorem 2.1 ([V2, Section 6]) in the case of f-semiample relative canonical bundles and constructed quasi-projective moduli spaces of polarized projective manifolds with semiample canonical bundles ([V2]). Since we use Viehweg's idea in this article, we state his result precisely. First we recall several definitions.

Definition 2.2 *Let Y be a quasi-projective scheme, let Y_0 be an open dense suchscheme and let $\mathcal{G}$ be a coherent sheaf on Y, We say that $\mathcal{G}$ is* **globally generated** *over Y_0, if the natural map $H^0(Y, \mathcal{G}) \otimes \mathcal{O}_Y \longrightarrow \mathcal{G}$ is surjective over Y_0.* □

For a coherent sheaf $\mathcal{F}$ and a positive integer a, $S^a(\mathcal{F})$ denotes the a-th symmetric power of $\mathcal{F}$. To measure the positivity of coherent sheaves, we shall introduce the following notion.

Definition 2.3 *Let Y be a quasi-projective reduced scheme, $Y_0 \subseteq Y$ an open dense subscheme and let $\mathcal{G}$ be locally free sheaf on Y, of finite constant rank. Then $\mathcal{G}$ is* **weakly positive** *over Y_0, if for an ample invertible sheaf $\mathcal{H}$ on Y and for a given number $\alpha > 0$ there exists some $\beta > 0$ such that $S^{\alpha\cdot\beta}(\mathcal{G}) \otimes \mathcal{H}^{\beta}$ is globally generated over Y_0.* □

The notion of weak positivity is a natural generalization of the notion of nefness of line bundles. Roughly speaking, the weak semipositivity of $\mathcal{G}$ over Y_0 means that $\mathcal{G} \otimes \mathcal{H}^{\varepsilon}$ is $\mathbb{Q}$-globally generated over Y_0 for every $\varepsilon > 0$.

Definition 2.4 *Let $\mathcal{F}$ be a locally free sheaf and let $\mathcal{A}$ be an invertible sheaf, both on a quasi-projective reduced scheme Y. We denote*

$$\mathcal{F} \succeq \frac{b}{a} \mathcal{A}, \tag{2.2}$$

if $S^a(\mathcal{F}) \otimes \mathcal{A}^{-b}$ is weakly positive over Y, where a, b are positive integers. □

For a normal variety X, we define the canonical sheaf ω_X of X by

$$\omega_X := i_* \mathcal{O}_{X_{reg}}(K_{X_{reg}}) \tag{2.3}$$

where X_{reg} denotes the regular part of X and $i : X_{reg} \longrightarrow X$ denotes the natural injection. The following notion introduced by Viehweg is closely related to the notion of logcanonical thresholds.

Definition 2.5 *Let (X, Γ) be a pair of normal variety X and an effective Cartier divisor Γ. Let $\pi : X' \longrightarrow X$ be a log resolution of (X, Γ) and let $\Gamma' := \pi^*\Gamma$. For a positive integer N we define*

$$\omega_X \left\{ \frac{-\Gamma}{N} \right\} = \pi_* \left(\omega_{X'} \left(- \left\lfloor \frac{\Gamma'}{N} \right\rfloor \right) \right) \tag{2.4}$$

and

$$\mathcal{C}_X(\Gamma, N) = \mathrm{Coker} \left\{ \omega_X \left\{ \frac{-\Gamma}{N} \right\} \longrightarrow \omega_X \right\}. \tag{2.5}$$

If X has at most rational singularities, one defines :

$$e(\Gamma) = \min\{ N > 0 \,|\, \mathcal{C}_X(\Gamma, N) = 0 \}. \tag{2.6}$$

If $\mathcal{L}$ is an invertible sheaf, X is proper with at most rational singularities and $H^0(X, \mathcal{L}) \neq 0$, then one defines

$$e(\mathcal{L}) = \sup \{ e(\Gamma) | \Gamma : \textit{ effective Cartier divisor with } \mathcal{O}_X(\Gamma) \simeq \mathcal{L} \}. \tag{2.7}$$

□

Now we state the result of E. Viehweg.

Theorem 2.6 *([V2, p.191, Theorem 6.22]) Let $f : X \longrightarrow Y$ be a flat surjective projective Gorenstein morphism of reduced connected quai-projective schemes. Assume that the sheaf $\omega_{X/Y}$ is f-semi-ample and that the fibers $X_y = f^{-1}(y)$ are reduced normal varieties with at most rational singularities. Then one has :*

1. **Functoriality**: *For $m > 0$ the sheaf $f_*\omega_{X/Y}^m$ is locally free of rank $r(m)$ and it commutes with arbitrary base change.*
2. **Weak semipositivity**: *For $m > 0$ the sheaf $f_*\omega_{X/Y}^m$ is weakly positive over Y.*

(3) **Weak semistability**: *Let $m > 1, e > 0$ and $\nu > 0$ be chosen so that $f_*\omega^m_{X/Y} \neq 0$ and*

$$e \geqq \sup\left\{\frac{k}{m-1}, e(\omega^k_{X_y}) \ ; \ \text{for } y \in Y\right\} \tag{2.8}$$

hold. Then

$$f_*\omega^m_{X/Y} \succeq \frac{1}{e \cdot r(k)} \det(f_*\omega^k_{X/Y}) \tag{2.9}$$

holds. □

Although Theorem 2.6 assumes the f-semiampleness of $\omega_{X/Y}$, the advantages of this generalization are :

- The base space is of arbitrary dimension.
- The semipositivity is more explicit than the one in Theorem 2.1.
- The comparison of the positivity of $f_*\omega^m_{X/Y}$ and $\det(f_*\omega^m_{X/Y})$.

3 Semipositivity and weak semistability of relative canonical bundles

In this section, we shall review some of the result is [T8].

3.1 Analytic Zariski decompositions

To state the result in [T8], we introduce the notion of analytic Zariski decompositions.

Definition 3.1 *Let M be a compact complex manifold and let L be a holomorphic line bundle on M. A singular hermitian metric h on L is said to be an* **analytic Zariski decomposition(AZD** *in short), if the followings hold.*

1. *Θ_h is a closed positive current.*
2. *for every $m \geq 0$, the natural inclusion:*

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL)) \tag{3.1}$$

is an isomorphim. □

Remark 3.2 *If an AZD exists on a line bundle L on a smooth projective variety M, L is pseudoeffective by the condition 1 above.* □

It is known that for every pseudoeffective line bundle on a compact complex manifold, there exists an AZD on F (cf. [T1, T2, D-P-S]). The advantage of the AZD is that we can handle pseudoeffective line bundle L on a compact complex manifold X as a singular hermitian line bundle with semipositive curvature current as long as we consider the ring $R(X, L) := \oplus_{m\geq 0} H^0(X, \mathcal{O}_X(mL))$.

3.2 Local semipositivity of relative canonical bundles

The following theorem is obtained in terms of the logarithmic plurisubharmonic variation properties of canonical measures.

Theorem 3.3 *Let $f : X \longrightarrow Y$ be an algebraic fiber space and let Y° be the complement of the discriminat locus of f in Y. Then we have the followings :*

(1) **Global generation**: *There exist positive integers b and m_0 such that for every integer m satisfying $b \,|\, m$ and $m \geqq m_0$, $f_*\mathcal{O}_X(mK_{X/Y})$ is globally generated over Y°.*

(2) **Weak semistability 1**: *Let r denote $\operatorname{rank} f_*\mathcal{O}_X(mK_{X/Y})$ and let $X^r := X \times_Y X \times_Y \cdots \times_Y X$ be the r-times fiber product over Y. Let $f^r : X^r \longrightarrow Y$ be the natural morphism.*

Let $\Gamma \in |mK_{X^r/Y} - f^{r*}\det f_*\mathcal{O}_X(mK_{X/Y})|$ be the effective divisor corresponding to the canonical inclusion :

(3.2)
$$f^{r*}(\det f_*\mathcal{O}_X(mK_{X/Y})) \hookrightarrow f^{r*}f^r_*\mathcal{O}_{X^r}(mK_{X^r/Y}) \hookrightarrow \mathcal{O}_{X^r}(mK_{X^r/Y}).$$

Then Γ does not contain any fiber $X^r_y (y \in Y^\circ)$ such that if we we define the number δ_0 by

$$\delta_0 := \sup\{\delta \,|\, (X^r_y, \delta \cdot \Gamma_y) \text{ is KLT for all } y \in Y^\circ\}, \tag{3.3}$$

then for every $\varepsilon < \delta_0$

$$f_*\mathcal{O}_X(mK_{X/Y}) \succeq \frac{m\varepsilon}{(1+m\varepsilon)r} \det f_*\mathcal{O}_X(mK_{X/Y}) \tag{3.4}$$

holds over Y°.

(3) **Weak semistability 2**: *There exists a singular hermitian metric $H_{m,\varepsilon}$ on $(1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot f^{r*}\det f_*\mathcal{O}_X(mK_{X/Y})^{**}$ such that*

   (a) $\sqrt{-1}\,\Theta_{H_{m,\varepsilon}} \geqq 0$ holds on X^r in the sense of current.

   (b) For every $y \in Y^\circ$, $H_{m,\varepsilon}|X^r_y$ is well defined and is an AZD (cf. Definition 3.1) of

$$(1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot f^{r*}\det f_*\mathcal{O}_X(mK_{X/Y})^{**}|X_y. \tag{3.5}$$

□

Remark 3.4 *The 3rd assertion implies the 2nd assertion.* □

4 Statement of the results

In this section, we shall state the main results.

4.1 Kähler-Einstein currents

To describe the main results, we need the notion of Kähler-Einstein currents.

Let X be a smooth projective variety defined over complex numbers. In 1977, T. Aubin and S.-T. Yau proved independently that if K_X is ample, there exists a unique Kähler-Einstein form ω_E on X such that

$$-\mathrm{Ric}_{\omega_E} = \omega_E$$

holds ([Au, Y1]). And the author and K. Sugiyama extended this result to the case of projective varieties of general type.

Theorem 4.1 *[T1, Su] Let X be a smooth projective variety of general type. Then there exists a unique closed positive current ω_E on X such that*

1. *There exists a nonempty Zariski open subset U of X such that $\omega_E|U$ is a C^∞ Kähler form on U,*
2. *$-\mathrm{Ric}_{\omega_E} = \omega_E$ holds on U,*
3. *We set $dV_E := (n!)^{-1}\omega_E^n (n = \dim X)$, then dV_E^{-1} is an AZD of K_X.*

□

Remark 4.2 *Actually uniqueness has been proven very recently (cf. [T7]).* □

Let $f : X \longrightarrow Y$ be a proper surjective projective morphism with connected fibers between complex manifolds. Let Y° be a the complement of the discriminant locus of $f : X \longrightarrow Y$. Suppose that K_X is f-big over Y°, i.e., $K_X|X_y$ is big for every $y \in Y^\circ$, where $X_y = f^{-1}(y)$. Then there exists a unique Kähler-Einstein current $\omega_{E,y}$ on X_y by Therem 4.1. Let n denote the relative dimension: $\dim X - \dim Y$ and we set

$$dV_{E,y} = \frac{1}{n!}\omega_{E,y}^n \tag{4.1}$$

and we define the relative volume form $dV_{X/Y}$ by

$$dV_{X/Y}|X_y = dV_{E,y} (y \in Y^\circ). \tag{4.2}$$

Then the following theorem is fundamental.

Theorem 4.3 *([T4]) Let $f : X \longrightarrow Y$ and $dV_{E.y}$ by as above. Then the hermitian metric $h_{X/Y}$ on $K_{X/Y}|f^{-1}(U^\circ)$,*

$$h_{X/Y}|X_y := (dV_{E,y})^{-1} \tag{4.3}$$

is a singular hermitian metric on $K_{X/Y}|f^{-1}(Y^\circ)$ and it extends to a singular hermitian metric on $K_{X/Y}$ with semipositive curvature current.

Corollary 4.4 *Let $f : X \longrightarrow Y$ be as in Theorem 4.3. For $m \geqq 1$, we set*

$$E_m := f_*\mathcal{O}_X(mK_{X/Y}). \tag{4.4}$$

We define a C^∞ hermitian metric h_m on $E_m|Y^\circ$ by

$$h_m(\sigma, \sigma') := \int_{X_y} h_{X/Y}^{m-1} \cdot \sigma \cdot \overline{\tau} \quad (\sigma, \tau \in E_{m,y}). \tag{4.5}$$

Then h_m is semipositive in the sense of Nakano. □

4.2 Schwarz type lemma

Theorem 4.5 *Let $f : X \longrightarrow Y$ be a proper surjective projective morphism with connected fibers between complex manifolds. Assume the followings:*

(1) f is smooth (a submersion).

(2) K_X is f-ample.

(3) Y admits a complete Kähler-Einstein form ω_Y such that $-\mathrm{Ric}_{\omega_Y} = \omega_Y$.

(4) There exists a complete Kähler-Einstein form ω_X on X such that $-\mathrm{Ric}_{\omega_X} = \omega_X$ holds on X.

Let $dV_{X/Y}$ be the relative Kähler-Einstein volume form defined as (4.2). Then

$$dV_X \geqq dV_{X/Y} \cdot f^*dV_Y \tag{4.6}$$

holds on X. □

To state an application of Theorem 4.5, we introduce the following notion.

Definition 4.6 *Let X be a smooth projective n-fold and let L be a line bundle on X. We set*

$$\mu(X, L) := n! \limsup_{m \longrightarrow \infty} m^{-n} h^0(X, \mathcal{O}_X(mL)) \tag{4.7}$$

and call it the volume of X with respect to L. □

As a corollary of Theorem 4.5, we obtain the following slight generalization of the result of Kawamata.

Corollary 4.7 *([Ka3]) Let $f : X \longrightarrow Y$ be a surjective projective morphism between projective manifolds of general type with connected fibers. Suppose that X, Y are of general type. Then*

$$\frac{\mu(X, K_X)}{(\dim X)!} \geqq \frac{\mu(F, K_F)}{(\dim F)!} \cdot \frac{\mu(Y, K_Y)}{(\dim Y)!} \tag{4.8}$$

holds. And the equality holds if and only if $f : X \longrightarrow Y$ is birationally isotrivial.

□

4.3 Boundedness of the families

Theorem 4.8 *Let $f : X \longrightarrow C$ be an effectively parametrized family of smooth canonically polarized varieties over a smooth quasiprojective curve C. Let m be a positive integer such that $f_*K_{X/C}^{\otimes m} \neq 0$. Then*

$$\deg \det f_*K_{X/C}^{\otimes m} \leqq \frac{1}{r\varepsilon} \deg K_C$$

holds, where ε is the threshold of $f_*K_{X/C}^{\otimes m}$ as in Theorem 6.2 and $r := \mathrm{rank}\, f_*K_{X/C}^{\otimes m}$.

□

In the case of the higher dimensional base, we have the following result.

Theorem 4.9 *Let $f : X \longrightarrow Y$ be an effectively parametrized family of smooth canonically polarized varieties over a smooth quasiprojective curve C. Let m be a positive integer such that $f_*K_{X/Y}^{\otimes m} \neq 0$. Then*

$$\det f_*K_{X/Y}^{\otimes m} - \frac{1}{r\varepsilon}\deg K_Y$$

is not pseudoeffective, where ε is the threshold of $f_*K_{X/C}^{\otimes m}$ as in Theorem 6.2 and $r := \operatorname{rank} f_*K_{X/Y}^{\otimes m}$. □

5 Proof of Theorem 4.5

In this section we shall prove Theorem 4.5. The proof is a simple application of the maximum principle.

Let ω_0 be the Kähler form on X defined by:

$$\omega_0 := -\mathrm{Ric}\,(dV_{X/Y}\cdot f^*dV_Y) = -\mathrm{Ric}\,dV_{X/Y} - \mathrm{Ric}\,f^*dV_Y. \tag{5.1}$$

In fact by Theorem 4.3, $-\mathrm{Ric}\,dV_{X/Y}$ is semipositive on X and strictly positive in the fiber direction. Hence ω_0 is a complete Kähler form on X. We consider the Monge-Ampère equation:

$$\log\frac{(\omega_0+\sqrt{-1}\partial\bar{\partial}u)^n}{\omega_0^n} = \log\frac{dV_{X/Y}\cdot f^*dV_Y}{\omega_0^n} + u \tag{5.2}$$

on X. We set

$$F := \log\frac{dV_{X/Y}\cdot f^*dV_Y}{\omega_0^n}. \tag{5.3}$$

By Theorem 4.3, we see that

$$F \leqq 0 \tag{5.4}$$

holds. By maximum principle, we have that

$$u \geqq 0 \tag{5.5}$$

holds.

6 Generalization of local semipositivity to KLT pairs of general type

It is not difficult to extend Theorem 6.2 to the case of KLT pairs of general type. In fact to prove Theorem 6.2, we need such a generalization.

Theorem 6.1 *([T8]) Let (X, D) be a KLT pair of log general type. Then there exists a closed positive current ω_E on X such that*

1. *There exists a nonempty Zariski open subset U of X such that $\omega_E|U$ is C^∞ and $-\mathrm{Ric}_{\omega_E} = \omega_E$ holds on U,*

2. $dV_E^{-1} = n!(\omega_E^n)^{-1}(n := \dim X)$ *is an AZD of* $K_X + D$, *i.e., for every* $\sigma \in H^0(X, \mathcal{O}_X(\lfloor m(K_X + D)\rfloor))$,

$$\int_X |\sigma|^2 \cdot dV_E^{-(m-1)} < \infty$$

holds. □

Theorem 6.2 *Let* $f : X \longrightarrow Y$ *be a proper surjective projective morphism with connected fibers between complex manifolds. And let* D *be an effective* $\mathbb{Q}$*-divisor such that*

$$Y^{\circ} := \{y \in Y | (X_y, D_y) \text{ is a KLT pair of log general type}\}$$

is nonempty. Assume the followings:

(1) f *is smooth (a submersion).*

(2) K_X *is* f*-ample.*

(3) Y *admits a complete Kähler-Einstein form* ω_Y *such that* $-\mathrm{Ric}_{\omega_Y} = \omega_Y$.

(4) There exists a complete Kähler-Einstein form ω_X *on* X *such that* $-\mathrm{Ric}_{\omega_X} = \omega_X$ *holds on* X.

Let $dV_{X/Y}$ *be the relative Kähler-Einstein volume form defined as (4.2). Then*

$$dV_X \geqq dV_{X/Y} \cdot f^* dV_Y \tag{6.1}$$

holds on X. □

7 Generalization to a family of minimal algebraic varieties with semiample canonical bundles

Let $f : X \longrightarrow Y$ be a smooth projective morphism with connected fibers such that every fiber is a minimal algebraic varieties with semiample canonical bundle. Theorem **??** can be generalized to this case by using the variation of canonical measures.

Theorem 7.1 *Let* $f : X \longrightarrow C$ *be an effectively parametrized family of smooth minimal algebraic varieties with semiample canonical bundles over a smooth quasiprojective curve* C. *Let* m *be a positive integer such that* $f_* K_{X/C}^{\otimes m} \neq 0$. *Then*

$$\deg \det f_* K_{X/C}^{\otimes m} \leqq \frac{1}{\varepsilon} \deg K_C$$

holds, where ε *is the threshold of* $f_* K_{X/C}^{\otimes m}$. □

In the case of the higher dimensional base, we have the following result.

Theorem 7.2 *Let $f : X \longrightarrow Y$ be an effectively parametrized family of minimal algebraic varieties with semiample canonical bundles over a smooth quasiprojective curve C. Let m be a positive integer such that $f_*K_{X/Y}^{\otimes m} \neq 0$. Then*

$$\det f_*K_{X/Y}^{\otimes m} - \frac{1}{\varepsilon}\deg K_Y$$

is not nef, where ε is the threshold of $f_*K_{X/C}^{\otimes m}$. □

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