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Optimal Execution Problem with Market Impact : Mathematical Formulation of the Model and Its Characterization by Viscosity Solution Theory

(株) 三菱UFJトラスト投資工学研究所 (MTEC) 加藤 恭 (Takashi Kato) *
Mitsubishi UFJ Trust Investment Technology Institute Co., Ltd. (MTEC)

Abstract

We study the optimal execution problem in the market model in consideration of market impact as a regular stochastic control problem. We focus on mathematical formulation and characterization as the viscosity solution of the corresponding nonlinear partial differential equation (so-called HJB.) First we introduce the outline of the theory of an optimal portfolio management problem and liquidity problems which are both important in mathematical finance. Then we formulate our optimal execution problem as the discrete-time model and describe the value function with respect to a trader's optimization problem. By shortening the intervals of execution times, we derive the value function of the continuous-time model and then we study some properties of them. We show that the properties of the continuous-time value function vary by the strength of market impact. Moreover we introduce some examples of this model, which tell us that the forms of the optimal execution strategies entirely change according to the amount of the security holdings.

1 Introduction

The optimal portfolio management problem in mathematical finance has been developed in [25], [26] and in other papers. They considered the problem such that how does a trader (or an investor) manage his/her portfolio of financial assets in the market to make his/her future wealth large. Such a problem is often considered as some stochastic control problem in continuous-time model. This is because the theory of stochastic calculus and stochastic differential equations (SDE) are suitable for describing fluctuations of prices of financial assets. Although a trader in the real market cannot trade continuously, considering the continuous-time market model is important to get some findings of theoretically appropriate trading policies and essence of portfolio selection problem. Discrete-time model is closer to the settings in the real market, and there are many trivial and noisy things which make the problem confused. So we consider the derivation of the continuous-time model from the discrete-time model by limit transition, and in some cases the derived model makes the situations clear.

The most significant and basic result of [25] and [26] is that in standard continuous-time market model the optimal trading strategy is to fix the portfolio ratio equal to some constant proportion (so-called the Merton proportion) under some ideal conditions. Later we review the outline of them. These classical financial theories assumed that the assets in the market are perfectly liquid. But in the real market we face various liquidity risks. One of liquidity

*E-mail: kato@mtec-institute.co.jp
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problems is market impact (MI), that is the effect of the investment behavior of traders on security prices. Such problems are often discussed in the framework of the optimal execution problems, where a trader has a certain amount of security holdings (shares of a security held) and tries to execute until the time horizon.

Recently there have been various studies made about the optimization problem with MI, but the standard framework has not been fixed yet. In this paper, we try to construct the framework of such a model. We formulate the optimal execution problem in discrete-time model first, and then derive the continuous-time model by taking the limit. By doing so, we may see the essence of liquidity problems, especially MI.

This paper is organized by an introduction of standard optimal portfolio management problem and the summary of [18]-[21] and [15]. Section 2 is an overview of an optimal investment problem and the Merton problem. The Merton problem is the basic of optimization problems in mathematical finance, but more realistic problems like liquidity problems are abstracted. Section 3 refers to recent studies of liquidity problems and introduce the mapping of this paper. In Section 4 we introduce our model. We formulate mathematically a trader's optimization problem in discrete-time model, and give some assumptions to derive the continuous-time model. In Section 5 we give our main results. We show that value functions in discrete-time model converge to the one in continuous-time model. Then we study some properties of the continuous-time value function : continuity, semi-group property and the characterization as the viscosity solution of Hamilton-Jacobi-Bellman equation (HJB.) Moreover we have the uniqueness result of the viscosity solution of HJB when MI is strong (in some meanings to be discussed later.) In Section 6 we also consider the case that a trader needs to sell up entire shares of the security. We show that such a sell-out condition does not influence the form of continuous-time value function in our model. Section 7 treats the model of MI with noise as an extension. In Section 8 we treat some examples of our model. We conclude this paper in Section 9. Almost all proofs are omitted, but you can refer to [18] and [15]. As the exception, Section 10 refers rough proofs of some results related with viscosity solutions.

2 Outline of Optimal Investment Problems

In this section we go back to the classical theory of an optimal investment problem, which is important and fundamental in mathematical finance. As I mentioned first, such a theory was basically constructed in [25] and [26] and the model considered in them is called "Merton theory" or "Merton problem."

The real market consists of many financial assets, but to make the point simple the model which we consider in this paper is assumed to consist of only 2 assets : the one is a risk-free asset (namely cash) and the other one is a risky asset (namely a security.) The price of cash is always equal to 1, which means that a risk-free rate is equal to zero. The price of a security fluctuates stochastically. To describe uncertainty, we prepare a (complete) stochastic basis (a filtered probability space which satisfies usual conditions, see [16]) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ for $T > 0$ and fix it.

We consider a single trader (or investor) who has an initial endowment $w_0 \geq 0$ as cash at $t = 0$ (initial time.) His/her purpose is to enlarge the wealth at the time horizon $t = T$ by repeating buying and selling the security. But he/she does not know the future price of the security, so he/she cannot maximize the value of terminal wealth itself.

As a tool for measuring the "happiness" of uncertain future wealth, we often use the expected utility of a trader. This is defined as the function of the form $\mathrm{E}[u(W_T)]$, where $W_T = W_T(\omega)$ is a trader's wealth at the time horizon T, which is given as an $\mathcal{F}_T$-measurable random variable, and $u : \mathbb{R} \longrightarrow \mathbb{R}$ is a deterministic function. By mathematical reason, we need some moment condition such that $\mathrm{E}[|u(W_T)|] < \infty$.

Let x_t (respectively, φ_t) be an amount of cash holdings (respectively, security holdings) of a trader at time t. Moreover let S_t be a price of the security at time t. Since we do not know the price S_t until time t, the stochastic process $(S_t)_t$ should be assumed to be $(\mathcal{F}_t)_t$-adapted. If we do not consider the liquidity problem, that is, a trader can buy and sell the security of any volume and at any time, then a trader's wealth W_t at time t is given by the sum of cash holdings and the value of the security held i.e. $W_t = x_t + \varphi_t S_t$. A trader's problem is to select an appropriate trading strategy $(x_t, \varphi_t)_{0\le t\le T}$ with given initial values (x_0, φ_0) to maximize $\mathrm{E}[u(W_T)]$.

The fluctuation of the process of the security price is often represented by the stochastic process of Ito-type. Usually a security price should take positive value at any time, we often consider the (real valued) log-price process $(X_t)_t$ first and then define $S_t = \exp(X_t)$. We assume that $(X_t)_t$ is a solution of the following SDE

$$dX_t = \mu_t dt + \sigma_t dB_t, \quad X_0 = \log s_0,$$

where $(\mu_t)_t$ and $(\sigma_t)_t$ are $(\mathcal{F}_t)_t$-progressively measurable process such that

$$\int_0^T (|\mu_t| + |\sigma_t|^2)dt < \infty \quad \text{a.s.}$$

and $(B_t)_t$ is an $(\mathcal{F}_t)_t$-Brownian motion. Here $s_0 > 0$ is the initial price of the security. Then Ito's formula implies that $(S_t)_t$ satisfies the following SDE

$$dS_t = S_t(\tilde{\mu}_t dt + \sigma_t dB_t),$$

where $\tilde{\mu}_t = \mu_t + \frac{1}{2}\sigma_t^2$.

Now we give the meaningful class of a trader's trading strategies. An $(\mathcal{F}_t)_t$-progressively measurable stochastic process $(x_t, \varphi_t)_t$ is called an admissible strategy if and only if both

$$\int_0^T (|\varphi_t S_t \tilde{\mu}_t| + |\varphi_t S_t \sigma_t|^2)dt < \infty \quad \text{a.s.} \tag{2.1}$$

and

$$dW_t = \varphi_t dS_t \tag{2.2}$$

hold (we recall that $W_t = x_t + \varphi_t S_t$.) The equality (2.2) is called the "self-financing condition." This means that a trader's wealth does not change other than investing a security.

When the market is fully liquid, it is convenient to consider a weight (proportion) process of investing a security as a trading strategy instead of $(x_t, \varphi_t)_t$. Their relation is described as

$$\pi_t = \frac{\varphi_t S_t}{x_t + \varphi_t S_t} = 1 - \frac{x_t}{W_t}.$$

Then (2.2) is rewritten as

$$\frac{dW_t}{W_t} = \pi_t \frac{dS_t}{S_t}.$$

As a substitute of (2.1), we assume the following condition

$$\int_0^T (|\pi_t \tilde{\mu}_t| + |\pi_t \sigma_t|^2) dt < \infty \quad \text{a.s.}$$

We denote W_t^π when we shall emphasize that the wealth process is given by the strategy $(\pi_t)_t$. A trading strategy which satisfies the condition above is called an admissible strategy. Here we also assume that an admissible strategy satisfies $\pi_t \in [0,1]$ for all t, which means that a trader does not short a security and borrow money.

Our problem is to find an optimal strategy $(\pi_t^*)_t$ such that

$$V(x_0, \varphi_0; u) = \mathrm{E}[u(W_T^{\pi^*})] = \sup_{(\pi_t)_t:\text{admissible}} \mathrm{E}[u(W_T^\pi)].$$

Here we notice that if $x_0 + \varphi_0 s_0 = \tilde{x}_0 + \tilde{\varphi}_0 s_0$, it follows that

$$V(x_0, \varphi_0; u) = V(\tilde{x}_0, \tilde{\varphi}_0; u),$$

thus we can replace our problem with $\hat{V}(w; u) = V(x_0, \varphi_0; u)$ with $w = x_0 + \varphi_0 s_0$. The solution of such a problem is given under wide assumptions by many papers (see [17].) Typically, when $\tilde{\mu}_t \equiv \mu$ and $\sigma_t \equiv \sigma$ are positive constants which satisfy $\mu < \sigma^2$ and the utility function is given by $u(w) = \log w$ (log utility,) we have the following theorem (we call the optimization problem under these settings the Merton problem.)

Theorem 1. $\hat{V}(w; u) = \log w + \dfrac{\mu^2 T}{2\sigma^2}$ *holds and an optimal strategy is given by* $\pi_t^* \equiv \dfrac{\mu}{\sigma^2}$.

This theorem implies that an optimal strategy in this case is to keep the portfolio ratio equal to the constant $\frac{\mu}{\sigma^2}$, which is called a Merton proportion. This is one of the most fundamental results in mathematical finance and dynamic portfolio management problem.

To prove Theorem 1, there are roughly divided into two approaches : "a PDE approach" and "a martingale approach."

Roughly saying, a martingale approach is the method to solve the problem by changing the probability measure to let the wealth process a martingale and by using the martingale representation theorem (Theorem 2.7.2 in [27].) For details of a martingale approach, see [9], [10], [11] and [17]. A martingale approach is useful to solve the optimization problem for various forms of utility functions without the assumption that a security price follows a Markov process, but we need some idea to apply for liquidity problems (for an example of applying a martingale approach to the problem of transaction costs, see [6].)

In this paper we mainly use a PDE approach. In this approach, we generalize the function $\hat{V}(w; u)$ to a "value function" $V_t(w; u)$ which is the function defined as the same as $\hat{V}(w; u)$ replacing $[0, T]$ with $[0, t]$. This definition differs a little from the one in the standard arguments, especially the direction of a time scale is opposite, but this makes no essential problem. Then we characterize the value function as a solution of a certain partial differential equation (PDE.) Such a PDE is often called the Hamilton-Jacobi-Bellman equation (HJB.)

By solving HJB, we can derive the explicit form of the value function and deduce an optimal strategy. But HJB often has a non-linear form and usually it is difficult to see the smoothness of the value function, so applying the classical theory of PDE is a little hard. To consider such a (second order) non-linear PDE, the theory of viscosity solutions was established. This theory is congenial to our optimization problem and also be applicable for liquidity problems. In this section we only introduce the outline of how to apply the PDE approach to the Merton problem intuitively.

We define the value function as

$$V_t(w;u) = \sup_{(\pi_r)_{0\le r\le t}:\text{admissible}} \mathrm{E}[u(W_t^\pi)], \quad t \in (0,T]$$

and $V_0(w;u) = u(w)$. As you can see, it holds that $V_T(w;u) = \hat{V}(w;u)$. Then we can show the following semi-group property (or the Bellman principle) :

$$V_{t+r}(w;u) = V_t(w;V_r(\cdot;u)), \quad 0 \le r,t \text{ with } t+r \le T. \tag{2.3}$$

For the proof of this property, see [24], [27] or [28]. Later we will see the similar property for our optimal execution problem by using Nishio's method in [28].

If we show (2.3), then we can derive the corresponding HJB intuitively by the following way. Let $\varepsilon > 0$ be a small number and let $(\pi_r^*)_r$ be an optimal strategy for $V_\varepsilon(w;V_t(\cdot;u))$. By (2.3), we have

$$V_{t+\varepsilon}(w;u) - V_t(w;u) = \mathrm{E}[V_t(W_\varepsilon^{\pi^*};u) - V_t(w;u)].$$

If we assume that the value function is smooth with respect to t and w, then we can apply Ito's formula and we get for any admissible strategy $(\pi_r)_r$

$$\begin{aligned} V_t(W_\varepsilon^\pi;u) - V_t(w;u) &= \int_0^\varepsilon \frac{\partial}{\partial w}V_t(W_r^\pi;u)dW_r^\pi + \int_0^\varepsilon \frac{1}{2}\frac{\partial^2}{\partial w^2}V_t(W_r^\pi;u)d\langle W^\pi\rangle_r \\ = \int_0^\varepsilon \left\{\mu\pi_r W_r^\pi \frac{\partial}{\partial w}V_t(W_r^\pi;u) + \frac{\sigma^2}{2}(\pi_r W_r^\pi)^2\frac{\partial^2}{\partial w^2}V_t(W_r^\pi;u)\right\}dr &+ \int_0^\varepsilon \sigma W_r^\pi \frac{\partial}{\partial w}V_t(W_r^\pi;u)dB_r, \end{aligned} \tag{2.4}$$

thus

$$\mathrm{E}[V_t(W_\varepsilon^{\pi^*};u) - V_t(w;u)] \ge \int_0^\varepsilon \mathrm{E}\left[\mu\pi_r W_r^\pi \frac{\partial}{\partial w}V_t(W_r^\pi;u) + \frac{\sigma^2}{2}(\pi_r W_r^\pi)^2\frac{\partial^2}{\partial w^2}V_t(W_r^\pi;u)\right]dr$$

under suitable moment conditions which make the last term in the right-hand side of (2.4) a martingale. Dividing by ε and letting $\varepsilon \to 0$, we get

$$\frac{\partial}{\partial t}V_t(w;u) \ge \mu\pi_0 w\frac{\partial}{\partial w}V_t(w;u) + \frac{\sigma^2}{2}\pi_0^2 w^2\frac{\partial^2}{\partial w^2}V_t(w;u).$$

Since $(\pi_r)_r$ is arbitrary, we obtain

$$\frac{\partial}{\partial t}V_t(w;u) \ge \sup_{\theta\in[0,1]}\left\{\mu\theta w\frac{\partial}{\partial w}V_t(w;u) + \frac{\sigma^2}{2}\theta^2 w^2\frac{\partial^2}{\partial w^2}V_t(w;u)\right\}. \tag{2.5}$$

On the other hand, if we take $(\pi_r)_r$ equal to $(\pi_r^*)_r$ in (2.4), the same calculation gives us

$$\begin{aligned}\frac{\partial}{\partial t}V_t(w;u) &= \mu\pi_0^* w\frac{\partial}{\partial w}V_t(w;u)+\frac{\sigma^2}{2}(\pi_0^*)^2w^2\frac{\partial^2}{\partial w^2}V_t(w;u)\\ &\le \sup_{\theta\in[0,1]}\left\{\mu\theta w\frac{\partial}{\partial w}V_t(w;u)+\frac{\sigma^2}{2}\theta^2w^2\frac{\partial^2}{\partial w^2}V_t(w;u)\right\}.\end{aligned}\tag{2.6}$$

By (2.5) and (2.6), we get the following HJB

$$\frac{\partial}{\partial t}V_t(w;u)-\sup_{\theta\in[0,1]}\left\{\mu\theta w\frac{\partial}{\partial w}V_t(w;u)+\frac{\sigma^2}{2}\theta^2w^2\frac{\partial^2}{\partial w^2}V_t(w;u)\right\}=0.\tag{2.7}$$

Of course the above discussion is not strict. In many cases it is hard to prove the differentiability of the value function. The existence of an optimal strategy is not guaranteed. Moreover HJB has the form of non-linear PDE and sometimes we suffer from handling. So we consider viscosity solutions of HJB instead of classical solutions.

A viscosity solution is one of weak solutions of differential equations and is suitable for considering non-linear PDE. As the end of this section, we introduce the definition of viscosity solutions of HJB (2.7).

Definition 1. *The continuous function $v:(0,T]\times(0,\infty)\longrightarrow\mathbb{R}$ is called a viscosity subsolution (respectively, supersolution) of (2.7) if there is a smooth function $\hat{v}\in C^{1,2}((0,T]\times(0,\infty))$ such that $v-\hat{v}$ has a local maximum (respectively, minimum) at $(\bar{t},\bar{w})$, then it holds that*

$$\frac{\partial}{\partial t}\hat{v}(\bar{t},\bar{w})-\sup_{\theta\in[0,1]}\left\{\mu\theta\bar{w}\frac{\partial}{\partial w}\hat{v}(\bar{t},\bar{w})+\frac{\sigma^2}{2}\theta^2\bar{w}^2\frac{\partial^2}{\partial w^2}\hat{v}(\bar{t},\bar{w})\right\}\le 0 \text{ (respectively, } \ge 0).$$

Moreover v is called a viscosity solution if v is both viscosity subsolution and supersolution.

It is easy to see that a classical solution is a viscosity solution. So the definition of viscosity solutions gives an extension of the one of classical solutions. Moreover, in fact we can prove the uniqueness of viscosity solutions of (2.7). Thus, if we find a classical or viscosity solution $v(t,w)$ of (2.7) with $v(0,w)=u(w)$, then it is equal to the value function $V_t(w;u)$. We can check easily that $\hat{V}_t(w;u)=\log w+\dfrac{\mu^2}{2\sigma^2}t$ is a classical solution of (2.7) when $u(w)=\log w$ and this gives the explicit form of the value function.

For more details of viscosity solution theory, see [5], [23] and [27].

3 Liquidity Problems and Market Impact

The Merton problem is based on the ideal market where there is no liquidity problem. So a trader can buy or sell a security at his/her will when the market is favorite. To follow the Merton's optimal strategy (that is to keep the portfolio ratio equals to $\frac{\mu}{\sigma^2}$,) he/she needs to trade continuously. But the continuous trading is unrealistic. In the real market any trading generates transaction costs. Moreover a trader may not complete the transaction satisfactorily (uncertainty of trading time.) Considering such realistic problems, the Merton's optimal strategy is no longer optimal.

Another important liquidity problem and our main interest of this paper is market impact (MI.) MI makes a problem when we consider especially the optimal execution problem rather than investment problem. If a trader face a situation where he/she must sell (execute) the shares of a security, then MI may bring the serious shrink of proceeds of execution. So it is important to consider the optimal execution problem with MI.

[2] and [4] studied the optimal execution problem in the discrete-time market model with MI, and computed the optimal execution strategy explicitly in the linear MI model. [12], [30], and [31] studied the optimization problem with MI in continuous-time model as the singular / impulse control. [8] studied such problems in the framework for mean-variance analyses. [29] treated the infinite time horizon case. Also studied was the optimal execution problem in limit order book market ([1]). [22] studied the optimal execution problem with temporary market impact and bid-ask spread cost as an impulse control problem and characterized the value function as a constrained viscosity solution for the corresponding quasi-variational inequality.

As mentioned above, there have been various approaches to consider an optimal execution problem and the effect of MI. The purpose of this paper is to construct the framework of such a problem as a standard model. To construct a model, discrete-time model is significant to describe the realistic phenomenon exactly, but sometimes it is hard to get the clean model by complex noises. On the other hand continuous-time model often makes problems clear, but the abrupt construction of continuous-time model may overlook the essence of the problems. So first we consider the discrete-time model of an optimal execution problem with MI and then derive the continuous-time model as their limit.

We consider the case when MI function is convex with respect to the execution volume of a trader, whereas some empirical studies tell us that MI function is concave ([3] etc.) But [7] pointed out that "much of this variation comes about because these studies actually measure different things. Some of them measure the market impact of a single trade made in an order book, some measure the aggregate impact of sequential trades in an order book, some of them measure block trades, and many of them measure a mixture of all three." They asserted that these are analyzed separately. The MI function which we define in the next section is the one of a single trade, not the aggregate one. Moreover it is important to consider the case when MI is convex, and it is interesting that we can observe the very effect of MI which affects the trader's execution policy.

4 The Model

In this section we present the details of the model. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a stochastic basis and let $(B_t)_{0\le t\le T}$ be a standard one-dimensional $(\mathcal{F}_t)_t$-Brownian motion. Here $T > 0$ means the time horizon. For brevity we assume $T = 1$.

Recall that we only consider the market which consists of one risk-free asset (namely cash) and one risky asset (namely a security.) The price of cash is always equal to 1, and the price of a security fluctuates according to a certain stochastic flow, and is influenced by sales of a trader.

First we consider the discrete-time model with time interval $1/n$. We consider a single trader who has an endowment $\Phi_0 > 0$ shares of a security. This trader executes the shares Φ_0 over a time interval $[0, 1]$, but his/her sales affect the prices of a security. We assume that the trader executes only at time $0, 1/n, \ldots, (n-1)/n$.

Now we describe the effect of the trader's execution. For $l = 0, \dots, n$, we denote by S^n_l the price of the security at time l/n and $X^n_l = \log S^n_l$. Let $s_0 > 0$ be an initial price (i.e. $S^n_0 = s_0$) and $X^n_0 = \log s_0$. If the trader sells the amount ψ^n_l at time l/n, the log-price changes to $X^n_l - g_n(\psi^n_l)$, where $g_n : [0, \infty) \longrightarrow [0, \infty)$ is a non-decreasing and continuously differentiable function which satisfies $g_n(0) = 0$, and he/she gets the amount of cash $\psi^n_l S^n_l \exp(-g_n(\psi^n_l))$ as proceeds of the execution.

After the trading at time l/n, X^n_{l+1} and S^n_{l+1} are given by

$$X^n_{l+1} = Y\Big(\frac{l+1}{n}; \frac{l}{n}, X^n_l - g_n(\psi^n_l)\Big), \quad S^n_{l+1} = \exp(X^n_{l+1}), \tag{4.1}$$

where $Y(t; r, x)$ is the solution of the following SDE

$$\begin{cases} dY(t; r, x) = \sigma(Y(t; r, x))dB_t + b(Y(t; r, x))dt, \quad t \geq r, \\ Y(r; r, x) = x \end{cases} \tag{4.2}$$

and $b, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$ are Borel functions. We assume that b and σ are bounded and Lipschitz continuous. Then for each $r \geq 0$ and $x \in \mathbb{R}$ there exists a unique solution of (4.2).

At the end of the time interval $[0, 1]$, The trader has the amount of cash W^n_n and the amount of the security φ^n_n, where

$$W^n_{l+1} = W^n_l + \psi^n_l S^n_l \exp(-g_n(\psi^n_l)), \quad \varphi^n_{l+1} = \varphi^n_l - \psi^n_l \tag{4.3}$$

for $l = 0, \dots, n-1$ and $W^n_0 = 0$, $\varphi^n_0 = \Phi_0$. We say that an execution strategy $(\psi^n_l)^{n-1}_{l=0}$ is admissible if $(\psi^n_l)_l \in \mathcal{A}^n_n(\Phi_0)$ holds, where $\mathcal{A}^n_k(\varphi)$ is the set of strategies $(\psi^n_l)^{k-1}_{l=0}$ such that ψ^n_l is $\mathcal{F}_{l/n}$-measurable, $\psi^n_l \geq 0$ for each $l = 0, \dots, k-1$, and $\sum_{l=0}^{k-1} \psi^n_l \leq \varphi$.

A trader whose execution strategy is in $\mathcal{A}^n_n(\Phi_0)$ is permitted to leave the unsold shares of the security, and there will be no penalty if he/she cannot finish the liquidation until the time horizon. In Section 6, we consider the case when a trader must finish the liquidation.

The investor's problem is to choose an admissible strategy to maximize the expected utility $\mathrm{E}[u(W^n_n, \varphi^n_n, S^n_n)]$, where $u \in \mathcal{C}$ is his/her utility function and $\mathcal{C}$ is the set of non-decreasing, non-negative and continuous functions on $D = \mathbb{R} \times [0, \Phi_0] \times [0, \infty)$ such that

$$u(w, \varphi, s) \leq C_u(1 + w^{m_u} + s^{m_u}), \quad (w, \varphi, s) \in D \tag{4.4}$$

for some constants $C_u > 0$ and $m_u \in \mathbb{N}$.

For $k = 1, \dots, n$, $(w, \varphi, s) \in D$ and $u \in \mathcal{C}$, we define the (discrete-time) value function $V^n_k(w, \varphi, s; u)$ by

$$V^n_k(w, \varphi, s; u) = \sup_{(\psi^n_l)^{k-1}_{l=0} \in \mathcal{A}^n_k(\varphi)} \mathrm{E}[u(W^n_k, \varphi^n_k, S^n_k)]$$

subject to (4.1) and (4.3) for $l = 0, \dots, k-1$ and $(W^n_0, \varphi^n_0, S^n_0) = (w, \varphi, s)$. (For $s = 0$, we set $S^n_l \equiv 0$.) For $k = 0$, we denote $V^n_0(w, \varphi, s; u) = u(w, \varphi, s)$. Then our problem is the same as $V^n_n(0, \Phi_0, s_0; u)$. We consider the limit of the value function $V^n_k(w, \varphi, s; u)$ as $n \to \infty$.

Let $h : [0, \infty) \longrightarrow [0, \infty)$ be a non-decreasing continuous function. We introduce the following condition.

$[A]$ $\lim_{n \to \infty} \sup_{\psi \in [0,\Phi_0]} \left| \frac{d}{d\psi} g_n(\psi) - h(n\psi) \right| = 0.$

Let $g(\zeta) = \int_0^\zeta h(\zeta')d\zeta'$ for $\zeta \in [0, \infty)$. Under the condition $[A]$, we see that $\varepsilon_n \longrightarrow 0$, where

$$\varepsilon_n = \sup_{\psi \in (0,\Phi_0]} \left| \frac{g_n(\psi)}{\psi} - \frac{g(n\psi)}{n\psi} \right|. \tag{4.5}$$

Now we define the function which gives the limit of the discrete-time value functions. For $t \in [0, 1]$ and $\varphi \in [0, \Phi_0]$ we denote by $\mathcal{A}_t(\varphi)$ the set of $(\mathcal{F}_r)_{0 \le r \le t}$-progressively measurable process $(\zeta_r)_{0 \le r \le t}$ such that $\zeta_r \ge 0$ for each $r \in [0, t]$, $\int_0^t \zeta_r dr \le \varphi$ almost surely and $\sup_{r,\omega} \zeta_r(\omega) < \infty$. For $t \in [0, 1]$, $(w, \varphi, s) \in D$ and $u \in \mathcal{C}$, we define $V_t(w, \varphi, s; u)$ by

$$V_t(w, \varphi, s; u) = \sup_{(\zeta_r)_r \in \mathcal{A}_t(\varphi)} \mathrm{E}[u(W_t, \varphi_t, S_t)] \tag{4.6}$$

subject to

$$dW_r = \zeta_r S_r dr, \quad d\varphi_r = -\zeta_r dr, \quad dS_r = \hat{\sigma}(S_r)dB_r + \hat{b}(S_r)dr - g(\zeta_r)S_r dr \tag{4.7}$$

and $(W_0, \varphi_0, S_0) = (w, \varphi, s)$, where $\hat{\sigma}(s) = s\sigma(\log s), \hat{b}(s) = s\left\{b(\log s) + \frac{1}{2}\sigma(\log s)^2\right\}$ for $s > 0$ and $\hat{\sigma}(0) = \hat{b}(0) = 0$. When $s > 0$, we obviously see that the process of the log-price of the security $X_r = \log S_r$ satisfies

$$dX_r = \sigma(X_r)dB_r + b(X_r)dr - g(\zeta_r)dr.$$

We remark that $V_0(w, \varphi, s; u) = u(w, \varphi, s)$. We notice that $V_t(w, \varphi, s; u) < \infty$ for each $t \in [0, 1]$ and $(w, \varphi, s) \in D$.

5 Main Results

In this section we present main results of this paper. First we give the convergence theorem for value functions.

Theorem 2. *For each $(w, \varphi, s) \in D$, $t \in [0, 1]$ and $u \in \mathcal{C}$ it holds that*

$$\lim_{n \to \infty} V^n_{[nt]}(w, \varphi, s; u) = V_t(w, \varphi, s; u), \tag{5.1}$$

where $[nt]$ is the greatest integer less than or equal to nt.

Theorem 2 implies that an optimal execution problem in continuous-time model is derived as the limit of the ones in discrete-time model. We call $V_t(w, \varphi, s; u)$ a continuous-time value function. We regard a stochastic process $(\zeta_r)_r$ as a trader's execution strategy. The value of ζ_r means instantaneous sales (in other words, execution speed) at time r.

As for the continuity of $V_t(w, \varphi, s; u)$, we have the following theorem.

Theorem 3. *Let $u \in \mathcal{C}$.*
(i) *If $h(\infty) = \infty$, then $V_t(w, \varphi, s; u)$ is continuous in $(t, w, \varphi, s) \in [0, 1] \times D$.*
(ii) *If $h(\infty) < \infty$, then $V_t(w, \varphi, s; u)$ is continuous in $(t, w, \varphi, s) \in (0, 1] \times D$ and $V_t(w, \varphi, s; u)$ converges to $Ju(w, \varphi, s)$ uniformly on any compact subset of D as $t \downarrow 0$, where*

$$Ju(w, \varphi, s) = \begin{cases} \sup\limits_{\psi \in [0,\varphi]} u\Big(w + \dfrac{1 - e^{-h(\infty)\psi}}{h(\infty)} s, \varphi - \psi, se^{-h(\infty)\psi}\Big) & (h(\infty) > 0) \\ \sup\limits_{\psi \in [0,\varphi]} u(w + \psi s, \varphi - \psi, s) & (h(\infty) = 0). \end{cases}$$

As you can see, the continuity in t at the origin is according to the state of the function h at the infinity point. When $h(\infty) < \infty$, the value function is not always continuous at $t = 0$ and has the right limit $Ju(w, \varphi, s)$. $Ju(w, \varphi, s)$ implies the utility of the profit of the execution of a trader who sells a part of the shares of a security ψ by dividing infinitely in infinitely short time (enough to neglect the fluctuation of the price of a security) and makes the amount $\varphi - \psi$ remain.

Next we study the semi-group property (Bellman principle) of the family of non-linear operators corresponding to the continuous-time value function. We define an operator Q_t : $\mathcal{C} \longrightarrow \mathcal{C}$ by $Q_t u(w, \varphi, s) = V_t(w, \varphi, s; u)$. We easily see that Q_t is well-defined. Then we have the following.

Theorem 4. *For each $r, t \in [0, 1]$ with $t + r \leq 1$, $(w, \varphi, s) \in D$ and $u \in \mathcal{C}$ it holds that $Q_{t+r} u(w, \varphi, s) = Q_t Q_r u(w, \varphi, s)$.*

By using Theorem 4, we can characterize the continuous-time value function as a viscosity solution of the corresponding HJB. We define a function $F : \mathscr{S} \longrightarrow [-\infty, \infty)$ by

$$F(z, p, X) = -\sup_{\zeta \geq 0} \left\{ \frac{1}{2} \hat{\sigma}(z_s)^2 X_{ss} + \hat{b}(z_s) p_s + \zeta \left(z_s p_w - p_\varphi \right) - g(\zeta) z_s p_s \right\},$$

where $\mathscr{S} = U \times \mathbb{R}^3 \times S^3$, $U = D \setminus \partial D$, S^3 is the space of symmetric matrices in $\mathbb{R}^3 \otimes \mathbb{R}^3$ and

$$z = \begin{pmatrix} z_w \\ z_\varphi \\ z_s \end{pmatrix} \in D,\ p = \begin{pmatrix} p_w \\ p_\varphi \\ p_s \end{pmatrix} \in \mathbb{R}^3,\ X = \begin{pmatrix} X_{ww} & X_{w\varphi} & X_{ws} \\ X_{\varphi w} & X_{\varphi\varphi} & X_{\varphi s} \\ X_{sw} & X_{s\varphi} & X_{ss} \end{pmatrix} \in S^3.$$

Although the function F may take $-\infty$, we can define a viscosity solution of the following HJB as usual :

$$\frac{\partial}{\partial t} v + F(z, \mathcal{D}v, \mathcal{D}^2 v) = 0 \ \text{ on } (0, 1] \times U, \tag{5.2}$$

where $\mathcal{D}$ denotes the differential operator with respect to $z = (w, \varphi, s)$. We introduce the definitions of viscosity solutions of the HJB above.

Definition 2. (i) *A continuous function $v : (0, 1] \times U \longrightarrow \mathbb{R}$ is a viscosity subsolution of (5.2) if there is a smooth function $\hat{v} \in C^{1,2}((0, 1] \times U)$ such that $v - \hat{v}$ has a local maximum at $(\bar{t}, \bar{z})$, then it holds either $F(\hat{v}(\bar{t}, \bar{z}), \mathcal{D}\hat{v}(\bar{t}, \bar{z}), \mathcal{D}^2 \hat{v}(\bar{t}, \bar{z})) = -\infty$ or*

$$\frac{\partial}{\partial t} \hat{v}(\bar{t}, \bar{z}) + F(\hat{v}(\bar{t}, \bar{z}), \mathcal{D}\hat{v}(\bar{t}, \bar{z}), \mathcal{D}^2 \hat{v}(\bar{t}, \bar{z})) \leq 0.$$

(ii) *A continuous function* $v : (0,1] \times U \longrightarrow \mathbb{R}$ *is a viscosity supersolution of* (5.2) *if there is a smooth function* $\hat{v} \in C^{1,2}((0,1] \times U)$ *such that* $v - \hat{v}$ *has a local minimum at* $(\bar{t}, \bar{z})$, *then it holds both*
$F(\hat{v}(\bar{t},\bar{z}), \mathcal{D}\hat{v}(\bar{t},\bar{z}), \mathcal{D}^2\hat{v}(\bar{t},\bar{z})) > -\infty$ *and*

$$\frac{\partial}{\partial t}\hat{v}(\bar{t},\bar{z}) + F(\hat{v}(\bar{t},\bar{z}), \mathcal{D}\hat{v}(\bar{t},\bar{z}), \mathcal{D}^2\hat{v}(\bar{t},\bar{z})) \geq 0.$$

(iii) v *is called a viscosity solution of* (5.2) *if* v *is both viscosity subsolution and supersolution.*

Here we remark that (5.2) is rewritten as

$$\frac{\partial}{\partial t}v(t,w,\varphi,s) - \sup_{\zeta \geq 0} \mathscr{L}^{\zeta} v(t,w,\varphi,s) = 0, \quad (t,w,\varphi,s) \in (0,1] \times U, \tag{5.3}$$

where

$$\begin{aligned} \mathscr{L}^{\zeta} v(t,w,\varphi,s) &= \frac{1}{2}\hat{\sigma}(s)^2 \frac{\partial^2}{\partial s^2} v(t,w,\varphi,s) + \hat{b}(s)\frac{\partial}{\partial s}v(t,w,\varphi,s) \\ &\quad + \zeta\left(s\frac{\partial}{\partial w}v(t,w,\varphi,s) - \frac{\partial}{\partial \varphi}v(t,w,\varphi,s)\right) - g(\zeta)s\frac{\partial}{\partial s}v(t,w,\varphi,s). \end{aligned} \tag{5.4}$$

Now we introduce the following theorem which we will give the proof in Section 10.

Theorem 5. *Assume* h *is strictly increasing and* $h(\infty) = \infty$. *Moreover we assume*

$$\liminf_{\varepsilon \downarrow 0} \frac{V_t(w,\varphi,s+\varepsilon;u) - V_t(w,\varphi,s;u)}{\varepsilon} > 0 \tag{5.5}$$

for any $t \in (0,1]$ *and* $(w,\varphi,s) \in U$. *Then* $V_t(w,\varphi,s;u)$ *is a viscosity solution of* (5.2).

Finally we give the uniqueness result of viscosity solutions of (5.3).

Theorem 6. *Assume that* $\hat{\sigma}$ *and* $\hat{b}$ *are both Lipschitz continuous. Moreover we assume the conditions in Theorem 5 and the growth condition* $\liminf_{\zeta \to \infty} \dfrac{h(\zeta)}{\zeta} = 0$. *If the polynomial growth function* $v : [0,1] \times D \longrightarrow \mathbb{R}$ *is a viscosity solution of* (5.3) *and satisfies the following boundary conditions*

$$\begin{aligned} v(0,w,\varphi,s) &= u(w,\varphi,s), \quad (w,\varphi,s) \in D, \\ v(t,w,0,s) &= \mathrm{E}\left[u\left(w,0,Z\left(t;0,s\right)\right)\right], \quad (t,w,s) \in [0,1] \times \mathbb{R} \times [0,\infty), \\ v(t,w,\varphi,0) &= u(w,\varphi,0), \quad (t,w,\varphi) \in [0,1] \times \mathbb{R} \times [0,\Phi_0], \end{aligned} \tag{5.6}$$

then it holds that $V_t(w,\varphi,s;u) = v(t,w,\varphi,s)$, *where*

$$Z\left(t;r,s\right) = \exp\left(Y\left(t;r,\log s\right)\right) \ (s > 0), \quad 0 \ (s = 0). \tag{5.7}$$

The proof is also in Section 10. In Section 8.2, we will present an example where assumptions in Theorem 5 and Theorem 6 are fulfilled.

6 Sell-Out Condition

In this section we consider the optimal execution problem under the "sell-out condition." A trader has a certain shares of a security at the initial time, and he/she must liquidate all of them until the time horizon. Then the spaces of admissible strategies are reduced to the following :

$$\mathcal{A}_k^{n,\mathrm{SO}}(\varphi) = \left\{ (\psi_l^n)_l \in \mathcal{A}_k^n(\varphi) \ ; \ \sum_{l=0}^{k-1} \psi_l^n = \varphi \right\}, \quad \mathcal{A}_t^{\mathrm{SO}}(\varphi) = \left\{ (\zeta_r)_r \in \mathcal{A}_t(\varphi) \ ; \ \int_0^t \zeta_r dr = \varphi \right\}.$$

Now we define value functions with the sell-out condition by

$$V_k^{n,\mathrm{SO}}(w,\varphi,s;U) = \sup_{(\psi_l^n)_l \in \mathcal{A}_k^{n,\mathrm{SO}}(\varphi)} \mathrm{E}[U(W_k^n)], \quad V_t^{\mathrm{SO}}(w,\varphi,s;U) = \sup_{(\zeta_r)_r \in \mathcal{A}_t^{\mathrm{SO}}(\varphi)} \mathrm{E}[U(W_t)]$$

for a continuous, non-decreasing and polynomial growth function $U : \mathbb{R} \longrightarrow \mathbb{R}$. Then we have the following theorem.

Theorem 7. *It holds that $V_t^{\mathrm{SO}}(w,\varphi,s;U) = V_t(w,\varphi,s;u)$, where $u(w,\varphi,s) = U(w)$.*

By Theorem 7, we see that the sell-out condition $\int_0^t \zeta_r dr = \varphi$ makes no change for the (value of) value function in continuous-time model when a trader wants to maximize the expected utility of only the terminal cash holdings (that is the proceeds of his/her execution.) Thus, although the value function in the discrete-time model may vary whether the sell-out condition exists or not, in the continuous-time model we may not worry about such a condition when we treat the utility functions which are independent of φ and s. Moreover we obtain the following theorem which is the similar result of Theorem 2.

Theorem 8. *For each $(w,\varphi,s) \in D$ it holds that*

$$\lim_{n\to\infty} V_{[nt]}^{n,\mathrm{SO}}(w,\varphi,s;U) = V_t^{\mathrm{SO}}(w,\varphi,s;U) \quad (= V_t(w,\varphi,s;U).)$$

7 Random MI Model

Until now we consider MI functions as deterministic functions. This means that we can get the information of MI beforehand. But in the real market it is hard to estimate the effect of MI. Moreover it often happens that the concentration of unexpected orders will causes the overfluctuation of the price. In this section we consider an optimal execution problem with random MI. We also consider the discrete-time model first and derive the continuous-time model by taking the limit. We set the random MI function as

$$g_k^n(\psi,\omega) = c_k^n(\omega) g_n(\psi), \quad \psi \in [0,\Phi_0], \ \omega \in \Omega,$$

where $g_n(\psi)$ is the same as in Section 4 and c_k^n, $k = 0,1,2,\ldots$, are i.i.d. positive random variables. The discrete-time value function $\hat{V}_k^n(w,\varphi,s;u)$ is defined by the same way as in Section 4 by replacing $g_n(\psi)$ of (4.1) and (4.3) with $g_k^n(\psi,\omega)$.

For deterministic part of MI $g_n(\psi)$, we also assume the condition [A]. Moreover, for noise part of MI $c_k^n(\omega)$, we assume the following conditions.

$[B1]$ For any $n \in \mathbb{N}$ and $x \geq 0$ it holds that $\gamma_n > 0$ and

$$\frac{h(x/\gamma_n)}{n} \longrightarrow 0, \quad n \to \infty, \tag{7.1}$$

where $\gamma_n = \text{essinf} c_k^n$.

$[B2]$ Let μ_n is the distribution of $\frac{c_0^n + \ldots + c_{n-1}^n}{n}$. Then μ_n has a weak limit μ as $n \to \infty$.

$[B3]$ There is a sequence of infinitely divisible distributions $(p_n)_n$ on $\mathbb{R}$ such that $\mu_n = \mu * p_n$ and / either

[B3-a] $n \int_{\mathbb{R}} x^2 p_n(dx) \longrightarrow 0, \quad n \to \infty$

or

[B3-b] There is a sequence $(K_n)_n \subset (0, \infty)$ such that $\lim_{n\to\infty} K_n = 0$, $p_n((-\infty, -K_n)) = 0$ (or $p_n((K_n, \infty)) = 0$) and $\int_{\mathbb{R}} x p_n(dx) \longrightarrow 0, \quad n \to \infty.$

Let us give some remarks for condition $[B1]$. Since c_k^n, $k = 0, 1, 2, \ldots$, are identically distributed, γ_n is independent of k. Moreover, if $h(\infty) < \infty$, then (7.1) is always fulfilled. if $h(\infty) = \infty$, we have the following example :

$$h(\zeta) = \alpha \zeta^p, \quad \gamma_n = \frac{1}{n^{1/p - \delta}} \quad (p, \delta > 0, \ \delta < 1/p).$$

Since μ is an infinitely divisible distribution, there is some Lévy process $(L_t)_{0 \leq t \leq 1}$ on a certain probability space such that L_1 is distributed by μ. The process $(L_t)_t$ is independent of any Brownian motion, because $(L_t)_t$ is obviously non-decreasing and non-negative. Without loss of generality, we may assume that $(L_t)_t$ and $(B_t)_t$ are defined on the same filtered space. Let ν is a Lévy measure of $(L_t)_t$. We assume the following moment condition for ν.

$[C]$ $\int_{(0,\infty)} (z + z^2) \nu(dz) < \infty.$

Now we present the function which corresponds to the limit of discrete-time value functions. Let $\hat{\mathcal{A}}_t(\varphi)$ be the set of $(\zeta_r)_r \in \mathcal{A}_t(\varphi)$ such that $(\zeta_r)_r$ is $(\mathcal{F}_r)_r$-adapted and càglàd (i.e. right continuous and has a left limit at each point.) For $t \in [0, 1], (w, \varphi, s) \in D$ and $u \in \mathcal{C}$, we define $\hat{V}_t(w, \varphi, s; u)$ by

$$\hat{V}_t(w, \varphi, s; u) = \sup_{(\zeta_r)_r \in \hat{\mathcal{A}}_t(\varphi)} \mathrm{E}[u(W_t, \varphi_t, S_t)]$$

subject to

$$dW_r = \zeta_r S_r dr, \quad d\varphi_r = -\zeta_r dr, \quad dX_r = \sigma(X_r) dB_r + b(X_r) dr - g(\zeta_r) dL_r, \quad S_r = \exp(X_r) \tag{7.2}$$

and $(W_0, \varphi_0, S_0) = (w, \varphi, s)$. Then we have the following.

Theorem 9. *For each $(w, \varphi, s) \in D$, $t \in [0, 1]$ and $u \in \mathcal{C}$ it holds that*

$$\lim_{n\to\infty} \hat{V}_{[nt]}^n(w, \varphi, s; u) = \hat{V}_t(w, \varphi, s; u).$$

By this theorem, we see that the function $\hat{V}_t(w, \varphi, s; u)$ corresponds to the continuous-time model of an optimal execution problem with random MI. The term $g(\zeta_r)dL_r$ represents MI with noise. Let $L_t = \gamma t + \int_{(0,\infty)} zN(dr, dz)$ be the Lévy decomposition of $(L_t)_t$. Then $g(\zeta_r)dL_r$ can be divided into the following two terms :

$$g(\zeta_{r-})dL_r = \gamma g(\zeta_r)dr + g(\zeta_{r-}) \int_{(0,\infty)} zN(dr, dz).$$

The last term in the right-hand side refers to the effect of noise of MI. This means that noise of MI appears as a jump of Lévy process.

We also have the continuity and semi-group property of continuous-time value function as the same form in Section 5. As for the characterization as the viscosity solution of HJB, we may also have the similar results as in Section 5, and this is a further development.

8 Examples

In this section we consider two examples of our model. Let $b(x) \equiv -\mu$ and $\sigma(x) \equiv \sigma$ for some constants $\mu, \sigma \geq 0$ and suppose $\tilde{\mu} = \mu - \sigma^2/2 > 0$. We assume that a trader has a risk-neutral utility function $u(w, \varphi, s) = w$. We remark that we can replace the stochastic control problem $V_t(w, \varphi, s; u)$ with the deterministic control problem $f(t, \varphi)$, where

$$\begin{aligned} f(t, \varphi) &= \sup_{(\zeta_r)_r \in \mathcal{A}_t^{\mathrm{det}}(\varphi)} \int_0^t \zeta_r \exp\left(-\tilde{\mu} r - \int_0^r g(\zeta_v) dv\right) dr, \\ \mathcal{A}_t^{\mathrm{det}}(\varphi) &= \{(\zeta_r)_r \in \mathcal{A}_t(\varphi) \ ; \ (\zeta_r)_r \text{ is deterministic}\}. \end{aligned}$$

Indeed we have the following.

Proposition 1. *It holds that $V_t(w, \varphi, s; u) = w + sf(t, \varphi)$.*

By Proposition 1, we see that $\dfrac{\partial}{\partial s} V_t(w, \varphi, s) = f(t, \varphi) > 0$ for $t, \varphi > 0$.

8.1 Log-Linear Impact

Let $(\alpha_n)_{n \in \mathbb{N}} \subset (0, \infty)$ be a sequence which has a limit $\alpha \in (0, \infty)$ as $n \to \infty$ and let $g_n(\psi) = \alpha_n \psi$. Then the condition $[A]$ is satisfied with $h(\zeta) \equiv \alpha$ (and thus $g(\zeta) = \alpha\zeta$.) We have the following.

Theorem 10. *It holds that*

$$V_t(w, \varphi, s; u) = w + \frac{1 - e^{-\alpha\varphi}}{\alpha} s \tag{8.1}$$

for each $t \in (0, 1]$ and $(w, \varphi, s) \in D$.

We notice that the right-hand side of (8.1) is equal to $Ju(w, \varphi, s)$ and converges to $w + \varphi s$ as $\alpha \downarrow 0$, which is the profit gained by choosing the execution strategy of so-called block liquidation such that a trader sells all shares φ at $t = 0$ when there is no market impact.

Theorem 10 implies that the optimal strategy in this case is to execute all shares dividing infinitely in infinitely short time at $t = 0$. This is almost the same as a block liquidation at the initial time, and a trader does not delay the execution time (although MI lowers the profit of the execution.) Therefore we cannot see the essential influence of the MI in this example.

8.2 Log-Quadratic Impact

In this subsection we consider the case of strictly convex MI function. Let $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence and $g_n(\psi) = \alpha_n \psi^2$. We suppose $\lim_{n\to\infty} |\alpha_n - n\alpha| = 0$ for some $\alpha \in (0,\infty)$. Then the condition $[A]$ is satisfied with $h(\zeta) = 2\alpha\zeta$ and $g(\zeta) = \alpha\zeta^2$. We remark that the continuous-time value function in this example is the unique viscosity solution of (5.2) with boundary conditions (5.6).

Now we extend the set of admissible strategies such that

$$\begin{aligned}\tilde{\mathcal{A}}_t(\varphi) &= \Big\{(\zeta_r)_{0\le r\le t}\ ;\ (\mathcal{F}_r)_r\text{-adapted},\ \zeta_r \ge 0,\ \int_0^t \zeta_r dr \le \varphi \\ &\qquad \text{and} \sup_{(r,\omega)\in[0,t-\varepsilon]\times\Omega} \zeta_r(\omega) < \infty \text{ for all } \varepsilon \in (0,t)\Big\}.\end{aligned}$$

We easily see that the value of $V_t(w,\varphi,s;u)$ does not change by replacing $\mathcal{A}_t(\varphi)$ with $\tilde{\mathcal{A}}_t(\varphi)$.

We define functions $\hat{v}^i(t,w,\varphi,s)$ and $\hat{\zeta}^i_t$, $i = 1,2$, by

$$\hat{v}^1(t,w,\varphi,s) = w + \frac{s\sqrt{1-e^{-2\tilde{\mu}t}}}{2\sqrt{\alpha\tilde{\mu}}},\quad \hat{\zeta}^1_t = \sqrt{\frac{\tilde{\mu}}{\alpha(1-e^{-2\tilde{\mu}(t-r)})}}$$

and

$$\hat{v}^2(t,w,\varphi,s) = w + \frac{s}{2\sqrt{\alpha\tilde{\mu}}}(1-e^{-2\sqrt{\alpha\tilde{\mu}}\varphi}),\quad \hat{\zeta}^2_t = \sqrt{\frac{\tilde{\mu}}{\alpha}}1_{[0,\varphi\sqrt{\alpha/\tilde{\mu}}]}(r).$$

Then we have the following.

Theorem 11.
(i) *If* $\dfrac{\mathrm{arctanh}\sqrt{1-e^{-2\tilde{\mu}t}}}{\sqrt{\alpha\tilde{\mu}}} \le \varphi$, *then* $V_t(w,\varphi,s;u) = \hat{v}^1(t,w,\varphi,s)$ *and the optimal strategy is given by* $(\hat{\zeta}^1_r)_r$.
(ii) *If* $\varphi \le \sqrt{\dfrac{\tilde{\mu}}{\alpha}}t$, *then* $V_t(w,\varphi,s;u) = \hat{v}^2(t,w,\varphi,s)$ *and the optimal strategy is given by* $(\hat{\zeta}^2_r)_r$.

This theorem implies that the form of optimal strategies and value functions vary according to the amount of the security holdings φ. If a trader has a little amount of securities, then we have the case (ii) and the optimal strategy is to sell up the entire shares of the security until the time $\sqrt{\dfrac{\tilde{\mu}}{\alpha}}t$. If he/she has so large amount, then we have the case (i) and a trader cannot finish the selling.

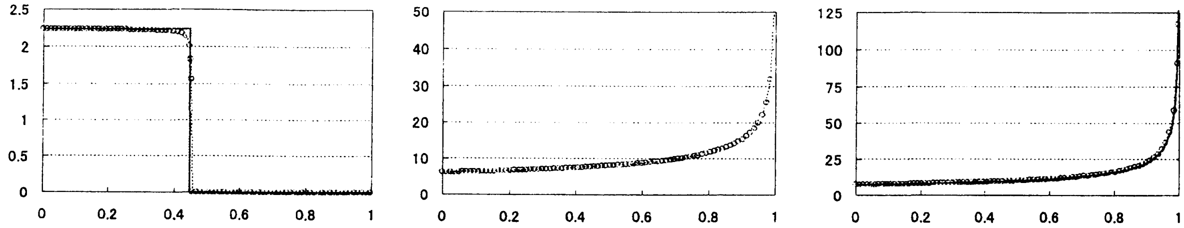

Figure 1: The forms of optimal execution strategies $(\zeta_r)_r$. Circle-marked dotted line means the numerical solution, and solid line means the analytical solution. Horizontal axis is time r. The left graph : $\varphi = 1$. The middle graph : $\varphi = 10$. The right graph : $\varphi = 100$.

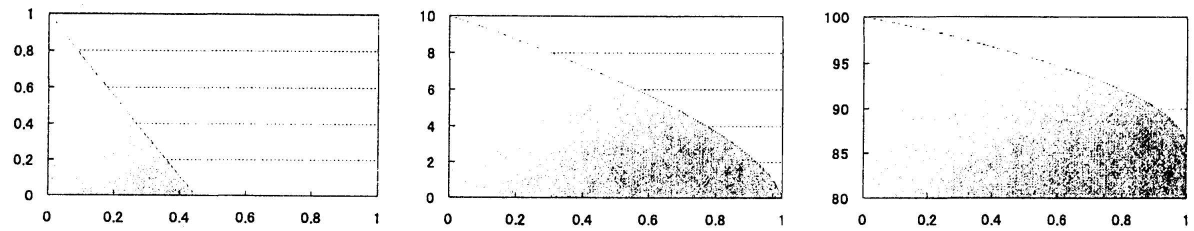

Figure 2: The forms of the amount of security holdings $(\varphi_r)_r$ corresponding with optimal strategies. Horizontal axis is time r. The left graph : $\varphi = 1$. The middle graph : $\varphi = 10$. The right graph : $\varphi = 100$.

We have not had the explicit form of $V_t(w, \varphi, s; u)$ on a whole space. So we try to solve this example numerically. $V_1(w, \varphi, s; u)$ is approximated by $V_n^n(w, \varphi, s; u)$ for enough large n, and we can assume that the optimal strategy is deterministic. We can get the value of $V_n^n(w, \varphi, s; u)$ numerically by the computer when n is not so large. Figure 1 describes the form of execution strategies and Figure 2 describes the form of corresponding processes of the amount of a security when we set $n = 500$, $w = 0, s = 0, \alpha = 0.01, \tilde{\mu} = 0.05, \sigma = 0$ and $\varphi = 1, 10$ and 100. We also get the form of the function $f(t, \varphi)$ of Proposition 1 numerically, which is described in Figure 3. If a pair (t, φ) is in the range (a) of Figure 4, then we have $f(t,\varphi) = \dfrac{\sqrt{1 - e^{-2\tilde{\mu}t}}}{2\sqrt{\alpha\tilde{\mu}}}$, and if (t, φ) is in the range (c), we have $f(t, \varphi) = \dfrac{1}{2\sqrt{\alpha\tilde{\mu}}}(1 - e^{-2\sqrt{\alpha\tilde{\mu}}\varphi})$. We have not had the form of $f(t, \varphi)$ analytically when (t, φ) is in the range (b).

9 Concluding Remarks

In this paper we study the optimal execution problem in consideration of MI. First we formulate the discrete-time model and then take the limit. We show that the discrete-time value functions converge to the continuous-time value function.

We mainly treat the case when MI function is convex. This is not only from the mathematical reason, but also from the financial viewpoint. In a Black-Scholes type market, an optimal execution strategy of a risk-neutral trader is a block liquidation when there is no MI. The form of the optimal strategy entirely changes when MI is quadratic. When MI is

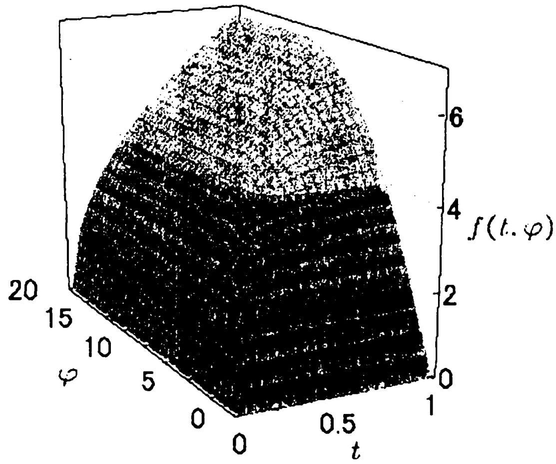

Figure 3: The form of the function $f(t, \varphi)$.

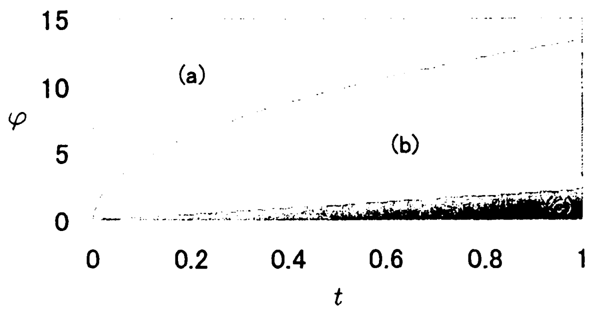

Figure 4: The region of pairs (t, φ).

not convex, especially linear, then a trader's optimal strategy is almost block liquidation.

As for types of MI, it is often said that MI can be divided into two parts : permanent impact and temporary impact (see [2] and [13].) As time passes, the temporary impact disappears and the price once pushed down transitorily is recovered. Our examples treat permanent impact only, but we can also consider temporary impact and price recovery effects. If the process of security prices follows some mean-reverting process, like Ornstein-Uhlenbeck (OU) process, then we may deal with the optimization problem with MI and price recovery. Now we are going to study such a problem and may have a quite similar result as in [1], in spite of the fact that there is a difference of whether the fluctuation of a security price is described as an arithmetic OU process or a geometric OU process.

It is also meaningful to characterize the continuous-time value function as the solution of corresponding HJB. We have shown that the value function is a viscosity solution under some strong assumptions. Such assumptions are not necessary when we consider only bounded strategies. But since the control region of our model is unbounded, we should argue deliberately about whether $F > -\infty$ or not.

In trading operations, a trader should execute while considering the fluctuation of the price of other assets (e.g. the rebalance of an index fund.) [14] studied the multi-dimensional version of this model to consider such a case. But in the case of rebalancing, it is necessary to consider not only selling but also buying the securities. We should formulate such a model of an optimal execution problem carefully, avoiding the opportunity of free-lunch when MI is large.

The complete solution of our example in Section 8.2 is another remaining task. This is a representative example where an trading policy is influenced vastly by MI, and will be pleasant to solve completely in future researches.

10 Appendix

10.1 Proof of Theorem 5

In Section 10.1 and Section 10.2 we always assume that h is strictly increasing and $h(\infty) = \infty$.

Let $L > 0$. We define the function $V_t^L(w, \varphi, s; u)$ as (4.6) replacing $\mathcal{A}_t(\varphi)$ with $\mathcal{A}_t^L(\varphi)$, where $\mathcal{A}_t^L(\varphi) = \{(\zeta_r)_r \in \mathcal{A}_t(\varphi) \ ; \ \zeta_r \leq L \text{ for any } r\}$. First we consider the characterization of $V_t^L(w, \varphi, s; u)$ as the viscosity solution of the corresponding HJB. We define a function $F^L : \mathscr{S} \longrightarrow \mathbb{R}$ by

$$F^L(z, p, X) = - \sup_{0 \leq \zeta \leq L} \left\{ \frac{1}{2}\hat{\sigma}(z_s)^2 X_{ss} + \hat{b}(z_s)p_s + \zeta\left(z_s p_w - p_\varphi\right) - g(\zeta) z_s p_s \right\}.$$

Then we have the following.

Proposition 2. *Assume $h(\infty) = \infty$. Then, for each $u \in \mathcal{C}$, the function $V_t^L(w, \varphi, s; u)$ is a viscosity solution of*

$$\frac{\partial}{\partial t} v + F^L(z, \mathcal{D}v, \mathcal{D}^2 v, v) = 0 \quad \text{on } (0, 1] \times U. \tag{10.1}$$

Since the control region $[0, L]$ is compact, we obtain Proposition 2 by the semi-group property of $(Q_t^L)_t$, which is defined by $V_t^L(w, \varphi, s; u)$, and the similar arguments in the proof of Theorem 5.4.1 in [27] .

Next we treat HJB (5.2). Let $\mathscr{U} = \{(z, p, X) \in \mathscr{S} \ ; \ F(z, p, X) > -\infty\}$. A direct calculation gives the following.

Proposition 3. *For $(z, p, X) \in \mathscr{U}$, it holds that*

$$F(z, p, X) = -\frac{1}{2}\hat{\sigma}(z_s)^2 X_{ss} - \hat{b}(z_s)p_s - \max\left\{\zeta^*(z, p)\left(z_s p_w - p_\varphi\right) - g(\zeta^*(z, p)) z_s p_s, 0\right\},$$

where $\zeta^*(z, p) = h^{-1}\left(\dfrac{z_s p_\varphi - p_w}{z_s p_s} \vee h(0)\right) 1_{\{p_s > 0\}}$. *In particular F is continuous on $\mathscr{U}$.*

Now we prove Theorem 5. We define an open set $\mathscr{R} = U \times \mathbb{R}^2 \times (0, \infty) \times S^3 \subset \mathscr{U}$. Since F is continuous on $\mathscr{R}$ and F^L converges to F monotonously, we see that this convergence is uniform on any compact set in $\mathscr{R}$ by Dini's theorem. Similarly, using Dini's theorem again, we see that V^L converges to V uniformly on any compact set in $(0, 1] \times U$. Then, by the similar arguments in the proof of Proposition 4.8 in [23], we obtain the assertion.

10.2 Proof of Theorem 6

Let $\tilde{U} \subset U$ be open and bounded. Let $\mathcal{P}^{2,\pm}_{(0,1]\times\tilde{U}}$ be parabolic variants of semijets and $\overline{\mathcal{P}}^{2,\pm}_{(0,1]\times\tilde{U}}$ be their closures (see [5].) By Proposition 3, we easily show the following lemma.

Lemma 1. *Suppose v is a subsolution (resp., supersolution) of (5.2). Then it holds that*

$$a + F(z, p, X) \leq 0 \ (\textit{resp.}, \ \geq 0)$$

for any $(a, z, p, X) \in (0, 1] \times \tilde{U} \times \mathbb{R}^3 \times S^3$ with $(a, p, X) \in \overline{\mathcal{P}}^{2,+}_{(0,1]\times\tilde{U}} v(z)$ (resp., $(a, p, X) \in \overline{\mathcal{P}}^{2,-}_{(0,1]\times\tilde{U}} v(z)$.)

Now we consider the comparison principle on a bounded domain.

Proposition 4. *Suppose v (resp., v') be a subsolution (resp., supersolution) of (5.2) on $(0,1]\times\tilde{U}$. Moreover suppose $v(0,z)\le 0\le v'(0,z)$ for $z\in\tilde{U}$ and $v\le v'$ on $(0,1]\times\partial\tilde{U}$. Then it holds that $v\le v'$ on $(0,1]\times\tilde{U}$.*

By Lemma 1 and the similar arguments in the proof of Theorem 8.12 in [5], we see that for proving Proposition 4 it suffices to show the following Proposition 5.

Proposition 5. *The function F satisfies the following structure condition*

$$F(z',\alpha(z-z'),Y)-F(z,\alpha(z-z'),X)\le\rho\left(\alpha|z-z'|^2+|z-z'|\right).$$

for $\alpha>1$, $\rho\in C([0,\infty);[0,\infty))$ with $\rho(0)=0$, $z,z'\in U$, $X,Y\in S^3$ with $F(z',\alpha(z-z'),Y)\ge 0$ and

$$-3\alpha\begin{pmatrix} I & O\\ O & I\end{pmatrix}\le\begin{pmatrix} X & O\\ O & -Y\end{pmatrix}\le 3\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}, \tag{10.2}$$

where $I\in\mathbb{R}^3\otimes\mathbb{R}^3$ denotes a unit matrix.

Proof. The condition $F(z',\alpha(z-z'),Y)\ge 0$ implies $(z',\alpha(z-z'),Y)\in\mathscr{U}$, thus it holds that either (i) $z_s>z'_s$ or (ii) $z_s=z'_s$ and $z'_s(p_w-p'_w)-(p_\varphi-p'_\varphi)>0$. In each case we have $F(z,\alpha(z-z'),X)>-\infty$ and

$$\begin{aligned}
&F(z',\alpha(z-z'),Y)-F(z,\alpha(z-z'),X)\\
\le\ &\frac{1}{2}(\hat{\sigma}^2(z_s)X_{ss}-\hat{\sigma}^2(z'_s)Y_{ss})+|\hat{b}(z_s)-\hat{b}(z'_s)|\alpha|z_s-z'_s|\\
&+\alpha\sup_{\zeta\ge 0}\left\{-(z_s-z'_s)^2g(\zeta)+(z_s-z'_s)(z_w-z'_w)\zeta\right\}.
\end{aligned}\tag{10.3}$$

Since the condition (10.2) implies

$$\hat{\sigma}^2(z_s)X_{ss}-\hat{\sigma}^2(z'_s)Y_{ss}\le 3\alpha(\hat{\sigma}(z_s)-\hat{\sigma}(z'_s))^2$$

and, $\hat{\sigma}$ and $\hat{b}$ are both Lipschitz continuous and linear growth, we have

$$\frac{1}{2}(\hat{\sigma}^2(z_s)X_{ss}-\hat{\sigma}^2(z'_s)Y_{ss})+|\hat{b}(z_s)-\hat{b}(z'_s)|\alpha|z_s-z'_s|\le C_0\alpha|z_s-z'_s|^2$$

for some $C_0>0$.

Next we estimate the last term of the right-hand side of (10.3). If $z_s=z'_s$, it is obvious that this term is equal to zero. Then we consider the case of $z_s>z'_s$. Since $\liminf_{\zeta\to\infty}\dfrac{h(\zeta)}{\zeta}>0$, we see that there are some $\beta>0$ and $\zeta_0>0$ such that $g(\zeta)\ge\beta\zeta^2$ for any $\zeta\ge\zeta_0$. Thus

$$\begin{aligned}
&\sup_{\zeta\ge 0}\left\{-(z_s-z'_s)^2g(\zeta)+(z_s-z'_s)(z_w-z'_w)\zeta\right\}\\
\le\ &-(g(\zeta_0)+\zeta_0)|z-z'|^2+\sup_{\zeta\ge 0}\left\{-(z_s-z'_s)^2\beta\zeta^2+(z_s-z'_s)(z_w-z'_w)\zeta\right\}\\
\le\ &(g(\zeta_0)+\zeta_0)|z-z'|^2+|z_s-z'_s||z_w-z'_w|\times\left(\frac{z_w-z'_w}{2\beta(z_s-z'_s)}\vee 0\right)\ \le\ C_2|z-z'|^2
\end{aligned}$$

for some $C_1,C_2>0$. Thus we obtain the assertion. ■

Then we obtain Theorem 6 by the similar arguments in the proof of Theorem 7.7.2 in [27], but we should do the calculation carefully. Note especially that the divergence speed of $h^{-1}(\zeta)$ is slower than the linear function ζ (as $\zeta \to \infty$) and that the function $q(w,\varphi,s) = (w^2+s^2+1)^m$ $(m \in \mathbb{N})$ satisfies

$$\sup_{(w,\varphi,s)\in U}\left\{\zeta\left(s\frac{\partial}{\partial w}q(w,\varphi,s) - \frac{\partial}{\partial \varphi}q(w,\varphi,s)\right) - g(\zeta)s\frac{\partial}{\partial s}q(w,\varphi,s)\right\} \leq C_0 q(w,\varphi,s)$$

for some $C_0 > 0$.

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