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周期的 $\delta^{(1)}$ 型点相互作用に従う 1 次元シュレディンガー作用素の退化したスペクトラルギャップについて

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1. Introduction and main result

In this article, we consider the one-dimensional Schrödinger operators with periodic point interactions and discuss its spectrum. In our previous works [10, 12, 13], we discussed the coexistence problem. In this article, we especially introduce the main results in [13] and describe the outline of the proof.

In order to explain the motivation of our research, we describe backgrounds. The one-dimensional Schrödinger operators with periodic point interactions play an impertant role in solid state physics and have been studied in numerous works [1, 2, 3, 4, 5, 6, 7, 8, 15, 16] so far. Especially, it is notable that R. Kronig and W. Penney introduced the one-dimensional Schrödinger operators with periodic δ-interactions. Let $\delta(\cdot)$ be the Dirac delta function supported at the origin. The following operator is nowadays called the Kronig–Penney Hamiltonian.

$$L_1 := -\frac{d^2}{dx^2} + \beta \sum_{l\in\mathbf{Z}} \delta(x-2\pi l) \quad \text{in} \quad L^2(\mathbf{R}), \quad \beta \in \mathbf{R}\setminus\{0\}.$$

One can prove that a function y from the $\mathrm{Dom}(L_1)$ satisfies that $y \in W^{2,2}(\mathbf{R}\setminus 2\pi\mathbf{Z})$ and the following boundary conditions at $x \in 2\pi\mathbf{Z}$:

$$\begin{pmatrix} y(x+0) \\ \frac{dy}{dx}(x+0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} y(x-0) \\ \frac{dy}{dx}(x-0) \end{pmatrix}.$$

This operators is the Hamiltonian for an electron in a one-dimensional crystal. The δ-interaction was widely generalized by P. Šeba in 1986 (see also [2, 3] and [1, Section K.1.4]). He investigated the family of the self-adjoint extensions of the second derivation operator $L^{00} = -d^2/dx^2$ with $\mathrm{Dom}(L^{00}) = \{\psi \in W^{2,2}(\mathbf{R}) |\quad \psi(0) = \psi'(0) = 0\}$. Since this operator has the deficiency indeces $(2,2)$, there is a four-parameter family of self-adjoint

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extensions. In particular, the family of the connected types of self-adjoint extension is given by

$$\{L(\theta, A)| \quad \theta \in \mathbf{R}, \quad A \in SL(2, \mathbf{R})\},$$

where

$$(L(\theta, A)y)(x) = -\frac{d^2y}{dx^2}(x), \qquad x \in \mathbf{R} \setminus \{0\},$$

$$\mathrm{Dom}(L(\theta, A)) = \left\{ y \in W^{2,2}(\mathbf{R} \setminus \{0\}) \,\middle|\, \begin{pmatrix} y(+0) \\ \frac{dy}{dx}(+0) \end{pmatrix} = e^{i\theta} A \begin{pmatrix} y(-0) \\ \frac{dy}{dx}(-0) \end{pmatrix} \right\}$$

for $\theta \in \mathbf{R}$, $A \in SL(2, \mathbf{R})$. The generalized point interaction corresponds to the boundary condition of this operator. In order to express the potential of the operator $L(\theta, A)$, P. Kurasov introduced the *distribution theory for the discontinuous test functions* in 1996. Let $D_x\delta = \delta^{(1)}$ be the derivative of the Dirac delta function in the sense of this disctibution theory. According to [7], one can prove that

$$L(0, A_0) = -D_x^2 + \beta\delta^{(1)},$$

where $\beta \in \mathbf{R} \setminus \{-2, 2\}$ and

$$A_0 = \begin{pmatrix} \frac{2+\beta}{2-\beta} & 0 \\ 0 & \frac{2-\beta}{2+\beta} \end{pmatrix}.$$

In this article, we especially summarize the results of the spectral analysis for the second derivation operator $-D_x^2$ perturbed by the periodic $\delta^{(1)}$-interactions. For $\beta_1, \beta_2, \beta_3 \in \mathbf{R} \setminus \{2, -2\}$, $\beta_3 \neq 0$ and $0 < \kappa_1 < \kappa_2 < 2\pi$, we consider the operator

$$H = -D_x^2 + \sum_{l \in \mathbf{Z}} (\beta_1 \delta^{(1)}(x - \kappa_1 - 2\pi l) + \beta_2 \delta^{(1)}(x - \kappa_2 - 2\pi l) + \beta_3 \delta^{(1)}(x - 2\pi l)) \quad \text{in} \quad L^2(\mathbf{R}).$$

We define the domain of H as

$$\mathrm{Dom}(H) = \left\{ \psi \in L^2(\mathbf{R}) \,\middle|\, \begin{array}{c} \text{there exists some } f \in L^2(\mathbf{R}) \text{ such that} \\ (H\psi, \varphi)_{L^2(\mathbf{R})} = (f, \varphi)_{L^2(\mathbf{R})} \text{ for all } \quad \varphi \in \mathcal{D} \end{array} \right\},$$

where $\mathcal{D} = C_0^\infty(\mathbf{R})$.

We next introduce the precise definition of the operator H. For that purpose, we describe the distribution theory for the discontinuous test functions. We put $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_1 = \{\kappa_1\} + 2\pi\mathbf{Z}$, $\Gamma_2 = \{\kappa_2\} + 2\pi\mathbf{Z}$ and $\Gamma_3 = 2\pi\mathbf{Z}$. For $t \in \Gamma$, we define the set K_t as the set of all functions with compact support on $\mathbf{R}$ such that those derivatives of any order outside the point t are uniformly bounded. Furthermore, we put $K = \cup_{t \in \Gamma} K_t$. Let K' be the set of the distribution corresponding to K. This implies that $f \in K'$ is a linear form on K such that for every compact set $B \subset \mathbf{R}$, there exist constants $C > 0$ and $n \in \mathbf{N} \cup \{0\}$ satisfying

$$|f(\varphi)| \le C \sum_{\alpha \le n} \sup_{x \neq t} \left| \left(\frac{d}{dx} \right)^\alpha \varphi \right|, \quad \varphi \in K_t, \quad t \in \Gamma, \quad \mathrm{supp}(\varphi) \subset B.$$

For a disctibution $f \in K'$ and a test function $\varphi \in K$, we define the derivative $D_x f = f^{(1)}$ as

$$(D_x f)(\varphi) = -f\left(\frac{d\varphi}{dx}\right),$$

where $d\varphi/dx$ stands for the derivative of φ on $\mathbf{R}\setminus\Gamma$ in the classical sense. Moreover, we define the delta function supported at $t \in \Gamma$ in K' as

$$(\delta(x-t))(\varphi) = \frac{\varphi(t+0)+\varphi(t-0)}{2}$$

for $\varphi \in K$. The derivative of Delta function in K' is calculated as

$$(\delta^{(1)}(x-t))(\varphi) = -\frac{(\frac{d\varphi}{dx})(t+0)+(\frac{d\varphi}{dx})(t-0)}{2}$$

for $\varphi \in K_t$. The relationship between derivatives D_x and d/dx can be given by using the derivation of the constant disctibution 1. The derivation of 1 is the distribution defined by the formula

$$(\beta(x-t))(\varphi) = \varphi(t+0) - \varphi(t-0)$$

for $t \in \Gamma$ and $\varphi \in K_t$. The derivative $D_x\beta(x-t) = \beta^{(1)}(x-t)$ of this distribution is defined the equation

$$(\beta^{(1)}(x-t))(\varphi) = -\left(\frac{d\varphi}{dx}(t+0) - \frac{d\varphi}{dx}(t-0)\right) \quad \text{for} \quad \varphi \in K_t \quad \text{and} \quad t \in \Gamma.$$

Next, we describe the difference between the generalized and classical derivatives. We define

$$K_{t,loc} = \left\{ f \in C^\infty(\mathbf{R}\setminus\{t\}) \middle| \quad f \text{ is bounded}, \quad \left|\frac{d^n}{dx^n} f(t\pm 0)\right| < \infty \right\}$$

for $t \in \Gamma$. For every $\psi \in K_{t,loc}$, $\psi' = (d/dx)\psi$ stands for the classical derivative, $D_x\psi = \psi^{(1)}$ the derivative calculated as a distribution. As proved in [7, Lemma 4.5], the difference between the classical deivative $(d/dx)\psi$ and the generalized derivative $D_x\psi = \psi^{(1)}$ for $\psi \in K_{t,loc}$ is illustrated by the formula

$$D_x\psi = \frac{d}{dx}\psi + (\beta(x-t))(\psi)\delta(x-t) + (\delta(x-t))(\psi)\beta(x-t),$$

$$\begin{aligned} D_x^2\psi = & \frac{d^2}{dx^2}\psi + (\delta(x-t))D_x\beta(x-t) - (D_x\delta(x-t))(\psi)\beta(x-t) \\ & +(\beta(x-t)(\psi))D_x\delta(x-t) - (D_x\beta(x-t)(\psi))\delta(x-t). \end{aligned} \tag{1.1}$$

We consider the product of any distribution $f \in K'$ and any function $\psi \in K_{t,loc}$ for $t \in \Gamma$ as

$$f\psi(\varphi) = \psi f(\varphi) = f(\psi\varphi)$$

for an arbitrary test function $\varphi \in K_t$. We also define the product $\delta^{(1)}(x-t)$ and $\psi \in L^2(\mathbf{R})$ as

$$(\delta^{(1)}(x-t)\psi)(\varphi) = (\psi\delta^{(1)}(x-t))(\varphi) = -(\delta(x-t))\left(\frac{d}{dx}(\psi\varphi)\right)$$

for $\varphi \in \mathcal{D}$ satisfying $\operatorname{supp}(\varphi) \cap \{t\} = \emptyset$ because $((d/dx)(\psi\varphi))(t \pm 0)$ exists. As in [7, (14)], we also have

$$\begin{aligned} \psi\delta^{(1)}(x-t) &= (\delta(x-t))(\psi)\delta^{(1)}(x-t) + \frac{(\beta^{(1)}(x-t))(\psi)}{4}\beta(x-t) \\ &\quad + (\delta^{(1)}(x-t))(\psi)\delta(x-t) + \frac{(\beta(x-t))(\psi)}{4}\beta^{(1)}(x-t) \end{aligned} \tag{1.2}$$

for $\psi \in K_{t,loc}$ and $t \in \Gamma$.

One can express the definition of the operator H by the boundary conditions on the lattice Γ. We define the operator T in $L^2(\mathbf{R})$ as follows:

$$(Ty)(x) = -\frac{d^2}{dx^2}y(x), \qquad x \in \mathbf{R} \setminus \Gamma,$$

$$\operatorname{Dom}(T) = \left\{ y \in W^{2,2}(\mathbf{R} \setminus \Gamma) \,\middle|\, \begin{array}{c} \begin{pmatrix} y(x+0) \\ \frac{dy}{dx}(x+0) \end{pmatrix} = A_j \begin{pmatrix} y(x-0) \\ \frac{dy}{dx}(x-0) \end{pmatrix} \\ \text{for} \quad x \in \Gamma_j, \quad j = 1,2,3 \end{array} \right\},$$

where

$$A_j = \begin{pmatrix} \frac{2+\beta_j}{2-\beta_j} & 0 \\ 0 & \frac{2-\beta_j}{2+\beta_j} \end{pmatrix}$$

for $j = 1, 2, 3$. By using (1.2), one can prove that $H = T$ (see [13, Theorem 1.1]). In a similar way to [10, Proposition 2.1], we can show the self-adjointness of H. Since H has 2π-periodic point interactions, we can make use of a direct integral decomposition for H (see [14, Section XIII.16]). For $\mu \in \mathbf{R}$, we define the Hilbert space

$$\mathcal{H}_\mu = \{u \in L^2_{\text{loc}}(\mathbf{R}) |\quad u(x+2\pi) = e^{i\mu}u(x) \quad \text{for almost every } x \in \mathbf{R}\}$$

equipped with the inner product

$$\langle u, v \rangle_{\mathcal{H}_\mu} = \int_0^{2\pi} u(x)\overline{v(x)}dx, \qquad u, v \in \mathcal{H}_\mu.$$

We define a fiber operator $H_\mu = H_\mu(A_1, A_2, A_3)$ in $\mathcal{H}_\mu$ as

$$(H_\mu y)(x) = -y''(x), \qquad x \in \mathbf{R} \setminus \Gamma,$$

$$\operatorname{Dom}(H_\mu) = \left\{ y \in \mathcal{H}_\mu \,\middle|\, \begin{array}{c} y \in H^2((0,2\pi) \setminus \{\kappa_1, \kappa_2\}), \\ \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in \Gamma_j, \quad j = 1,2,3 \end{array} \right\}.$$

We further define a unitary operator $\mathcal{U}$ from $L^2(\mathbf{R})$ onto $\int_0^{2\pi} \oplus \mathcal{H}_\mu d\mu$ as

$$(\mathcal{U}u)(x,\mu) = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} e^{il\mu} u(x - 2l\pi).$$

Then we have the direct integral representation of T:

$$\mathcal{U}T\mathcal{U}^{-1} = \int_0^{2\pi} \oplus H_\mu d\mu.$$

Let $\lambda_j(\mu)$ be the jth eigenvalue of H_μ counted with multiplicity for $j \in \mathbf{N}$. We put

$$\xi = \prod_{j=1}^{3} \left(\frac{2+\beta_j}{2-\beta_j} + \frac{2-\beta_j}{2+\beta_j} \right).$$

To define the spectral gaps of H, we now quote the basic properties (a)–(f) of $\sigma(H)$ from [11, Proposition 1.1].

(a) The function $\lambda_j(\cdot)$ is continuous on $[0, 2\pi]$.

(b) It holds that $\lambda_j(\mu) = \lambda_j(-\mu)$.

(c) If $\mu \notin \pi\mathbf{Z}$, then every eigenvalue of H_μ is simple.

(d) The spectrum of H is given by

$$\begin{aligned} \sigma(H) &= \bigcup_{\mu \in [0,\pi]} \sigma(H_\mu(A_1, A_2, A_3)) \\ &= \bigcup_{j=1}^{\infty} \lambda_j([0,\pi]) \\ &= \bigcup_{j=1}^{\infty} \bigcup_{\mu \in [0,\pi]} \{\lambda_j(\mu)\}. \end{aligned}$$

(e) If $\xi > 0$, then the function $\lambda_j(\cdot)$ is strictly monotone increasing (respectively, decreasing) function on $[0, \pi]$ for odd (respectively, even) j.

(f) If $\xi < 0$, then the function $\lambda_j(\cdot)$ is strictly monotone increasing (respectively, decreasing) function on $[0, \pi]$ for even (respectively, odd) j.

Here we define the spectral gaps of H. We define

$$G_j = \begin{cases} (\lambda_j(\pi), \lambda_{j+1}(\pi)) & \text{for } j \text{ odd}, \\ (\lambda_j(0), \lambda_{j+1}(0)) & \text{for } j \text{ even} \end{cases}$$

in the case where $\xi > 0$, while we put

$$G_j = \begin{cases} (\lambda_j(\pi), \lambda_{j+1}(\pi)) & \text{for } j \text{ even}, \\ (\lambda_j(0), \lambda_{j+1}(0)) & \text{for } j \text{ odd} \end{cases}$$

if $\xi < 0$. Then we refer to the open interval G_j as the jth gap of the spectrum of H. Furthermore, we put $B_j = \lambda_j([0, \pi])$. This closed interval B_j is called the jth band of the spectrum of H. The consecutive bands B_j and B_{j+1} are separated by an spectral gap G_j. If there exists $j \in \mathbf{N}$ such that $G_j = \emptyset$, i.e. the jth spectral gap is degenerate, then the corresponding bands B_j and B_{j+1} merge. The aim in this article is to determine the degenerate spectral gaps of H, namely, to clarify the following set:

$$\mathcal{B} := \bigcup_{j=1}^{\infty} B_j \cap B_{j+1}.$$

Furthermore, we determine the induces of the degenerate gaps of $\sigma(H)$, i.e., we analyze the following set:

$$\Lambda := \{j \in \mathbf{N} | \quad G_j = \emptyset\}.$$

For $j = 1, 2, 3$, we put

$$\alpha_j = \frac{2 + \beta_j}{2 - \beta_j}.$$

Remark 1.1. *Two of the following four statements does not simultaneously hold.*

$$\text{(A.1)} \qquad \alpha_1^2\alpha_2^2\alpha_3^2 - 1 = 0.$$
$$\text{(A.2)} \qquad \alpha_2^2\alpha_3^2 - \alpha_1^2 = 0.$$
$$\text{(A.3)} \qquad \alpha_1^2\alpha_2^2 - \alpha_3^2 = 0.$$
$$\text{(A.4)} \qquad \alpha_1^2\alpha_3^2 - \alpha_2^2 = 0.$$

In [13], we obtained the following three results.

Theorem 1.2. (*the single periodic $\delta^{(1)}$-interaction*) *If $\beta_1 = \beta_2 = 0$ is valid, then we have*

$$G_j \neq \emptyset$$

for $j \in \mathbf{N}$, i.e., $\Lambda = \emptyset$.

Theorem 1.3. (*the double periodic $\delta^{(1)}$-interactions*) *If $\beta_1 = 0$ and $\beta_2 \neq 0$, then the following statements hold true.*

(i) *If $\alpha_2\alpha_3 \neq \pm 1$ or $\alpha_2 \neq \pm\alpha_3$, then we have $\Lambda = \emptyset$.*

(ii) *We suppose that $\alpha_2\alpha_3 = \pm 1$. Then, $\Lambda = \emptyset$ if and only if $\kappa_2/\pi \notin \mathbf{Q}$. If $\kappa_2/2\pi = q/p$, $(p, q) \in \mathbf{N}^2$, and $\gcd(p, q) = 1$, then $\Lambda = \{pj | \quad j \in \mathbf{N}\}$.*

(iii) *We assume that $\alpha_2 = \pm\alpha_3$ and $\kappa_2 \neq \pi$. Then, $\Lambda = \emptyset$ if and only if $\kappa_2/\pi \notin \{q/p | \quad (p, q) \in \mathbf{N}^2, \quad \gcd(p, q) = 1, \quad q \in 2\mathbf{N} - 1\}$. If $\kappa_2/\pi = q/p$, $(p, q) \in \mathbf{N}^2$, $\gcd(p, q) = 1$ and $q \in 2\mathbf{N} - 1$, then we have*

$$\Lambda = \{p(2j - 1) | \quad j \in \mathbf{N}\}.$$

For the simplicity, we put $\tau_1 = \kappa_1$, $\tau_2 = \kappa_2 - \kappa_1$, $\tau_3 = 2\pi - \kappa_2$. Note that the following statements are equivalent:

(A) $\kappa_2/\kappa_1 \in \mathbf{Q}$ and $\kappa_1/\pi \in \mathbf{Q}$.

(B) there exists $(p_1, p_2, p_3) \in \mathbf{N}^3$ such that $\tau_1 : \tau_2 : \tau_3 = p_1 : p_2 : p_3$ and $\gcd(p_1, p_2, p_3) = 1$.

For $(p_1, p_2, p_3) \in \mathbf{N}^3$ satisfying $\gcd(p_1, p_2, p_3) = 1$, we put $p = p_1 + p_2 + p_3$.

Theorem 1.4. *(the triple periodic $\delta^{(1)}$-interactions) If $\beta_1 \neq 0$ and $\beta_2 \neq 0$, then we have the following two statements.*

(i) *Suppose that $(\alpha_1^2\alpha_2^2\alpha_3^2 - 1)(\alpha_2^2\alpha_3^2 - \alpha_1^2)(\alpha_1^2\alpha_2^2 - \alpha_3^2)(\alpha_1^2\alpha_3^2 - \alpha_2^2) = 0$. If $(\kappa_2/\kappa_1, \kappa_1/\pi) \notin \mathbf{Q}^2$, then we have $\Lambda = \emptyset$. If there exists $(p_1, p_2, p_3) \in \mathbf{N}^3$ such that $\tau_1 : \tau_2 : \tau_3 = p_1 : p_2 : p_3$ and $\gcd(p_1, p_2, p_3) = 1$, then we have*

$$\Lambda = \begin{cases} p\mathbf{N} & \text{if } \alpha_1^2\alpha_2^2\alpha_3^2 = 1, \\ \frac{p}{2}\mathbf{N} & \text{if } p_1, p_2 \in 2\mathbf{N} - 1, \quad p_3 \in 2\mathbf{N} \quad \text{and } \alpha_2^2\alpha_3^2 - \alpha_1^2 = 0, \\ \frac{p}{2}\mathbf{N} & \text{if } p_1, p_3 \in 2\mathbf{N} - 1, \quad p_2 \in 2\mathbf{N} \quad \text{and } \alpha_1^2\alpha_2^2 - \alpha_3^2 = 0, \\ \frac{p}{2}\mathbf{N} & \text{if } p_2, p_3 \in 2\mathbf{N} - 1, \quad p_1 \in 2\mathbf{N} \quad \text{and } \alpha_1^2\alpha_3^2 - \alpha_2^2 = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

(ii) *Suppose that $(\alpha_1^2\alpha_2^2\alpha_3^2 - 1)(\alpha_2^2\alpha_3^2 - \alpha_1^2)(\alpha_1^2\alpha_2^2 - \alpha_3^2)(\alpha_1^2\alpha_3^2 - \alpha_2^2) \neq 0$. Then, we have*

$$\mathcal{B} = \left\{ \lambda \in \mathbf{R} \setminus \{0\} \;\middle|\; \begin{array}{l} \cot \tau_1\sqrt{\lambda} \cot \tau_2\sqrt{\lambda} = \frac{\alpha_2^2\alpha_3^2 - \alpha_1^2}{\alpha_1^2\alpha_2^2\alpha_3^2 - 1}, \\ \cot \tau_1\sqrt{\lambda} \cot \tau_3\sqrt{\lambda} = \frac{\alpha_1^2\alpha_2^2 - \alpha_3^2}{\alpha_1^2\alpha_2^2\alpha_3^2 - 1}, \\ \cot \tau_2\sqrt{\lambda} \cot \tau_3\sqrt{\lambda} = \frac{\alpha_1^2\alpha_3^2 - \alpha_2^2}{\alpha_1^2\alpha_2^2\alpha_3^2 - 1} \end{array} \right\}.$$

Our problem is called the coexistence problem, which relates the properties of the solutions to the differential equation corresponding to H. To explain the concept of the coexistence problem, we consider the equations

$$-\frac{d^2}{dx^2} y(x, \lambda) = \lambda y(x, \lambda), \quad x \in \mathbf{R} \setminus \Gamma, \tag{1.3}$$

$$\begin{pmatrix} y(x+0, \lambda) \\ \frac{dy}{dx}(x+0, \lambda) \end{pmatrix} = A_j \begin{pmatrix} y(x-0, \lambda) \\ \frac{dy}{dx}(x-0, \lambda) \end{pmatrix}, \quad x \in \Gamma_j, \quad j = 1, 2, 3, \tag{1.4}$$

where $\lambda \in \mathbf{R}$ is a spectral parameter. Let $y_1(x, \lambda)$ and $y_2(x, \lambda)$ be the solutions to (1.3) and (1.4) subject to the initial conditions

$$y_1(+0, \lambda) = 1, \quad \frac{dy_1}{dx}(+0, \lambda) = 0,$$

and

$$y_2(+0, \lambda) = 0, \quad \frac{dy_2}{dx}(+0, \lambda) = 1,$$

respectively. The monodromy matrix $M(\lambda)$ is defined by

$$M(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} y_1(2\pi+0,\lambda) & y_2(2\pi+0,\lambda) \\ \frac{dy_1}{dx}(2\pi+0,\lambda) & \frac{dy_2}{dx}(2\pi+0,\lambda) \end{pmatrix}.$$

The function $D(\lambda) := \text{tr } M(\lambda)$ is called the discriminant of the spectrum of H. It holds that $\sigma(H) = \{\lambda \in \mathbf{R} | \ |D(\lambda)| \leq 2\}$. The sequence $\{\lambda_j\}_{j=0}^{\infty}$ is defined as the zeroes of $D(\lambda) \pm 2$ counted with the multiplicity. Then, we have $\lambda_{2j-2} < \lambda_{2j-1} \leq \lambda_{2j}$ for $j \in \mathbf{N}$. Moreover, we obtain $B_j = [\lambda_{2j-2}, \lambda_{2j-1}]$ for $j \in \mathbf{N}$. In addition, we have

$$\mathcal{B} = \{\lambda \in \mathbf{R} | \quad M(\lambda) = E \quad \text{or} \quad M(\lambda) = -E\}, \tag{1.5}$$

E being the 2×2 unit matrix. According to [9, Section VII], one says that the periodic solutions to (1.3) and (1.4) *coexist* if all the solution to (1.3) and (1.4) are periodic or anti-periodic. We note that the periodic solutions to (1.3) and (1.4) coexist if and only if $\lambda \in \mathcal{B}$. In this sense, the coexistence problem relates the properties of the solution to the differential equation corresponding to H. Therefore, the coexistence problem for the periodic Schrödinger operators has been investigated by numerous authors. Especially, we can find the result of the coexistence problem for the one-dimensional Schrödinger operators with periodic point interactions in [4, 5, 6, 10, 12, 16] and so on.

2. Outline of the proof

In this article, we give the outline of the proof of Theorem 1.4. For that purpose, we first introduce the rotation number for H. To look back on the definition of the rotation number, we consider the Schrödinger equations (1.3) and (1.4). Let $y(x, \lambda)$ denote a non-trivial solution of (1.3) and (1.4). The Prüfer transform $\omega = \omega(x, \lambda)$ of $y(x, \lambda)$ is defined by the polar coordinates (r, ω) of $((d/dx)y, y)$, namely, $(d/dx)y = r\cos\omega$ and $y = r\sin\omega$. The function $\omega(x, \lambda)$ satisfies the equation

$$\frac{d}{dx}\omega(x,\lambda) = \cos^2(x,\lambda) + \lambda\sin^2\omega(x,\lambda), \quad x \in \mathbf{R}\setminus\Gamma, \tag{2.1}$$

as well as the boundary conditions

$$\alpha_j^2 \cos\omega(x+0,\lambda)\sin\omega(x-0,\lambda) = \sin\omega(x+0,\lambda)\cos\omega(x-0,\lambda), \tag{2.2}$$

$$\text{sgn}(\sin\omega(x+0,\lambda)) = \text{sgn}(\alpha_j\sin\omega(x-0,\lambda)), \tag{2.3}$$

$$\text{sgn}(\cos\omega(x+0,\lambda)) = \text{sgn}(\alpha_j^{-1}\cos\omega(x-0,\lambda)) \tag{2.4}$$

for $x \in \Gamma_j$ and $j = 1, 2, 3$. Following [11, Theorem 1.2], we choose the branch of $\omega(x+0, \lambda)$ as

$$\omega(x+0,\lambda) - \omega(x-0,\lambda) \in [-\pi,\pi) \quad \text{for} \quad x \in \Gamma. \tag{2.5}$$

Let $\omega = \omega(x, \lambda, \omega_0)$ be the solution to (2.1)–(2.5) subject to the initial condition $\omega(+0, \lambda) = \omega_0 \in \mathbf{R}$. The rotation number for H is defined as

$$\rho(\lambda) = \lim_{k\to\infty} \frac{\omega(2k\pi+0,\lambda,\omega_0) - \omega_0}{2k\pi}, \tag{2.6}$$

where $k \in \mathbf{N}$. Let us cite [11, Theorem 1.2], in which the properties of $\rho(\lambda)$ are summarized.

Theorem B. *The function $\rho(\lambda)$ has the following properties.*

(a) *The limit on the right-hand side of* (2.6) *exists and is independent of the initial value ω_0.*

(b) *The function $\rho(\lambda)$ is continuous and non-decreasing on* $\mathbf{R}$.

(c) *We recall $B_j = [\lambda_{2j-2}, \lambda_{2j-1}]$ for $j \in \mathbf{N}$. Put $\ell = \sharp\{j \in \{1,2,3\} \mid \alpha_j < 0\}$, where $\sharp A$ stands for the number of the elements of a finite set of A. Then, we have*

$$\lambda_{2j-2} = \max\left\{\lambda \in \mathbf{R} \,\middle|\, \rho(\lambda) = \frac{j-1}{2} - \frac{\ell}{2}\right\},$$

$$\lambda_{2j-1} = \min\left\{\lambda \in \mathbf{R} \,\middle|\, \rho(\lambda) = \frac{j}{2} - \frac{\ell}{2}\right\}$$

for $j \in \mathbf{N}$.

From now on, we start the discussion on the proof of Theorem 1.4. We assume that $\beta_1 \neq 0$ and $\beta_2 \neq 0$. The elements of monodromy matrix can be directly calculated by $M(\lambda) = T_1(\lambda)A_1T_2(\lambda)A_2T_3(\lambda)A_3$, where

$$T_j(\lambda) = \begin{pmatrix} \cos\tau_j\sqrt{\lambda} & \frac{1}{\sqrt{\lambda}}\sin\tau_j\sqrt{\lambda} \\ -\sqrt{\lambda}\sin\tau_j\sqrt{\lambda} & \cos\tau_j\sqrt{\lambda} \end{pmatrix}$$

for $j = 1, 2, 3$. By using this formula, we have

$$\begin{aligned}
m_{11}(\lambda) &= \alpha_1\alpha_2\alpha_3\cos\tau_1\sqrt{\lambda}\cos\tau_2\sqrt{\lambda}\cos\tau_3\sqrt{\lambda} - \frac{\alpha_2\alpha_3}{\alpha_1}\sin\tau_1\sqrt{\lambda}\sin\tau_2\sqrt{\lambda}\cos\tau_3\sqrt{\lambda} \\
&\quad - \frac{\alpha_1\alpha_3}{\alpha_2}\cos\tau_1\sqrt{\lambda}\sin\tau_2\sqrt{\lambda}\sin\tau_3\sqrt{\lambda} - \frac{\alpha_3}{\alpha_1\alpha_2}\sin\tau_1\sqrt{\lambda}\cos\tau_2\sqrt{\lambda}\sin\tau_3\sqrt{\lambda}, \\
m_{21}(\lambda) &= -\frac{\alpha_1\alpha_2}{\alpha_3}\sqrt{\lambda}\cos\tau_1\sqrt{\lambda}\cos\tau_2\sqrt{\lambda}\sin\tau_3\sqrt{\lambda} + \frac{\alpha_2}{\alpha_1\alpha_3}\sqrt{\lambda}\sin\tau_1\sqrt{\lambda}\sin\tau_2\sqrt{\lambda}\sin\tau_3\sqrt{\lambda} \\
&\quad - \frac{\alpha_1}{\alpha_2\alpha_3}\sqrt{\lambda}\cos\tau_1\sqrt{\lambda}\sin\tau_2\sqrt{\lambda}\cos\tau_3\sqrt{\lambda} - \frac{\sqrt{\lambda}}{\alpha_1\alpha_2\alpha_3}\sin\tau_1\sqrt{\lambda}\cos\tau_2\sqrt{\lambda}\cos\tau_3\sqrt{\lambda}, \\
m_{12}(\lambda) &= \frac{\alpha_1\alpha_2\alpha_3}{\sqrt{\lambda}}\sin\tau_1\sqrt{\lambda}\cos\tau_2\sqrt{\lambda}\cos\tau_3\sqrt{\lambda} + \frac{\alpha_2\alpha_3}{\alpha_1\sqrt{\lambda}}\cos\tau_1\sqrt{\lambda}\sin\tau_2\sqrt{\lambda}\cos\tau_3\sqrt{\lambda} \\
&\quad - \frac{\alpha_1\alpha_3}{\alpha_2\sqrt{\lambda}}\sin\tau_1\sqrt{\lambda}\sin\tau_2\sqrt{\lambda}\sin\tau_3\sqrt{\lambda} + \frac{\alpha_3}{\alpha_1\alpha_2\sqrt{\lambda}}\cos\tau_1\sqrt{\lambda}\cos\tau_2\sqrt{\lambda}\sin\tau_3\sqrt{\lambda},
\end{aligned}$$

$$\begin{aligned}
m_{22}(\lambda) &= -\frac{\alpha_1\alpha_2}{\alpha_3}\sqrt{\lambda}\sin\tau_1\sqrt{\lambda}\cos\tau_2\sqrt{\lambda}\sin\tau_3\sqrt{\lambda} - \frac{\alpha_2}{\alpha_1\alpha_3}\cos\tau_1\sqrt{\lambda}\sin\tau_2\sqrt{\lambda}\sin\tau_3\sqrt{\lambda} \\
&\quad - \frac{\alpha_1}{\alpha_2\alpha_3}\sin\tau_1\sqrt{\lambda}\sin\tau_2\sqrt{\lambda}\cos\tau_3\sqrt{\lambda} + \frac{1}{\alpha_1\alpha_2\alpha_3}\cos\tau_1\sqrt{\lambda}\cos\tau_2\sqrt{\lambda}\cos\tau_3\sqrt{\lambda}.
\end{aligned}$$

We define $S_1 = \{p^2j^2/4| \quad j \in \mathbf{N}\}$ and $S_2 = \{p^2j^2/16| \quad j \in \mathbf{N}\}$. The degenerate spectral gap is characterized by the formula (1.5). By solving the equation $M(\lambda) = \pm E$, we obtain the following result. (Since we presicely discussed in [13], we here omit the proof of this part.)

Lemma 2.1. *Suppose that* $(\alpha_1^2\alpha_2^2\alpha_3^2 - 1)(\alpha_2^2\alpha_3^2 - \alpha_1^2)(\alpha_1^2\alpha_2^2 - \alpha_3^2)(\alpha_1^2\alpha_3^2 - \alpha_2^2) = 0$. *Then, we have*

$$\mathcal{B} = \begin{cases} S_1 & \text{if} \quad (B) \quad \text{and} \quad (A.1), \\ S_2 & \text{if} \quad (B), \quad p_1 \in 2\mathbf{N} - 1, \quad p_2 \in 2\mathbf{N} - 1, \quad p_3 \in 2\mathbf{N} \quad \text{and} \quad (A.2), \\ S_2 & \text{if} \quad (B), \quad p_1 \in 2\mathbf{N} - 1, \quad p_2 \in 2\mathbf{N}, \quad p_3 \in 2\mathbf{N} - 1 \quad \text{and} \quad (A.3), \\ S_2 & \text{if} \quad (B), \quad p_1 \in 2\mathbf{N}, \quad p_2 \in 2\mathbf{N} - 1, \quad p_3 \in 2\mathbf{N} - 1 \quad \text{and} \quad (A.4), \\ \emptyset & \text{otherwise.} \end{cases}$$

We prove Theorem 1.4 (i) by using this lemma. (Since we can find the proof of Theorem 1.4 (ii) in [13], we here omit it.)

Proof of Theorem 1.4 (i). We prove that if (A.1) and (B) are valid, then we have $\Lambda = p\mathbf{N}$. We prove this statement in only the case where $\alpha_1, \alpha_2, \alpha_3 > 0$, which implies $\ell = 0$. By the previous lemma, we see that $\mathcal{B} = S_1$. So, we calcurate the rotation number at $\mu_j = p^2j^2/4$ for $j \in \mathbf{N}$. For that purpose, we calculate $\omega(2\pi k + 0, \mu_j, \omega_0)$ for $k \in \mathbf{N}$. Since the rotation number does not depend on the initial value, we put $\omega_0 = 0$. It turns out that $\omega(x, \lambda, 0)$ corresponds to the Prüfer transform of $y_2(x, \lambda)$. For $x \in (0, \kappa_1)$, we have

$$y_2(x, \mu_j) = \frac{1}{\sqrt{\mu_j}} \sin \sqrt{\mu_j} x,$$

and

$$y_2'(x, \mu_j) = \cos \sqrt{\mu_j} x.$$

Therefore, we have

$$\omega(\kappa_1 - 0, \mu_j, 0) = \sqrt{\mu_j} \cdot \frac{2\pi p_1}{p} = p_1 \pi j \in \pi\mathbf{Z}.$$

Equations (2.2)–(2.4) imply that $\omega(\kappa_1 + 0, \mu_j, 0)$ satisfies the equations

$$\operatorname{sgn}(\sin \omega(\kappa_1 + 0, \mu_j, 0)) = \operatorname{sgn}(\alpha_1 \sin p_1 \pi j) = (-1)^{p_1 j},$$

and

$$\cos \omega(\kappa_1 + 0, \mu_j, 0) = 0.$$

Because of (2.5), we obtain

$$\omega(\kappa_1 + 0, \mu_j, 0) = p_1 \pi j.$$

Since $y_2(\kappa_1 + 0, \mu_j) = 0$ and $y_2(\kappa_1 + 0, \mu_j) = (-1)^{p_1 j}/\alpha_1$, we have

$$y_2(x, \mu_j) = \frac{(-1)^{p_1 j}}{\alpha_1 \sqrt{\mu_j}} \sin(x - \kappa_1)\sqrt{\mu_j},$$

and

$$y_2'(x, \mu_j) = \frac{(-1)^{p_1 j}}{\alpha_1} \cos(x - \kappa_1)\sqrt{\mu_j}$$

on (κ_1, κ_2). This implies that

$$\omega(\kappa_2 - 0, \mu_j, 0) = p_1 \pi j + \sqrt{\mu_j} \cdot (\kappa_2 - \kappa_1) = p_1 \pi j + p_2 \pi j.$$

In a similar way, we obtain

$$\omega(\kappa_2 + 0, \mu_j, 0) = (p_1 + p_2)\pi j$$

and

$$\omega(2\pi - 0 + 0, \mu_j, 0) = (p_1 + p_2 + p_3)\pi j = p\pi j.$$

Since the equation (2.1) is periodic in ω, we obtain

$$\omega(2\pi k + 0, \mu_j, 0) = p\pi jk$$

for $k \in \mathbf{N}$. This is why we have

$$\rho(\mu_j) = \lim_{k\to\infty} \frac{\omega(2k\pi + 0, \mu_j, 0)}{2k\pi} = \frac{pj}{2}.$$

By using Theorem B and $\ell = 0$, if turns out that the pj^{th} spectral gap is degenerate at μ_j for every $j \in \mathbf{N}$.

In a similar way, we can obtain the other results. □

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