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Spectral properties of Schrödinger operators with singular magnetic fields supported by a circle in $\mathbf{R}^3$

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1 Introduction

In 1959, Aharonov-Bohm [AB] asserted that an electrically shielded solenoid can affect the phase of an electron moving outside the solenoid; this phenomenon is called the Aharonov-Bohm effect. Since then, numerous experimental attempts to demonstrate the Aharonov-Bohm effect were performed. However, as far as they used a solenoid of finite length, they could not avoid the criticism that their experimental result is caused by the leaking magnetic field from the ends of the solenoid. To avoid this criticism, Tonomura et al. [To] made a decisive experiment using a toroidal magnetic field enclosed by a superconductive material in 1986. Historical reviews in these subjects are found in e.g. Peshkin-Tonomura [PT] and Afanasiev [A].

After the experiment of Tonomura et al., several authors studied the Schrödinger operators with toroidal magnetic field. Afanasiev [A] gives a numerical calculation for the scattering amplitude by the toroidal solenoids. Ballesteros-Weder [BW] consider magnetic fields supported on handle bodies K (the boundary sum of several tori), and study the inverse scattering problem by means of the high-velocity limit for the Schrödinger operators defined on the exterior region $\boldsymbol{R}^3 \setminus K$ with Dirichlet boundary conditions.

We consider the Schrödinger operators H_ϵ in $\boldsymbol{R}^3$ with magnetic fields supported in a torus of thickness ϵ, and consider the singular limit $\epsilon \to 0$. The result of this type is obtained particularly in the two-dimensional case; see e.g. Albeverio et al. [AGHH] for the scalar potential case, and Tamura [Ta] for the magnetic case.

We have shown in [IMS] that, if we choose the magnetic field and the vector potential appropriately, then H_ϵ converges in the norm resolvent sense to some operator H_0, which is the Schrödinger operator with a singular magnetic field supported on a circle.

We like to present some improvements of our result of [IMS]: First we show the choice of the three dimensional magnetic fields B_ϵ is arbitrary in the sense that, if B_ϵ are of the form given in the condition (A2) below, the two dimensional magnetic field b is arbitrary as far as it satisfies the assumption (A1) below. Second we show "the norm resolvent convergence" of the Schrödinger operators defined on the exterior region $\boldsymbol{R}^3 \setminus \mathcal{T}_\epsilon$ with Dirichlet boundary conditions as ϵ tends to zero, where $\mathcal{T}_\epsilon$ is the torus of thickness ϵ around a fixed circle and where we should interpret the meaning of the norm resolvent convergence appropriately since the operators considered are defined on different domains (see Theorem 1.2 below). The proof is very similar to that of [IMS] with some refinement of the argument. In the last section we give some result concerning spectral and scattering properties of the operator H_0 with singular magnetic fields: see Theorems 9.1 and 9.3 below.

Now let us explain the rigorous mathematical setting. We consider magnetic Schrödinger operators on $\boldsymbol{R}^3$

$$\mathcal{L}_\epsilon = (D - A_\epsilon)^2 = \sum_{j=1}^{3} (D_j - A_{\epsilon,j})^2,$$

where $0 \le \epsilon < \epsilon_0$ (ϵ_0 is some positive constant), $D_j = \frac{1}{i}\partial_j$, $\partial_j = \frac{\partial}{\partial x_j}$, $D = {}^t(D_1, D_2, D_3)$ and $A_\epsilon = {}^t(A_{\epsilon,1}, A_{\epsilon,2}, A_{\epsilon,3})$. The magnetic field $B = {}^t(B_1, B_2, B_3)$ corresponding to a vector potential $A = {}^t(A_1, A_2, A_3)$ is given by

$$B = \nabla \times A = \begin{pmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \end{pmatrix}.$$

We denote

$$B_\epsilon = \nabla \times A_\epsilon. \tag{1.1}$$

We shall define our magnetic fields as follows. Let $a > \epsilon_0$ be a constant. We introduce a local coordinate

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (a + \tau\cos\phi)\cos\theta \\ (a + \tau\cos\phi)\sin\theta \\ \tau\sin\phi \end{pmatrix}, \tag{1.2}$$

where $0 \le \tau < a$, $\phi \in \boldsymbol{R}/2\pi\boldsymbol{Z}$, $\theta \in \boldsymbol{R}/2\pi\boldsymbol{Z}$. [1] We denote the torus and the circle

$$\{\tau < \epsilon\} = \mathcal{T}_\epsilon, \qquad \{\tau = 0\} = \mathcal{C} \tag{1.3}$$

for $0 < \epsilon \le a$. If we fix two of the coordinates (τ, ϕ, θ) and vary the rest one, we have a linear orbit or a circular one. We denote the unit tangent vector of these orbits as

$$e_\tau = \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ \sin\phi \end{pmatrix}, \quad e_\phi = \begin{pmatrix} -\sin\phi\cos\theta \\ -\sin\phi\sin\theta \\ \cos\phi \end{pmatrix}, \quad e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

We shall need the following conditions:

[1] $\boldsymbol{R}/2\pi\boldsymbol{Z}$ is the quotient Lie group equipped with local coordinates $\boldsymbol{R}/2\pi\boldsymbol{Z} \ni \theta = r + 2\pi\boldsymbol{Z} \mapsto r \in (r_0, r_0 + 2\pi)$, $r_0 \in \boldsymbol{R}$. In these coordinates, the trigonometric functions and the derivatives are well-defined as $\sin\theta = \sin r$, $\cos\theta = \cos r$ and $\partial_\theta f = \partial_r f$.

(A1) Let $E := \{ (x_1, x_2) \in \boldsymbol{R}^2 \,|\, x_1^2 + x_2^2 < 1 \}$ and $b \in C_0^\infty(E; \boldsymbol{R})$ satisfying

$$\int_E b(x_1, x_2) dx_1 dx_2 = 2\pi\alpha, \quad \alpha \in \boldsymbol{R} \setminus \boldsymbol{Z}. \tag{1.4}$$

(A2) For $0 < \epsilon < \epsilon_0$, we assume $B_\epsilon \in C^\infty(\boldsymbol{R}^3; \boldsymbol{R})^3$, $\operatorname{supp} B_\epsilon$ is contained in the open torus $\mathcal{T}_\epsilon$ and in this torus

$$B_\epsilon = -\frac{1}{\epsilon^2} b\left(\frac{\tau\cos\phi}{\epsilon}, \frac{\tau\sin\phi}{\epsilon}\right) e_\theta. \tag{1.5}$$

(A3) For $\epsilon = 0$, we assume $B_\epsilon \in \mathcal{D}'(\boldsymbol{R}^3; \boldsymbol{R})^3$ (the vector-valued distributions on $\boldsymbol{R}^3$) and

$$B_0 = -2\pi\alpha\delta_{\mathcal{C}} e_\theta, \tag{1.6}$$

where $\delta_{\mathcal{C}}$ is the delta measure on the circle $\mathcal{C}$, that is,

$$\langle B_0, \varphi \rangle = -2\pi\alpha \left(\int_0^{2\pi} \varphi(a\cos\theta, a\sin\theta, 0)\, e_\theta\, a\, d\theta \right)$$

for any test function $\varphi \in C_0^\infty(\boldsymbol{R}^3)$, where $\langle \cdot, \cdot \rangle$ denotes the coupling of a distribution and a test function. [2]

(A4) For $0 < \epsilon < \epsilon_0$, $A_\epsilon \in C_0^\infty(\boldsymbol{R}^3; \boldsymbol{R})^3$. For $\epsilon = 0$, $A_0 \in C^\infty(\boldsymbol{R}^3 \setminus \mathcal{C}; \boldsymbol{R})^3 \cap L^1(\boldsymbol{R}^3; \boldsymbol{R})^3$, and $\operatorname{supp} A_0$ is compact in $\boldsymbol{R}^3$. A_ϵ satisfies (1.1) for $0 \le \epsilon < \epsilon_0$.

Remark 1.1 Let $\Pi = \{x_2 = 0, x_1 > 0\}$ (the half x_3x_1 plane). We have the flux Φ through the plane Π of B_ϵ equals $2\pi\alpha$ independently of $\epsilon > 0$:

$$\Phi = \int_{\Pi\cap\{\tau\le\epsilon\}} B_{\epsilon,2}\, dx_3 \wedge dx_1 = \oint_{\Pi\cap\{\tau=\epsilon\}} (A_{\epsilon,1}dx_1 + A_{\epsilon,3}dx_3) = 2\pi\alpha$$

by (A1), (A2) and by the Stokes theorem. The minus sign before $1/\epsilon^2$ in (1.5) and (1.6) is added since $(e_\tau,\ e_\phi,\ -e_\theta)$ makes a right-hand system.

For given magnetic fields $\{B_\epsilon\}_{0\le\epsilon<\epsilon_0}$ satisfying (A2) and (A3), we show in section 3 that there exist vector potentials $\{A_\epsilon\}_{0\le\epsilon<\epsilon_0}$ satisfying (A4).

Then we can define self-adjoint realizations of $\{\mathcal{L}_\epsilon\}_{0\le\epsilon<\epsilon_0}$ as follows. When $0 < \epsilon < \epsilon_0$, the vector potential A_ϵ has no singularity. Then it is well-known that $L_\epsilon = \mathcal{L}_\epsilon|_{C_0^\infty(\mathbf{R}^3)}$ is essentially self-adjoint (see e.g. [IK] or [LS]), so we define $H_\epsilon = \overline{L_\epsilon}$. When $\epsilon = 0$, our vector potential A_0 has strong singularities on the circle $\mathcal{C}$ so that A_0 does *not* belong to $L^2(\boldsymbol{R}^3)^3$ (see (3.3); see also Proposition 5.1). Then $L_0 = \mathcal{L}_0|_{C_0^\infty(\mathbf{R}^3\setminus\mathcal{C})}$ is positive, symmetric, but not essentially self-adjoint.[3] As a self-adjoint realization, we choose the Friedrichs extension of L_0, and denote it by H_0.

[2] $C_0^\infty(\Omega)$ denotes the space of C^∞ functions on $\boldsymbol{R}^d$ with compact support contained in an open set $\Omega \subset \boldsymbol{R}^d$

[3] In fact, we can prove the deficiency indices of L_0 are (∞, ∞).

We further define the operators H_ϵ^D for $0 < \epsilon \le \epsilon_0$ in the exterior region $\Omega(\epsilon) := \boldsymbol{R}^3 \setminus \mathcal{T}_\epsilon$ with Dirichlet boundary conditions on its boundary as the Friedrichs extension of the operator $L_\epsilon^D = \mathcal{L}_0|_{C_0^\infty(\Omega(\epsilon))}$. Note that A_0 is smooth outside of $\mathcal{C}$ so that A_0 is smooth in a neighborhood of $\overline{\Omega(\epsilon)}$.

The main results of this paper are the following:

Theorem 1.1 *Suppose that* b *satisfies (A1) and* $\{B_\epsilon\}_{0\le\epsilon<\epsilon_0}$ *are given by (A2) and (A3). Then there exist vector potentials* $\{A_\epsilon\}_{0\le\epsilon<\epsilon_0}$ *satisfying (A4) such that* H_ϵ *converges to* H_0 *in the norm resolvent sense, as* ϵ *tends to* 0.

Theorem 1.2 *Suppose* B_0 *is given by (A3). Then there exists a vector potential* A_0 *satisfying (A4) such that* $\chi_{\Omega(\epsilon)}^*(H_\epsilon^D + E)^{-1}\chi_{\Omega(\epsilon)}$ *converges to* $(H_0 + E)^{-1}$ *in the operator norm of* $L^2(\boldsymbol{R}^3)$ *as* ϵ *tends to* 0, *where* χ_Ω *is the restriction operator* $L^2(\boldsymbol{R}^3) \to L^2(\Omega)$ *and* χ_Ω^* *is its adjoint which is the extension operator* $L^2(\Omega) \to L^2(\boldsymbol{R}^3)$.

These result are analogies of the result by Tamura [Ta]. There remains a natural question:

If we add some scalar potential V_ϵ to H_ϵ, then the norm resolvent limit exists?
If it exists, what are the boundary conditions on $\mathcal{C}$ of the limit operator?

Tamura's result suggests the conclusion is true if we choose suitable V_ϵ and the boundary conditions depend on the existence of the zero-energy resonance. We will discuss this problem elsewhere in the future.

2 Torus Coordinate

Let us give several formulas for the coordinate (τ, ϕ, θ) defined in (1.2). By direct computation, we have

$$\frac{\partial x}{\partial \tau} = e_\tau, \quad \frac{\partial x}{\partial \phi} = \tau e_\phi, \quad \frac{\partial x}{\partial \theta} = (a + \tau\cos\phi)e_\theta. \tag{2.1}$$

Since $(e_\tau, e_\phi, e_\theta)$ is an orthogonal matrix, we have

$$\left|\det\left(\frac{\partial(x_1, x_2, x_3)}{\partial(\tau, \phi, \theta)}\right)\right| = \tau(a + \tau\cos\phi).$$

Thus we have

$$\int_{\mathcal{T}_{\epsilon_0}} u\, dx_1 dx_2 dx_3 = \int_{\mathcal{T}_{\epsilon_0}} u\tau(a + \tau\cos\phi)\, d\tau d\phi d\theta \tag{2.2}$$

for any function $u \in L^1(\mathcal{T}_{\epsilon_0})$. For the derivatives, we have by (2.1)

$$\partial_\tau u = \nabla u \cdot e_\tau, \quad \partial_\phi u = \tau \nabla u \cdot e_\phi, \quad \partial_\theta u = (a + \tau\cos\phi)\nabla u \cdot e_\theta,$$

where $\partial_\tau = \frac{\partial}{\partial\tau}$, etc. Thus we have in $\mathcal{T}_{\epsilon_0}$

$$\begin{aligned}\nabla u &= (\nabla u \cdot e_\tau)e_\tau + (\nabla u \cdot e_\phi)e_\phi + (\nabla u \cdot e_\theta)e_\theta \\ &= (\partial_\tau u)e_\tau + \left(\frac{1}{\tau}\partial_\phi u\right)e_\phi + \left(\frac{1}{a + \tau\cos\phi}\partial_\theta u\right)e_\theta.\end{aligned} \tag{2.3}$$

3 Vector potentials

There are many ways of constructing vector potentials giving the toroidal magnetic fields: see [A] and [BW]. Especially, Afanasiev gives compactly supported vector potentials by using the Riemann toroidal coordinate [A, section 2.2.6]. In this section, we shall give vector potentials giving the toroidal magnetic fields, by using the coordinate defined in section 1. The resulting vector potentials can be compactly supported for a suitable choice of the functions given in the following construction. And in fact we shall assume that A_0 is compactly supported throughout the paper as is stated in the condition (A4).

In section 2, the coordinate functions τ, ϕ, θ are defined only in the torus $\mathcal{T}_{\epsilon_0}$. However, (1.2) is still valid in $\boldsymbol{R}^3 \setminus (\mathcal{C} \cap X_3)$ and τ, ϕ, θ are smooth there, where $X_3 = \{x_1 = x_2 = 0\}$ is the x_3-axis. Mollifying the functions τ and ϕ near X_3, we can construct new functions τ_1 and ϕ_1 satisfying the following conditions:

(i) $\tau_1 \in C^\infty(\boldsymbol{R}^3 \setminus \mathcal{C}; \boldsymbol{R}_+)$, $\phi_1 \in C^\infty(\boldsymbol{R}^3 \setminus \mathcal{C}; \boldsymbol{R}/2\pi\boldsymbol{Z})$.

(ii) $\tau_1 = \tau$ and $\phi_1 = \phi$ in the torus $\mathcal{T}_{\epsilon_0} = \{\tau < \epsilon_0\}$.

(iii) $\tau_1 > \epsilon_0$ on $\boldsymbol{R}^3 \setminus \overline{\mathcal{T}_{\epsilon_0}}$.

(iv) Let $\Lambda_\kappa = \{|x_3| < \kappa(a^2 - x_1^2 - x_2^2)\}$ for $\kappa > 0$. $|\phi_1 - \pi| > \eta_0$ on $\boldsymbol{R}^3 \setminus \Lambda_\kappa$ for some $\kappa > 0$ and $\eta_0 > 0$, where we choose $0 \le \phi_1 < 2\pi$ as the branch of $\boldsymbol{R}/2\pi\boldsymbol{Z}$.

In the sequel, we use only new functions τ_1 and ϕ_1, so we omit the subscript 1 and write $\tau = \tau_1$ and $\phi = \phi_1$.

Let $\psi \in C^\infty(\boldsymbol{R}/2\pi\boldsymbol{Z}; \boldsymbol{R})$ satisfying

$$\int_0^{2\pi} \psi(s)ds = 2\pi\alpha. \tag{3.1}$$

Define the vector potential A_0 by

$$A_0 = \psi(\phi)\nabla\phi. \tag{3.2}$$

Then, with the use of (2.3), we have

$$A_0 = \psi(\phi)\nabla\phi = \frac{1}{\tau}\psi(\phi)e_\phi \quad \text{in } \mathcal{T}_{\epsilon_0} \tag{3.3}$$

and, by (2.2), $A_0 \in C^\infty(\boldsymbol{R}^3 \setminus \mathcal{C}; \boldsymbol{R})^3 \cap L^1_{\mathrm{loc}}(\boldsymbol{R}^3; \boldsymbol{R})^3$. We assume also

(v) $\operatorname{supp}\psi \subset (\pi - \eta_0, \pi + \eta_0)$ for some η_0 with $0 < \eta_0 << 1$.

Then, by the condition (iv) above, $\operatorname{supp} A_0 = \operatorname{supp}\psi(\phi)\nabla\phi \subset \overline{\Lambda_\kappa}$ and hence is compact.

We have also

$$B_0 = \nabla \times A_0 = \psi'(\phi)\nabla\phi \times \nabla\phi + \psi(\phi)(\nabla \times \nabla)\phi = 0 \quad \text{on } \boldsymbol{R}^3 \setminus \mathcal{C},$$

so $\operatorname{supp} B_0 \subset \mathcal{C}$. Let $\rho \in C^\infty(\boldsymbol{R};\boldsymbol{R})$ such that $0 \le \rho(r) \le 1$ for every r, $\rho(r) = 0$ for $r \le 1$ and $\rho(r) = 1$ for $r \ge 2$. Put $\rho_n(\tau) = \rho(n\tau)$ for $n = 1, 2, \ldots$. For any $\varphi \in C_0^\infty(\boldsymbol{R}^3)$, we have

$$
\begin{aligned}
& \langle B_0, \varphi\rangle \\
= & \int_{\mathbf{R}^3} A_0 \times \nabla\varphi dx = \lim_{n\to\infty} \int_{\mathbf{R}^3} \rho_n A_0 \times \nabla\varphi dx \\
= & \lim_{n\to\infty} \int_{\mathbf{R}^3} A_0 \times \nabla(\rho_n\varphi) dx - \lim_{n\to\infty} \int_{\mathbf{R}^3} \varphi A_0 \times \nabla\rho_n dx \\
= & \lim_{n\to\infty} \langle B_0, \rho_n\varphi\rangle \\
& - \lim_{n\to\infty} \int_{1/n}^{2/n} d\tau \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \; \varphi \frac{1}{\tau} \psi(\phi) e_\phi \times n\rho'(n\tau) e_\tau \tau (a + \tau\cos\phi) \\
= & -\left(\int_0^{2\pi} \psi(\phi) d\phi \int_0^{2\pi} \varphi(a\cos\theta, a\sin\theta, 0)\, e_\theta \, a d\theta \right) \\
= & -\langle 2\pi\alpha\delta_{\mathcal{C}} e_\theta, \varphi\rangle,
\end{aligned}
$$

where we used (2.2), (2.3) and (3.3) in the third equality, $\operatorname{supp} B_0 \subset \mathcal{C}$, $e_\phi \times e_\tau = e_\theta$, and the Lebesgue dominated convergence theorem in the fourth, and (3.1) in the last. Thus we see $B_0 = \nabla \times A_0$ satisfies the condition (A3).

As for the vector potentials A_ϵ for $\epsilon > 0$, we only give a remark that we can construct them by modifying A_0 in $\mathcal{T}_\epsilon$ so that A_ϵ satisfies (A4).

Let us discuss the gauge invariance for the potential A_0.

Proposition 3.1 *Let $\alpha_1, \alpha_2 \in \boldsymbol{R}$, $\psi_1, \psi_2 \in C^\infty(\boldsymbol{R}/2\pi\boldsymbol{Z};\boldsymbol{R})$ satisfying*

$$\int_0^{2\pi} \psi_j(s) ds = \alpha_j$$

for $j = 1, 2$. Let $A_j = \psi_j(\phi)\nabla\phi$. Assume

$$\alpha_1 - \alpha_2 \in \boldsymbol{Z}. \tag{3.4}$$

Then, there exists $\Phi \in C^\infty(\boldsymbol{R}^3 \setminus \mathcal{C}; \boldsymbol{C})$ such that $|\Phi(x)| = 1$ and

$$(D - A_1)\Phi u = \Phi(D - A_2)u \tag{3.5}$$

for $u \in C_0^\infty(\boldsymbol{R}^3 \setminus \mathcal{C})$.

Proof. Put

$$\Phi(x) = \exp\left(i \int_0^{\phi(x)} (\psi_1(s) - \psi_2(s)) ds \right).$$

The right hand side is independent of the choice of the representative of $\phi(x) \in \boldsymbol{R}/2\pi\boldsymbol{Z}$ by the assumption (3.4), and is smooth in $\boldsymbol{R}^3 \setminus \mathcal{C}$. The equation (3.5) can be checked by direct computation. □

By this proposition, there is some arbitrariness in the choice of the function ψ satisfying (3.1). The simplest choice is the constant function $\psi(\phi) = \alpha$, then

$$A_0 = \alpha \nabla \phi.$$

However, we have chosen ψ so that $\operatorname{supp} \psi \subset [\pi - \eta_0, \pi + \eta_0]$ for some small positive η_0, to obtain a compactly supported vector potential $A_0 = \psi(\phi)\nabla\phi$.

Especially in the torus $\mathcal{T}_{\epsilon_0}$, we have $\nabla\phi = (1/\tau)e_\phi$ by (2.3). So

$$A_\epsilon = A_0 = \frac{1}{\tau}\psi(\phi)e_\phi \tag{3.6}$$

for $\epsilon < \tau < \epsilon_0$. Then, for $0 < \epsilon < \epsilon_0$ and $u \in C_0^\infty(\boldsymbol{R}^3)$, we have by (2.2) and (2.3)

$$\begin{aligned} &\int_{\epsilon<\tau<\epsilon_0} |(D - A_\epsilon)u|^2 dx \\ = &\int_{\epsilon<\tau<\epsilon_0} (|D_\tau u|^2 + |\tau^{-1}(D_\phi - \psi(\phi))u|^2 \\ &\qquad + |(a + \tau\cos\phi)^{-1} D_\theta u|^2)\tau(a + \tau\cos\phi)d\tau d\phi d\theta, \end{aligned} \tag{3.7}$$

where $D_\tau = \frac{1}{i}\frac{\partial}{\partial\tau}$, etc. When $\epsilon = 0$, the equality (3.7) holds for $u \in C_0^\infty(\boldsymbol{R}^3 \setminus \{0\})$. By an integration by parts, we have the explicit form of the operator $\mathcal{L}_0$ in the torus $\mathcal{T}_{\epsilon_0}$ in terms of the coordinate (τ, ϕ, θ):

$$\begin{aligned} \mathcal{L}_0 \;=\; &\frac{1}{\tau(a + \tau\cos\phi)}(D_\tau \tau(a + \tau\cos\phi)D_\tau \\ &+ (D_\phi - \psi(\phi))\tau^{-1}(a + \tau\cos\phi)(D_\phi - \psi(\phi)) \\ &+ (a + \tau\cos\phi)^{-2}D_\theta^2). \end{aligned}$$

4 Hardy type inequality

The Hardy type inequality is first proved by Laptev and Weidl [LW], for the two-dimensional Aharonov-Bohm type magnetic field. An analogy of their result holds for our operators, as stated below.

Proposition 4.1 *Let $\alpha \in \boldsymbol{R}$. Put*

$$C_\alpha = (a + \epsilon_0)^{-1}(a - \epsilon_0)\min_{m\in\mathbf{Z}} |m - \alpha|^2.$$

Then, we have

$$\int_{\epsilon<\tau<\epsilon_0} |(D - A_\epsilon)u|^2\, dx \geq C_\alpha \int_{\epsilon<\tau<\epsilon_0} \frac{1}{\tau^2}|u|^2 dx \tag{4.1}$$

for any $0 < \epsilon < \epsilon_0$ and any $u \in C_0^\infty(\boldsymbol{R}^3)$. When $\epsilon = 0$, (4.1) holds for any $u \in C_0^\infty(\boldsymbol{R}^3 \setminus C)$.

Remark. The constant C_α is positive if and only if $\alpha \in \boldsymbol{R} \setminus \boldsymbol{Z}$.

Proof. By Proposition 3.1, we may assume ψ is the constant function $\psi = \alpha$. For $0 < \tau < \epsilon_0$, we have

$$0 < a - \epsilon_0 < a + \tau \cos\phi < a + \epsilon_0.$$

Thus we have by (2.2) and (3.7)

$$\int_{\epsilon<\tau<\epsilon_0} \frac{1}{\tau^2}|u|^2 dx \leq (a+\epsilon_0) \int_{\epsilon<\tau<\epsilon_0} \frac{1}{\tau}|u|^2 d\tau d\phi d\theta, \tag{4.2}$$

$$\int_{\epsilon<\tau<\epsilon_0} |(D - A_\epsilon)u|^2 \, dx \geq (a-\epsilon_0) \int_{\epsilon<\tau<\epsilon_0} |(D_\phi - \alpha)u|^2 \frac{1}{\tau} d\tau d\phi d\theta. \tag{4.3}$$

Using the Fourier expansion $u = \sum_{m\in\mathbf{Z}} u_m(\tau,\theta)e^{im\phi}$, we have

$$\begin{aligned}
\int_0^{2\pi} |(D_\phi - \alpha)u|^2 d\phi &= 2\pi \sum_{m\in\mathbf{Z}} |(m-\alpha)^2 u_m(\tau,\theta)|^2 \\
&\geq 2\pi \min_{m\in\mathbf{Z}} |m-\alpha|^2 \sum_{m\in\mathbf{Z}} |u_m(\tau,\theta)|^2 \\
&= \min_{m\in\mathbf{Z}} |m-\alpha|^2 \int_0^{2\pi} |u|^2 d\phi. \qquad (4.4)
\end{aligned}$$

Integrating (4.4) with respect to the measure $\frac{1}{\tau} d\tau d\theta$ on $(\tau,\theta) \in (\epsilon,\epsilon_0) \times (0, 2\pi)$ and combining it with (4.2) and (4.3), we have (4.1). □

5 Diamagnetic inequality

In this section we state known facts about the diamagnetic inequality (see [LS], [DIM]) which holds for very wide class of potentials and for general open set:

Proposition 5.1 *Suppose* $A \in (L^2_{\mathrm{loc}}(\boldsymbol{R}^d))^d$, $V \in L^1_{\mathrm{loc}}(\boldsymbol{R}^d), V \geq 0$, Ω *is an open set in* $\boldsymbol{R}^d$. *Define sesqui-linear form* $h_\Omega = h_{A,V,\Omega}$ *and* h^D_Ω *as* $h_\Omega(u,v) = ((D-A)u,(D-A)v) + (Vu,v)$ *with form domain* $\mathcal{Q}(h_\Omega) = \{u \in L^2(\Omega) | (D-A)u \in (L^2(\Omega))^3, V^{1/2}u \in L^2(\Omega)\}$, $h^D_\Omega =$ *the form closure of* $h_\Omega|_{C_0^\infty(\Omega)}$. *Denote* $H^D_\Omega = H^D_{A,V,\Omega}$ *the selfadjoint operator associated with* h^D_Ω. *Then we have the following:*

(1) For $\Omega = \boldsymbol{R}^d$, $C_0^\infty(\boldsymbol{R}^d)$ *is a form core for* $h_{\boldsymbol{R}^d}$, *i.e.* $h_{\boldsymbol{R}^d} = h^D_{\boldsymbol{R}^d}$.

(2) Let $E > 0$ *and* $f \in L^2(\Omega)$. *Then*

$$|(H^D_\Omega + E)^{-1} f(x)| \leq \chi_\Omega (K_0 + E)^{-1} \chi^*_\Omega |f|(x) \quad a.e.\ x \in \Omega$$

where χ_Ω *is the restriction operator* $L^2(\boldsymbol{R}^d) \to L^2(\Omega)$ *and* $K_0 = -\Delta$ *with domain* $H^2(\boldsymbol{R}^d)$.

6 Cauchy sequence

The following lemma says the resolvent of our operators forms a Cauchy sequence in the operator norm.

Lemma 6.1 *Let* $\{A_\epsilon\}_{0<\epsilon<\epsilon_0}$ *be the vector potentials defined in section 3 and* $\{H_\epsilon\}_{0<\epsilon<\epsilon_0}$ *the corresponding self-adjoint operators defined in section 1. Then, we have*

$$\lim_{\epsilon,\epsilon'\to 0} \|(H_\epsilon + 1)^{-1} - (H_{\epsilon'} + 1)^{-1}\| = 0$$

We omit the detail of the proof of Lemma 6.1 since it is very similar to that in [IMS]. We only give several propositions needed and would like only note that the use of the resolvent equation $(H_\epsilon + 1)^{-1} - (H_{\epsilon'} + 1)^{-1} = (H_\epsilon + 1)^{-1}(H_{\epsilon'} - H_\epsilon)(H_{\epsilon'} + 1)^{-1}$ and the function $L_\epsilon(\tau)$ given by (6.3) below is key to our proof.

Let $\chi \in C^\infty(\boldsymbol{R}; \boldsymbol{R})$ such that $0 \leq \chi(t) \leq 1$ and

$$\chi(t) = \begin{cases} 1 & (t \geq 2), \\ 0 & (t \leq 1). \end{cases}$$

Put $\chi_\epsilon(\tau) = \chi(\tau/\epsilon)$ for $\epsilon > 0$.

Proposition 6.2 *Assume* $\alpha \in \boldsymbol{R} \setminus \boldsymbol{Z}$. *Then, there exists* $C_1 > 0$ *dependent only on* α *and* ϵ_0 *(independent of* ϵ*), such that*

$$\left\| \frac{\chi_\epsilon(\tau)}{\tau} (H_\epsilon + 1)^{-\frac{1}{2}} \right\| \leq C_1$$

for any ϵ *with* $0 < 2\epsilon \leq \epsilon_0$.

This proposition is shown by using the Hardy type inequality.

Proposition 6.3 *Let* $M \in L^2(\boldsymbol{R}^3)$. *Then, we have*

$$\|M(H_\epsilon + 1)^{-1}\| \leq C_2 \|M\|_{L^2(\mathbf{R}^3)}, \tag{6.1}$$

where $C_2 = \left(\int_{\mathbf{R}^3} (\xi^2 + 1)^{-2} d\xi)\right)^{1/2} / (2\pi)^{3/2}$.

Proof. It is sufficient to show that

$$\|M(H_\epsilon + 1)^{-1}\|_{HS} \leq C_2 \|M\|_{L^2(\mathbf{R}^3)}, \tag{6.2}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. By the diamagnetic inequality, we have

$$|M(H_\epsilon + 1)^{-1} f| \leq |M|(-\Delta + 1)^{-1}|f| \quad \text{a.e.}$$

The operator $|M|(-\Delta + 1)^{-1}$ has the integral kernel $|M(x)|g(x - y)/(2\pi)^{3/2}$, where g is the inverse Fourier transform of the function $(\xi^2 + 1)^{-1}$. Thus (6.2) follows from the Plancherel theorem. □

Take $\eta \in C^\infty(\boldsymbol{R}_+)$ such that $0 \le \eta \le 1$ and

$$\eta(s) = \begin{cases} 0 & (s \ge \epsilon_0), \\ 1 & (s \le \epsilon_0/2). \end{cases}$$

For $0 < 4\epsilon < \epsilon_0$, put

$$L_\epsilon(\tau) = \eta(\tau) \int_\tau^{\epsilon_0/2} \frac{\chi_\epsilon(s)}{s} ds. \tag{6.3}$$

Then we have

$$|L_\epsilon(\tau)| \le \left|\log \frac{\epsilon_0}{2\tau}\right| \tag{6.4}$$

for $0 < \tau \le \epsilon_0$, and

$$L_\epsilon(\tau) \ge \begin{cases} \log(\epsilon_0/4\epsilon) & (0 < \tau < 2\epsilon), \\ \log(\epsilon_0/2\tau) & (2\epsilon \le \tau < \epsilon_0/2). \end{cases} \tag{6.5}$$

Proposition 6.4 *There exists a constant $C_3 > 0$ independent of ϵ and γ such that*

$$\|L_\epsilon^{2\gamma}(H_\epsilon + 1)^{-\gamma}\| \le C_3 \tag{6.6}$$

for $0 < 4\epsilon < \epsilon_0$ and $0 \le \gamma \le 1$.

We can show this proposition by the interpolation theorem from Proposition 6.3.

7 Form domain of H_0

We can specify explicitly the form domain of the operator H_0. Define a sesqui-linear form h_0 by

$$\begin{aligned} h_0(u,v) &= (\mathcal{L}_0 u, v) = ((D - A_0)u, (D - A_0)v), \\ Q(h_0) &= C_0^\infty(\boldsymbol{R}^3 \setminus \mathcal{C}). \end{aligned}$$

Let $\overline{h_0}$ be the closure of the form h_0. The operator H_0 is the self-adjoint operator associated with the form $\overline{h_0}$.

Proposition 7.1 *Suppose $\alpha \in \boldsymbol{R} \setminus \boldsymbol{Z}$. Then, we have*

$$Q(\overline{h_0}) = \left\{ u \in L^2(\boldsymbol{R}^3) \;\middle|\; (D - A_0)u \in L^2(\boldsymbol{R}^3)^3,\ \frac{1}{\tau} u \in L^2(\boldsymbol{R}^3) \right\},$$

where the distribution $Du = -i\nabla u$ is defined as an element of $\mathcal{D}'(\boldsymbol{R}^3 \setminus \mathcal{C})^3$.

For the proof of Proposition 7.1, we shall use a lemma.

Lemma 7.2 *Assume $u \in L^2(\boldsymbol{R}^3)$, $(D - A_0)u \in L^2(\boldsymbol{R}^3)^3$ and $\operatorname{supp} u \cap \mathcal{C} = \emptyset$. Then $u \in Q(\overline{h_0})$.*

Proposition 7.1 and Lemma 7.2 can be shown by using usual cutoff argument and making the approxmating sequence of functions (see [IMS]).

8 Sketch of the proof of the main theorems

By Lemma 6.1, there exists a bounded, self-adjoint operator R on $L^2(\boldsymbol{R}^3)$ such that

$$R = \lim_{\epsilon \to 0}(H_\epsilon + 1)^{-1}.$$

Thus the proof of Theorem 1.1 is completed if we prove

$$R = (H_0 + 1)^{-1}. \tag{8.1}$$

For the proof we use a series of lemmas which are shown with the use of the Hardy type inequality.

Lemma 8.1 *The operator R is injective.*

By Lemma 8.1, we can define a self-adjoint operator T by

$$T = R^{-1} - 1, \quad D(T) = \operatorname{Ran} R.$$

Then T is self-adjoint and $R = (T+1)^{-1}$. Thus it suffices to prove $T = H_0$.

Lemma 8.2 *For $u \in D(T)$, we have*

$$Tu = \mathcal{L}_0 u = (D - A_0)^2 u, \tag{8.2}$$

where $\mathcal{L}_0 u$ is defined as an element of $\mathcal{D}'(\boldsymbol{R}^3 \setminus \mathcal{C})$.

Lemma 8.3 *We have $D(T) \supset C_0^\infty(\boldsymbol{R}^3 \setminus \mathcal{C})$.*

Lemma 8.4 *Suppose $\alpha \in \boldsymbol{R} \setminus \boldsymbol{Z}$. Then, we have $D(T) \subset Q(\overline{h_0})$.*

Lemma 8.3 and Lemma 8.3 implies T is a self-adjoint extension of $\mathcal{L}_0|_{C_0^\infty(\mathbf{R}^3 \setminus \mathcal{C})}$. Since the Friedrichs extension H_0 is the unique self-adjoint extension of L_0 with the property $D(H_0) \subset Q(\overline{h_0})$, we have $H = T_0$ by Lemma 8.4. Thus Theorem 1.1 is proved. The proof of Theorem 1.2 is quite similar.

9 Spectral and Scattering theory

In this section, we study the spectral properties of H_0 and develop the scattering theory for the pair (H_0, K_0), where K_0 denotes the free hamiltomian $K_0 = -\Delta$ with $D(K_0) = H^2(\boldsymbol{R}^3)$.

Theorem 9.1 *The operator H_0 has no eigenvalue.*

Proof. It suffices to show that H_0 has no nonnegative eigenvalue, since H_0 is nonnegative. First assume that there exist $u \in D(H_0)$ and $\lambda > 0$ such that $H_0 u = \lambda u$. Then, since A_0 is smooth except the circle $\mathcal{C} = \{\tau = 0\}$, it follows from Lemma 8.3 and the elliptic regularity that u is smooth except $\mathcal{C}$. Moreover, since A_0 has a compact support, u satisfies the Helmholtz equation $(\Delta + \lambda)u = 0$ on $\{|x| > R\}$(some large $R > 0$), which implies u must vanish on that exterior region by [M, Lemma 8.4]. Thus, noting the unique continuation property for the eliptic equations (e.g. [H]), we have $u = 0$ in $L^2(\boldsymbol{R}^3)$. Next assume that there exists $u \in D(H_0)$ such that $H_0 u = 0$. Then, we have

$$0 = (H_0 u, u) = \overline{h_0}(u, u) = ||(D - A_0)u||^2.$$

Hence $Du(x) = 0$ on $\{|x| > R\}$(some large $R > 0$), which implies u must vanish on that exterior region, since $u \in L^2(\boldsymbol{R}^3)$. Thus, the unique continuation property again shows $u = 0$ in $L^2(\boldsymbol{R}^3)$. □

Let us proceed to the scattering problems for the pair (H_0, K_0). The wave operators $W_\pm(H_0, K_0)$ are defined by

$$W_\pm(H_0, K_0) = s - \lim_{t \to \pm\infty} e^{itH_0} e^{-itK_0},$$

if they exist. We use the Enss method and know the following ([P, p.106, Theorem 8.1; p.108, Proposition 8.1]).

Theorem 9.2 *Let H be a self-adjoint operator on $L^2(\boldsymbol{R}^d)$ such that*

(s1) $(H - z)^{-1} - (K_0 - z)^{-1}$ *is compact.*

(s2) *The function $h(R) = ||j_R(K_0 + i)^{-1} - (H + i)^{-1}(K_0 + i)j_R(K_0 + i)^{-1}||$ is integrable on $(0, \infty)$, where $j_R(x) = \varphi(\frac{|x|}{R})$ and $\varphi \in C^\infty(\boldsymbol{R}; \boldsymbol{R})$ is taken such that*

$$0 \le \varphi(s) \le 1 (\forall s \in \boldsymbol{R}), \quad \varphi(s) = 0(|s| \le 1), \quad = 1(|s| \ge 2).$$

Then :

(i) $\sigma_{ess}(H) = [0, \infty)$ *, where $\sigma_{ess}(H)$ denotes the essential spectrum of H.*

(ii) *H has empty singular continuous spectrum.*

(iii) *The wave operators $W_\pm(H, K_0)$ exist and are complete.*

We apply the above theorem to obtain the following.

Theorem 9.3 *We have :*

(i) $\sigma(H_0) = \sigma_{abs}(H_0) = \sigma_{ess}(H_0) = [0, \infty)$, *where $\sigma(H_0)$ and $\sigma_{abs}(H_0)$ denote the spectrum of H_0 and the absolutely continuous spectrum of H_0, respectively.*

(ii) *The wave operators $W_\pm(H_0, K_0)$ exist and are complete.*

Proof. We first show **(s1)** for (H_0, K_0) with $z = -1$. We write

$$\begin{aligned}(H_0+1)^{-1}-(K_0+1)^{-1} &= \{(H_0+1)^{-1}-(H_\epsilon+1)^{-1}\}+\{(H_\epsilon+1)^{-1}-(K_0+1)^{-1}\}\\ &= I_\epsilon + J_\epsilon.\end{aligned}$$

In view of Theorem 1.1, $I_\epsilon \to 0$ in the operater norm as $\epsilon \downarrow 0$. On the other hand, the resolvent equation reads as

$$\begin{aligned}J_\epsilon &= (H_\epsilon+1)^{-1}\{D\cdot A_\epsilon + 2A_\epsilon\cdot D - |A_\epsilon|^2\}(K_0+1)^{-1}\\ &= (H_\epsilon+1)^{-1}V(x,\partial)(K_0+1)^{-1},\end{aligned}$$

which implies J_ϵ is compact, since $V(x,\partial)(K_0+1)^{-1}$ is compact by (A4). So, **(s1)** holds, since the set of compact operators is closed in $\boldsymbol{B}(L^2(\boldsymbol{R}^3))$.

Next, we show that **(s2)** holds for (H_0, K_0). It suffcies to show that $\boldsymbol{B}(L^2(\boldsymbol{R}^3))$-valued function

$$B(R) = j_R(K_0+i)^{-1} - (H_0+i)^{-1}(K_0+i)j_R(K_0+i)^{-1}$$

vanishes for large R. Take $R_0 > 0$ such that $A_0(x) = 0$ for $|x| > R_0$ and put $u = (K_0+i)^{-1}f, v = (H_0-i)^{-1}g$ for $f, g \in L^2(\boldsymbol{R}^3)$. Then we have

$$\begin{aligned}(B(R)f,g) &= (\{j_R(K_0+i)^{-1}-(H_0+i)^{-1}(K_0+i)j_R(K_0+i)^{-1}\}f,g)\\ &= (j_Ru,(H_0-i)v) - ((K_0+i)(j_Ru),v)\\ &= (j_Ru,H_0v)-(K_0(j_Ru),v).\end{aligned}$$

Now, let us show that for $R > R_0$

$$(j_Ru, H_0v) = (K_0(j_Ru), v).$$

In fact, since $u \in D(K_0) = H^2(\boldsymbol{R}^3)$, we can find a sequence $\{u_n\} \subset H^2(\boldsymbol{R}^3)$ such that $u_n \to u$ in $H^2(\boldsymbol{R}^3)$ as $n \to \infty$. Then, noting that $j_Ru_n \in C_0^\infty(\{|x| > R_0\}) \subset C_0^\infty(\boldsymbol{R}^3 \setminus C)$, we have by Lemma 7.2

$$\begin{aligned}(j_Ru,H_0v) &= \lim_{n\to\infty}(j_Ru_n,H_0v)\\ &= \lim_{n\to\infty}((D-A_0)^2(j_Ru_n),v)\\ &= \lim_{n\to\infty}(K_0(j_Ru_n),v)\\ &= (K_0(j_Ru),v).\end{aligned}$$

The above argument shows $B(R) = 0$ for $R > R_0$, and hence **(s2)** holds. Therefore the second part of the theorem has been proven. The first part follows from Theorems 9.1, 9.2 **(i)**, **(ii)**. □

参考文献

[A] G. N. Afanasiev; *Topological effects in quantum mechanics*, Fundamental Theories of Physics, 107. Kluwer Academic Publishers Group, Dordrecht, 1999.

[AB] Y. Aharonov and D. Bohm; Significance of electromagnetic potentials in the quantum theory, *Phys. Rev.* **115** (1959) 485–491.

[AGHH] S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden; *Solvable models in quantum mechanics. Second edition. With an appendix by Pavel Exner.* AMS Chelsea Publishing, Providence, RI, 2005.

[BW] M. Ballesteros and R. Weder; High-velocity estimates for the scattering operator and Aharonov-Bohm effect in three dimensions, *Comm. Math. Phys.* **285** (2009), 345–398.

[DIM] S. Doi, A. Iwatsuka and T. Mine; The uniqueness of the integrated density of states for the Schrödinger operators with magnetic fields, *Math. Z.* **237** (2001), 335–371.

[H] L. Hörmander; Uniqueness theorems for second order elliptic differential equations, *Comm. in Partial Differential Equations.* **8** (1), (1983), 21–64.

[IK] T. Ikebe and T. Kato; Uniqueness of the self-adjoint extension of singular elliptic differential operators, *Arch. Rational Mech. Anal.* **9** (1962), 77–92.

[IMS] A. Iwatsuka, T. Mine and S. Shimada; Norm resolvent convergence to Schrödinger operators with infinitesimally thin toroidal magnetic fields, *Spectral and Scattering Theory for Quantum Magnetic Systems* (CIRM, Marseille, 2008), Contemporary Mathematics **500** (2009), 139–151.

[LW] A. Laptev and T. Weidl; Hardy inequalities for magnetic Dirichlet forms, *Mathematical results in quantum mechanics* (Prague, 1998), *Oper. Theory Adv. Appl.* **108** (1999), 299–305.

[LS] H. Leinfelder and C. G. Simadar, Schrödinger operators with singular magnetic vector potentials, *Math. Z.* **176** (1981), 1–19.

[M] S. Mizohata; *The theory of partial differential equations*, Cambridge University Press, New York, 1973.

[P] P. A. Perry; *Scattering theory by the Enss method*, Harwood academic publishers gmbh, 1983.

[PT] M. Peshkin and A. Tonomura; *The Aharonov-Bohm effect*, Lecture Notes in Physics **340**, Springer-Verlag, Berlin, 1989.

[Ta] H. Tamura; Norm resolvent convergence to magnetic Schrödinger operators with point interactions, *Rev. Math. Phys.* **13** (2001), no. 4, 465–511.

[To] A. Tonomura, N. Osakabe, T. Matsuda, T. Kawasaki, J. Endo, S. Yano and H. Yamada; Evidence for Aharonov-Bohm effect with magnetic field completely shielded from electron wave, *Phys. Rev. Lett.* **56**, No. 8 (1986), 792–795.