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Integral equations and approximation of $p-$Laplace equations

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1 Introduction

In this note I partially describe the contents of the lecture that I gave at the conference. The lecture was based on a recent joint work with H. Ishii [11].

We consider the Dirichlet problem of integral equation

$$(\mathrm{E}_{\sigma})\left\{\begin{array}{ll} M_{\sigma}[u]=f & \text{in } \Omega\\ u=g & \text{for } x\in\partial\Omega, \end{array}\right.$$

where Ω is a bounded domain of $\mathbb{R}^{n}$ and $f\in C(\overline{\Omega})$ and $g\in C(\partial\Omega)$ are given functions. Let $p>1$ and $p>\sigma$. The operator M_{σ} is defined as

$$M[\phi](x)=\mathrm{p.v.}\int_{B(0,\mathrm{dist}(x,\partial\Omega))}G(\phi(x+z)-\phi(z))\frac{p-\sigma}{|z|^{n+\sigma}}dz,$$

for bounded measurable functions ϕ on Ω, where G is a function on $\mathbb{R}$ given by $G(r)=|r|^{p-2}r$. We establish the existence and uniqueness result for (E_{σ}) and convergence of solution of (E_{σ}) as $\sigma\rightarrow p$ to the corresponding Dirichlet problem for p-Laplace equation

$$(\mathrm{E}_{\infty})\left\{\begin{array}{ll} \nu\Delta_{p}u=f & \text{in } \Omega\\ u=g & \text{for } x\in\partial\Omega, \end{array}\right.$$

where $\nu=\nu_{n,p}$ is a constant given by

$$\nu=\frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{n+p}{2})}.$$

2 Solvability of equation (E_σ)

First it is to be noted that we solve this problem in viscosity sense, and to establish the existence of solution the Perron method is employed, and it is necessary to establish stability properties of subsolutions beforehand.

Theorem 2.1 *Let $\mathcal{S}_0$ be a nonempty subset of subsolutions of* (E_σ). *Assume that the family $\mathcal{S}_0$ is uniformly bounded on Ω. Define the bounded function u on Ω by $u(x) = \sup\ \{v(x)|v \in \mathcal{S}_0\}$. Then u is a subsolution of* (E_σ).

It is natural to check that the half relaxed limit of subsolutions is also a subsolution.

Theorem 2.2 *Let $\{u_k\}$ be a sequence of subsolutions of* (E_σ). *Assume that the collection $\{u_k\}$ is uniformly bounded on Ω. Define the bounded function u on Ω by*

$$u(x) = \lim_{j\to\infty} \sup \left\{u_k(y)|y \in B(x,\ j^{-1}) \cap \Omega,\ k \geq j\right\}$$

Then u is a subsolution of (E_σ).

These theorems are proved through some appropriate estimates of the operators M_σ.

To formulate a basic existence result (Perron method) for (E_σ), we let $g^- \in \mathrm{LSC}(\Omega)$ and $g^+ \in \mathrm{USC}(\Omega)$ be a subsolution and a supersolution of (E_σ), respectively. Assume furthermore that $g^\pm$ are bounded in Ω and $g^- \leq g^+$ in Ω. Set

$$u(x) = \sup \left\{v(x)|\ v \text{ is a subsolution of } (\mathrm{E}_\sigma),\ g^- \leq v \leq g^+ \text{ in } \Omega\right\} \quad (1)$$

Theorem 2.3 *The function given by* (**??**) *is a solution of* (E_σ).

The uniqueness of solution is a consequence of the comparison theorem.

Theorem 2.4 *Let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ be a subsolution and a supersolution of* (E_σ), *respectively. Assume that $u \leq v$ on $\partial\Omega$ and u and v are bounded on $\overline{\Omega}$. Then $u \leq v$ in Ω.*

To conclude the existence of solution, it is not enough only to have Perron method, because in it the existence of sub and supersolution which satisfy the comparison principle is assumed. We need to construct such functions. And for this purpose we impose two following additional assumptions.

(H1) The set Ω satisfies the *uniform exterior sphere condition.* That is, there is an $R > 0$ and for each $x \in \partial\Omega$, a point $y \in \mathbb{R}^n$ such that

$$B(y, R) \cap \overline{\Omega} = \{x\}.$$

(H2) There exist constants $\epsilon_0 \in (0, 1)$ and $C_0 > 0$ such that

$$|f(x)| \leq C_0(\mathrm{dist}(x, \partial\Omega))^{\epsilon_0(p-1)-\sigma} \quad \text{for all } x \in \Omega.$$

With **(H1)** and **(H2)** assumed, we have

Theorem 2.5 *There exist functions* $\psi^- \in \mathrm{USC}(\overline{\Omega})$ *and* $\psi^+ \in \mathrm{LSC}(\overline{\Omega})$ *such that* ψ^+ *(resp.,* ψ^-*) is a supersolution (resp., subsolution) of* (E_σ), $\psi^- \leq \psi^+$ *on* $\overline{\Omega}$ *and* $\psi^\pm = g$ *on* $\partial\Omega$. *Moreover, the functions* $\psi^\pm$ *can be chosen independently of* σ.

It is important that this construction of barrier functions is independent of σ, that is, when thinking the asymptoic behaviour of solutions as $\sigma \longrightarrow p+$ later, the solutions are dominated by the barrier functions and so do not diverge to $\pm\infty$.

As a consequence of all these theorems above, we conclude

Theorem 2.6 *There exists a unique solution of* (E_σ).

3 p-Laplace equation in the limit as $\sigma \longrightarrow p$

For each σ, we have a unique solution of (E_σ), which we write u_σ. We now turn our attention to the asymptotic behavior of $u_\sigma$as we let $\sigma \longrightarrow p$. And we insist that the sequence $\{u_\sigma\}$ converges to the solution of the corresponding Dirichlet problem of p-Laplace equation (E_∞).

The existence and uniqueness of solution of (E_∞) must be checked, and actually

Theorem 3.1 *There is a unique weak solution of* (E_∞).

Theorem 3.2 *Let* $v \in W^{1,p}_{loc} \cap C(\overline{\Omega})$ *be the unique weak solution of* (E_∞). *Then*

$$\lim_{\sigma \to p-} u_\sigma(x) = v(x) \quad \textit{uniformly on } \overline{\Omega}.$$

Outline of proof. Here we give the fundamental calculation on which Theorem3.2 is based. Let $u \in C^2(\mathbb{R}^d)$. We compute

$$I := \lim_{\sigma \to p-} \int_{|z|<1} G(u(x+z) - u(x)) K_\sigma(z) dz$$

where $K_\sigma(z) = \frac{p-\sigma}{|z|^{n+\sigma}}$. Put $q := Du(x)$, $A := D^2u(x)$. For simplicity we assume that $q \neq 0$ and $q = |q|e_n$. Here e_k denotes the k-th basis of $\mathbb{R}^n$. If $0 < \delta \ll 1$, and $|z| < \delta$,

$$\begin{aligned} G(u(x+z) - u(x)) &= G(q \cdot z + \frac{1}{2}Az \cdot z) = G(q \cdot z) G(1 + \frac{Az \cdot z}{2q \cdot z}) \\ &= G(q \cdot z)\Big(G(1) + G'(1 + \theta \frac{Az \cdot z}{2q \cdot z}) \frac{Az \cdot z}{2q \cdot z}\Big) \\ &\approx G(q \cdot z) + G'(1) Az \cdot z |q \cdot z|^{p-2}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{|z|<\delta} G(u(x+z) - u(x)) K(z) dz &\approx \int_{|z|<\delta} G(q \cdot z) K(z) dz \\ &+ G'(1) \int_{|z|<\delta} Az \cdot z |q \cdot z|^{p-2} K(z) dz \\ &= G'(1)|q|^{p-2}(p-\sigma) \int_{|z|<\delta} \frac{Az \cdot z |z_n|^{p-2}}{|z|^{n+\sigma}} dz \\ &= G'(1)|q|^{p-2}(p-\sigma) \sum_{j=1}^{n} a_{j,j} \int_{|z|<\delta} \frac{|z_j|^2 |z_n|^{p-2}}{|z|^{n+\sigma}} dz. \end{aligned}$$

Next we compute the integral part of the last term.

$$\begin{aligned} (p-\sigma) \int_{|z|<\delta} \frac{|z_j|^2 |z_n|^{p-2}}{|z|^{n+\sigma}} dz &= \begin{cases} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2})^{n-2}}{\Gamma(\frac{p+n}{2})} \delta^{p-\sigma} & (j \neq n) \\ \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})^{n-1}}{\Gamma(\frac{p+n}{2})} \delta^{p-\sigma} & (j = n) \end{cases} \\ &:= \begin{cases} \gamma \delta^{p-\sigma} & (p \neq n) \\ \gamma' \delta^{p-\sigma} & (p = n) \end{cases} . \end{aligned}$$

And

$$\frac{\gamma'}{\gamma} = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})^{n-1}}{\Gamma(\frac{3}{2})\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2})^{n-2}} = p-1.$$

Therefore

$$\begin{aligned} &(p-\sigma)\sum_{j=1}^{n} a_{j,j}\int_{|z|<\delta}\frac{|z_j|^2|z_n|^{p-2}}{|z|^{n+\sigma}}dz \\ =\ & (\gamma\sum_{j=1}^{n-1} a_{j,j} + \gamma' a_{n,n})\delta^{p-\sigma} \\ =\ & \gamma(\triangle u(x) + (p-2)\partial_{n,n}u(x))\delta^{p-\sigma}, \end{aligned}$$

where $a_{i,j}$ denotes the (i,j)-entry of the matrix A.

On the other hand,

$$\begin{aligned} & \triangle_p u(x) = \operatorname{div}(|Du(x)|^{p-2}Du(x)) \\ =\ & (p-2)|Du(x)|^{p-4}D^2u(x)Du(x)\cdot Du(x) \\ +\ & |Du(x)|^{p-2}\triangle u(x) \\ =\ & |Du(x)|^{p-2}\Big(\triangle u(x) + (p-2)\partial_{n,n}u(x)\Big). \end{aligned}$$

Hence we get

$$I = \nu_{n,p}\triangle_p u(x),$$

where

$$\nu_{n,p} = \frac{1}{2}G'(1)\gamma = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})^{n-1}}{\Gamma(\frac{p+n}{2})}.$$

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