数理解析研究所講究録
第 1695 巻 2010 年 1-4

Hamilton-Jacobi equations and Euclidean Sobolev inequality

Yasuhiro FUJITA*

1 Introduction

The result of this note is a special case of [3], and the readers should refer to it for more detailed results and their proofs.

Let Ω be a bounded and Lebesgue measurable set in $\mathbb{R}^n$. Let $0<\alpha<\beta<\infty$. Then, as is well-known, the following inequality holds:

$$|\Omega|^{-1/\alpha}\|f\|_{\alpha,\Omega}\leq|\Omega|^{-1/\beta}\|f\|_{\beta,\Omega}\leq\|f\|_{\infty,\Omega}, \qquad f\in L^{\infty}(\Omega) \tag{1.1}$$

where $|\Omega|$ is the Lebesgue measure of Ω and $\|\cdot\|_{\beta,\Omega}$ is the $L^{\beta}(\Omega)$-norm $(0<\beta<\infty)$ with respect to the Lebesgue measure in $\mathbb{R}^n$. Furthermore, this inequality is optimal in the sense that all inequalities in (1.1) are reduced to equalities when f is a constant function on Ω. This inequality show a norm-monotone property of $\{|\Omega|^{-1/\beta}\|f\|_{\beta,\Omega}\}_{0<\alpha<\infty}$.

However, as far as we know, there is no inequality corresponding to (1.1) when a bounded and Lebesgue measurable set Ω in $\mathbb{R}^n$ is replaced by the whole domain $\mathbb{R}^n$. A reason for it is that when $\Omega=\mathbb{R}^n$, we have $|\Omega|^{-1/\beta}=0$ for all $0<\beta<\infty$.

The goal of this note is to provide an inequality corresponding to (1.1) when a bounded and Lebesgue measurable set Ω in $\mathbb{R}^n$ is replaced by the whole domain $\mathbb{R}^n$. This inequality is obtained by using the Euclidean logarithmic Sobolev inequality and Hamilton-Jacobi equations. We use the inequalities obtained by [4, 5], and minimize this inequality with respect to some parameter, and finally get the desired inequality by letting another parameter tend to ∞.

2 Preliminaries

In this section, we collect some results of [4, 5]. For $p\geq 1$, we denote by $W^{1,p}(\mathbb{R}^n)$ the space of all weakly differentiable functions f on $\mathbb{R}^n$ such that f and $|Df|$ are in $L^p(\mathbb{R}^n)$. Throughout this note, the integral without its domain is understood as the one over $\mathbb{R}^n$.

---

*Department of Mathematics, University of Toyama, Toyama 930-8555, Japan. Supported in part by Grant-in-Aid for Scientific Research, No. 21540168, 18204009, JSPS.

Lemma 2.1 *Let $p \geq 1$. Then, we have the following Euclidean logarithmic Sobolev inequality:*

$$(2.1)\quad \int |f|^p \log |f|^p \, dx \leq \frac{n}{p} \log \left(L_p \int |Df|^p \, dx \right) \quad \textit{for } f \in W^{1,p}(\mathbb{R}^n) \textit{ with } \int |f|^p \, dx = 1.$$

Here,

$$(2.2)\qquad L_p = \frac{p}{n} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left(\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p-1}{p}+1\right)} \right)^{p/n},$$

and this is the best possible constant satisfying (2.1).

We denote by $\|\cdot\|_\alpha$ the $L^\alpha(\mathbb{R}^n)$-norm with respect to the Lebesgue measure in $\mathbb{R}^n$.

Lemma 2.2 *Let $p > 1$. For $f \in Lip(\mathbb{R}^n)$, let $u \in Lip(\mathbb{R}^n \times [0,\infty))$ be a viscosity subsolution of the Hamilton-Jacobi equation*

$$(2.3)\qquad u_t(x,t) + \frac{1}{p}|Du(x,t)|^p = 0 \quad \textit{in } \mathbb{R}^n \times (0,\infty), \qquad u = f \quad \textit{on } \mathbb{R}^n \times \{0\}.$$

If there is a constant $\alpha > 0$ such that $e^f \in L^\alpha(\mathbb{R}^n)$, then $e^{u(\cdot,t)} \in L^\beta(\mathbb{R}^n)$ for any $\beta \in (\alpha,\infty)$ and $t \in (0,\infty)$. Furthermore, we have

$$(2.4)\qquad \|e^{u(\cdot,t)}\|_\beta \leq \|e^f\|_\alpha \left(\frac{nL_p e^{p-1}(\beta-\alpha)}{p^p t} \right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \frac{\alpha^{\frac{n}{\alpha\beta}\left(\frac{\alpha}{p}+\frac{\beta}{q}\right)}}{\beta^{\frac{n}{\alpha\beta}\left(\frac{\beta}{p}+\frac{\alpha}{q}\right)}}, \qquad t > 0,$$

where $q > 1$ is the exponent conjugate of p, i.e., $(1/p) + (1/q) = 1$.

3 A result

Let $\theta > 0$. For $\alpha > 0$, we set

$$(3.1)\qquad \mathcal{L}_{\alpha,\theta} = \left\{ f \in \mathrm{Lip}(\mathbb{R}^n) \, : \, \mathrm{Lip}(f) \leq \theta, \ e^f \in L^\alpha(\mathbb{R}^n) \right\},$$

where $\mathrm{Lip}(f)$ is the Lipschitz constant of f, i.e., $\mathrm{Lip}(f) = \sup_{x \neq y} |f(x) - f(y)|/|x-y|$. Let us denote by ω_{n-1} the surface area of the unit ball in $\mathbb{R}^n$. We set

$$(3.2)\qquad k_n = \left(\frac{1}{\omega_{n-1}(n-1)!} \right)^{1/n}.$$

Now, we state our result of this note and give a sketch of its proof.

Theorem 3.1 *Let $\alpha, \theta > 0$. For $f \in \mathcal{L}_{\alpha,\theta}$, we have the following inequality:*

$$(3.3)\qquad \|e^f\|_\infty \leq \|e^f\|_\beta \, (k_n\theta\beta)^{n/\beta} \leq \|e^f\|_\alpha \, (k_n\theta\alpha)^{n/\alpha}, \qquad \alpha \leq \beta \leq \infty.$$

Inequality (3.3) is optimal in the sense that equality holds when $f(x) = C - \theta|x|$ for some constant $C \in \mathbb{R}$.

Remark. Note that $\lim_{\beta\to\infty}(k_n\theta\beta)^{n/\beta} = 1$. Hence, the family $\{\|e^f\|_\beta\ (k_n\theta\beta)^{n/\beta}\}_{\alpha<\beta<\infty}$ interpolates continuously and monotonically between $\|e^f\|_\alpha\ (k_n\theta\alpha)^{n/\alpha}$ and $\|e^f\|_\infty$.

Sketch of Proof. Let $f \in \mathcal{L}_{\alpha,\theta}$. Then, the function $v(x,t) = f(x) - (\theta^p t/p)$ is a subsolution of (2.3), so that $v \le u$ on $\mathbb{R}^n \times [0,\infty)$ by [7]. By Lemma 2.2, we have, for any $\beta \in (\alpha,\infty)$ and $t \in (0,\infty)$,

$$\text{(3.4)} \quad \|e^f\|_\beta \le \|e^f\|_\alpha\, e^{\theta^p t/p}\, t^{-\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \times \left(\frac{nL_p e^{p-1}(\beta-\alpha)}{p^p}\right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \frac{\alpha^{\frac{n}{\alpha\beta}\left(\frac{\alpha}{p}+\frac{\beta}{q}\right)}}{\beta^{\frac{n}{\alpha\beta}\left(\frac{\beta}{p}+\frac{\alpha}{q}\right)}}, \qquad t>0,$$

where $q > 1$ is the exponent conjugate of p, i.e., $(1/p) + (1/q) = 1$. By minimizing the right-hand side of (3.4) with respect to the t-variable, we have

$$\begin{aligned}\text{(3.5)} \quad \|e^f\|_\beta &\le \|e^f\|_\alpha \left(\frac{\theta^p e}{n^{\frac{\beta-\alpha}{\alpha\beta}}}\right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \times \left(\frac{nL_p e^{p-1}(\beta-\alpha)}{p^p}\right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \frac{\alpha^{\frac{n}{\alpha\beta}\left(\frac{\alpha}{p}+\frac{\beta}{q}\right)}}{\beta^{\frac{n}{\alpha\beta}\left(\frac{\beta}{p}+\frac{\alpha}{q}\right)}} \\ &= \|e^f\|_\alpha \left(\frac{\theta e L_p^{1/p}}{p}\right)^{\frac{n}{\alpha}-\frac{n}{\beta}} \times \alpha^{\frac{n}{\alpha}}\, \beta^{-\frac{n}{\beta}}.\end{aligned}$$

Hence, we obtain

$$\text{(3.6)} \qquad \|e^f\|_\beta \left(k_p^{(n)}\theta\beta\right)^{n/\beta} \le \|e^f\|_\alpha \left(k_p^{(n)}\theta\alpha\right)^{n/\alpha},$$

where

$$\begin{aligned}\text{(3.7)} \qquad k_p^{(n)} &= \frac{eL_p^{1/p}}{p} \\ &= \left(\frac{n}{eq}\right)^{1/q} \left[\Gamma\left(\frac{n}{q}+1\right)\right]^{-1/n} \frac{e}{n\sqrt{\pi}} \left[\Gamma\left(\frac{n}{2}+1\right)\right]^{1/n}.\end{aligned}$$

Now, letting p tend to ∞ in (3.7), i.e., letting q tend to 1 in (3.7), we conclude that

$$\begin{aligned}\lim_{p\to\infty} k_p^{(n)} &= \lim_{q\to 1} \left(\frac{n}{eq}\right)^{1/q} \left[\Gamma\left(\frac{n}{q}+1\right)\right]^{-1/n} \frac{e}{n\sqrt{\pi}} \left[\Gamma\left(\frac{n}{2}+1\right)\right]^{1/n} \\ &= \frac{1}{\sqrt{\pi}} \frac{1}{(n!)^{1/n}} \left[\Gamma\left(\frac{n}{2}+1\right)\right]^{1/n} = \left(\frac{1}{\omega_{n-1}(n-1)!}\right)^{1/n} = k_n.\end{aligned}$$

The proof is completed. □

References

[1] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhäuser, Boston, 1997.

[2] S. G. BOBKOV, I. GENTIL AND M. LEDOUX, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pures Appl. 80 (2001), pp. 669–696.

[3] Y. FUJITA, *Norm-monotone inequalities over the Euclidean n space*, preprint.

[4] I. GENTIL, *Ultracontractive bounds on Hamilton—Jacobi equations*, Bull. Sci. Math. 126 (2002), pp. 507–524.

[5] I. GENTIL, *The general optimal L^p-Euclidean logarithmic Sobolev inequality by Hamilton-Jacobi equations*, J. Funct. Anal. 202 (2003) pp. 591–599.

[6] H. ISHII, *Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space*, Annales de l'IHP-Analyse non lineaire 25 (2008), pp. 231-266

[7] H. ISHII, *Asymptotic solutions of Hamilton-Jacobi equations with semi-periodic Hamiltonians*, to appear in Comm. PDE.