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Markoff spectra, geodesics, palindromes

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We investigated the Markoff spectrum for $\mathbb{Q}(i)$ from the point of view of geometry and combinatorics. The investigation made into the Markoff spectrum for $\mathbb{Q}$ is a model of this work. The object of geometric study is a simple closed geodesic on a special once punctured torus. That of combinatorial study is a word representing a matrix whose axis projects to the simple closed geodesic on the once punctured torus. Through the word, we obtain a continued fraction expansion of the matrix. Both the word and the continued fraction expansion mean codings of the simple closed geodesic. In this expository note, using such geodesics and words, we show and compare characterizations of the Markoff spectra for $\mathbb{Q}$ and for $\mathbb{Q}(i)$. Some results are obtained in collaboration with I.R.Aitchison in a geometric aspect and with B.Rittaud in a combinatorial aspect.

1 Markoff spectra

Let $f(x,y)=ax^2+bxy+cy^2$ be an indefinite quadratic form with real coefficients and with discriminant $D(f)=b^2-4ac$. Define $m(f)=\inf_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}}|f(x,y)|$. The *Markoff spectrum for* $\mathbb{Q}$ is defined as the set

$$\mathcal{M}=\left\{\sqrt{D(f)}/m(f)\,\middle|\,(a,b,c)\in\mathbb{R}^3, D(f)>0\right\}.$$

The *Markoff spectrum for* $\mathbb{Q}(i)$ is defined in the same way:

$$\mathcal{M}_1=\left\{\sqrt{|D(f)|}/m_1(f)\,\middle|\,(a,b,c)\in\mathbb{C}^3, D(f)\neq 0\right\}$$

where $m_1(f)=\inf_{(x,y)\in\mathbb{Z}[i]^2-\{(0,0)\}}|f(x,y)|$.

We recall that the discrete parts of these spectra are described by using solutions of Diophantine equations.

Markoff triples are integral solutions (p,q,r) of *Markoff's equation* $p^2+q^2+r^2=3pqr$. In this note we suppose $1\le p\le q\le r$. We easily verify that $(1,1,1)$ and $(1,1,2)$ are Markoff triples. The latter is the unique triple derived from the former. These are the only Markoff triples that p,q,r are not distinct. All the Markoff triples consisting of distinct positive integers are obtained by building an infinite binary tree starting from $(1,2,5)$. Here a binary tree means that each node of a tree has two children. To build the tree we use inductively the following operation: for a Markoff triple $(p,q,r)\neq(1,1,1),(1,1,2)$ we take

$$(p,r,3pr-q)\ \text{as its left child;}\quad (q,r,3qr-p)\ \text{as its right child.}$$

The tree constructed by this operation is called the *Markoff tree.* It is known that all the Markoff triples, except for $(1,1,1)$ and $(1,1,2)$, occur only once in the Markoff tree (see [CF]).

A member of the Markoff triples is called a *Markoff number.* Let K denote the set of the Markoff numbers:

$$K = \{1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, \ldots\}.$$

A.Markoff proved the following theorem (see [CF]).

Theorem 1.1. *The set* $\mathcal{M} \cap (0,3)$ *is described as* $\left\{ \sqrt{9-(4/k^2)} \,\middle|\, k \in K \right\}$.

Note that $\mathcal{M} \cap [3, \infty)$ is not discrete.

Vulakh-Schmidt (VS) quadruples are integral solutions $(x_1, x_2; y_1, y_2)$ of Vulakh's equations: $x_1 + x_2 = 2y_1y_2$, $2x_1x_2 = y_1^2 + y_2^2$ (see [V] and [S]). In this note we suppose $1 \leq x_1 \leq x_2$, $1 \leq y_1 \leq y_2$. We easily verify that $(1,1;1,1)$ and $(1,5;1,3)$ are VS quadruples. The former is the only quadruple that $x_1 = x_2$ or $y_1 = y_2$. All the VS quadruples are obtained by building an infinite ternary tree starting from $(1,1;1,1)$. Here a ternary tree means that each node of a tree has three children. To build the tree we use inductively the following operation: for a VS quadruple $(x_1, x_2; y_1, y_2)$ we take

$$\begin{array}{ll}
(x_1, 2y_2(4x_1y_2 - y_1) - x_1; y_2, 4x_1y_2 - y_1) & \text{as its left child,} \\
(x_2, 2y_2(4x_2y_2 - y_1) - x_2; y_2, 4x_2y_2 - y_1) & \text{as its center child,} \\
(x_2, 2y_1(4x_2y_1 - y_2) - x_2; y_1, 4x_2y_1 - y_2) & \text{as its right child.}
\end{array}$$

(See §5.2 in [S].)

Let us define a set $\mathcal{N}(\Lambda)$ as the set of members x_1, x_2 and a set $\mathcal{N}(M)$ as the set of members y_1, y_2 of VS quadruples:

$$\mathcal{N}(\Lambda) = \{1, 5, 29, 65, 169, 349, 901, 985, 4549, 5741, \ldots\},$$

$$\mathcal{N}(M) = \{1, 3, 11, 17, 41, 59, 99, 153, 339, 571, 577, \ldots\}.$$

The following theorem is showed in [V] and [S].

Theorem 1.2. *The discrete part of* $\mathcal{M}_1$ *is described as*

$$\left\{ \sqrt{4 - \frac{1}{\lambda^2}} \,\middle|\, \lambda \in \mathcal{N}(\Lambda) \right\} \cup \left\{ \sqrt{3\sqrt{41}/5} \right\}.$$

The subset of the Markoff spectrum for $\mathbb{Q}(i)$ described by using $\mathcal{N}(\Lambda)$ is called the *VS spectrum.*

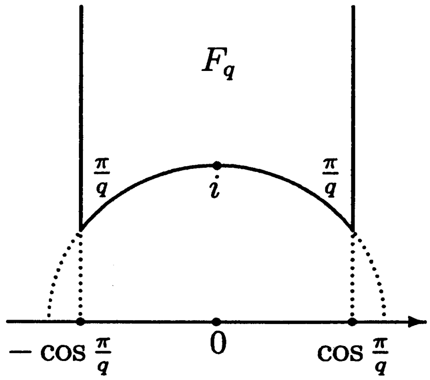

Figure 1: Fundamental domain of G_q.

2 Simple closed geodesics

Let $\mathbb{H}^2 = \{z = x+iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane endowed with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. A geodesic in $\mathbb{H}^2$ is a semicircle or a ray perpendicular to the real axis. The group $\mathrm{PSL}(2,\mathbb{R})$ acts on the upper half-plane $\mathbb{H}^2$ by fractional linear transformations. We always identify a matrix $g \in \mathrm{PSL}(2,\mathbb{R})$ with the fractional linear transformation induced by g.

Hecke groups G_q are the groups generated by two matrices T_q and S, where

$$T_q = \begin{pmatrix} 1 & 2\cos(\pi/q) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for integers $q \geq 3$. They are discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. A fundamental domain of G_q is represented as follows:

$$F_q = \left\{ x + iy \in \mathbb{H}^2 \,\middle|\, x^2 + y^2 \geq 1,\ |x| \leq \cos\left(\frac{\pi}{q}\right) \right\}$$

(see Figure 1). For $q = 3$ we have the modular group $\mathrm{PSL}(2,\mathbb{Z})$. We use in this note G_3 and G_4.

A *Fricke group* is a free group generated by two hyperbolic elements C, D of $\mathrm{PSL}(2,\mathbb{R})$ such that the commutator $[C, D]$ is parabolic. The quotient space of $\mathbb{H}^2$ by a Fricke group is identified with a once punctured torus. Let us write $X = \mathrm{tr}(C)$, $Y = \mathrm{tr}(D)$ and $Z = \mathrm{tr}(CD)$. It is known that $\langle C, D\rangle$ is a Fricke group if and only if the triple (X, Y, Z) satisfies Fricke's equation $X^2 + Y^2 + Z^2 = XYZ$ and $X, Y, Z > 2$.

Let us introduce two special Fricke groups (see [C2]). We consider a free group $\Gamma_3^o = \langle A_3, B_3\rangle$ generated by

$$A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

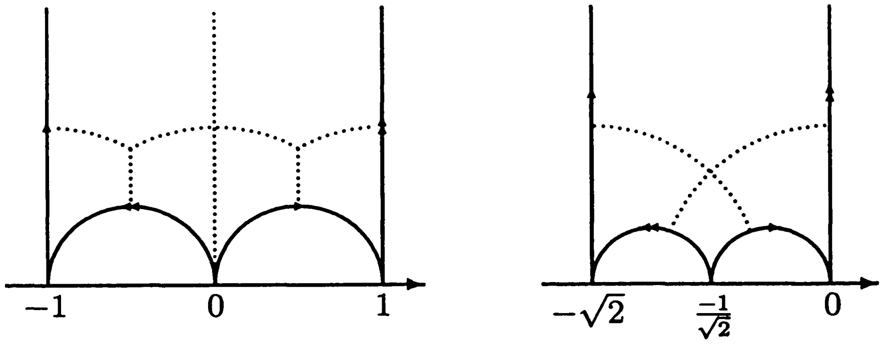

Figure 2: Fundamental domains of Γ_3° and Γ_4°. In the left, the sides with one pile are identified by A_3; the sides with two piles are identified by B_3. In the right, the sides with one pile are identified by A_4; the sides with two piles are identified by B_4.

The quotient space $\mathbb{H}^2/\Gamma_3^\circ$ is a once punctured torus denoted by $\mathbb{T}_3^\circ$. The group Γ_3° has $(3,3,3)$. This is a torsion-free normal subgroup of G_3 with index 6. Equivalently, $\mathbb{T}_3^\circ$ is a six-fold cover of the modular surface. (See Figure 2.)

Let $\Gamma_4^\circ = \langle A_4, B_4 \rangle$ be a free group generated by

$$A_4 = \begin{pmatrix} 2\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B_4 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

This is a torsion-free normal subgroup of G_4 with index 4. The quotient space $\mathbb{T}_4^\circ = \mathbb{H}^2/\Gamma_4^\circ$ is also a once punctured torus, which is a four-fold cover of a fundamental region of G_4. The group Γ_4° has $(2\sqrt{2}, 2\sqrt{2}, 4)$. (See Figure 2.)

In general, if we abelianize a Fricke group $\langle C, D \rangle$, then the commutator $[C, D]$ becomes the identity; geometrically, the cusp of the quotient space disappears. We thus have the closed torus corresponding to a once punctured torus. The groups Γ_3° and Γ_4° are special in the following sense (see [C1], [C2], [A]): by abelianization the closed torus $\mathbb{C}/\langle 1, \rho \rangle$ is obtained from $\mathbb{T}_3^\circ$ and $\mathbb{C}/\langle 1, i \rangle$ is obtained from $\mathbb{T}_4^\circ$, where 1, ρ, and i mean the translations on $\mathbb{C}$: $z \mapsto z+1$, $z \mapsto z+\rho$ (with $\rho = e^{\frac{2}{3}\pi i}$) and $z \mapsto z+i$, respectively.

In order to interpret geometrically the Markoff spectrum for $\mathbb{Q}$, H.Cohn introduced in [C1] triples of matrices each of which is a generator of Γ_3°. Here "a generator of a group" is used in the following sense: let $\langle C, D \rangle$ be a free group; an element C' is called a *generator* of $\langle C, D \rangle$ if there exists a D' satisfying $\langle C, D \rangle = \langle C', D' \rangle$. We proposed in [AR2] an algorithm constructing an infinite binary tree of triples of matrices such that every matrix in the tree is defined by using two special matrices $N_1 = B_3^{-1}$ and $N_2 = B_3^{-1} A_3^{-1} B_3^{-1}$.

Let us show the construction of the tree. Each node of the tree has type I or II. The type is determined in the following way: the root has type I; the left child of a node has the same type as the parent; the right child has the other type. We assign a triple of matrices $(N_1, N_2, N_5 = N_2N_1)$ to the root, where

$$N_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \quad N_5 = \begin{pmatrix} 12 & 7 \\ 5 & 3 \end{pmatrix}.$$

Let (N, N', N'') be a triple of matrices in a node of the tree. The children of (N, N', N'') are defined by using the following algorithm.

Algorithm CM.

- If the node is of type I, then the left child is $(N, N'', N''N)$ and the right child is $(N', N'', N'N'')$.
- If the node is of type II, then the left child is (N, N'', NN'') and the right child is $(N', N'', N''N')$.

We call the tree thus obtained the *Cohn-Markoff tree* and a matrix appearing in it a *Cohn-Markoff (CM) matrix.*

We can prove the following theorem by induction (see [AR2]).

Theorem 2.1. *Each Cohn-Markoff matrix has a form*

$$N_k = \begin{pmatrix} a & b \\ k & d \end{pmatrix} \in G_3 \quad \textit{and} \quad \mathrm{tr}(N_k) = 3k,$$

where k is a Markoff number.

Note that the theorem contains the following assertion: if (N_p, N_q, N_r) is a triple of matrices in the Cohn-Markoff tree, the $(2,1)$-entry of the third matrix N_rN_p (or N_pN_r) of its left child is $3pr - q$; the $(2,1)$-entry of the third matrix N_qN_r (or N_rN_q) of its right child is $3qr - p$.

Recall that the *Euclidean height* of a geodesic $\tilde{\gamma}$ in $\mathbb{H}^2$ is defined by $|\eta - \xi|/2$ if η and ξ are finite or by ∞ otherwise, where η and ξ are the two endpoints of $\tilde{\gamma}$. We now state Cohn's theorem:

Theorem 2.2. *The discrete part of the Markoff spectrum for $\mathbb{Q}$ is given by the Euclidean heights of the lifts of the simple closed geodesics on the once punctured torus $\mathbb{T}_3^\circ$.*

Proof. It is known that the outer automorphism group of the free group on C and D is generated by the three operations: exchanging C and D; replacing C by C^{-1}; replacing C by CD. Combining this fact with Algorithms CM, we have each CM matrix is a generator of Γ_3°. Thanks to the following proposition by Nielsen (see [H]), the axis of each CM matrix projects to a simple closed geodesic on $\mathbb{T}_3^\circ$.

Proposition 2.1. *Let $\langle C, D\rangle$ be a Fricke group. A geodesic $\tilde{\gamma}$ in $\mathbb{H}^2$ is the axis of a generator of $\langle C, D\rangle$ if and only if $\tilde{\gamma}$ projects to a simple closed geodesic on the once punctured torus $\mathbb{H}^2/\langle C, D\rangle$.*

Moreover, we can directly compute the Euclidean height of the axis of a CM matrix N_k: $(1/2)\sqrt{9-(4/k^2)}$. $\square$

In order to interpret geometrically the Markoff spectrum for $\mathbb{Q}(i)$, we established in [AA] an algorithm building an infinite ternary tree of quadruples of matrices such that each matrix in the tree is defined by using A_4 and B_4. Let us show the construction of the tree. Each node has type I or II. The type is determined by the rules: the root has type I; the left and right children of a node have the same type as the parent, the center child has the other type. We assign a quadruple of matrices $(\Lambda_1, \Lambda_5; M_1, M_3)$ to the root, where

$$\Lambda_1 = \begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} = B_4 A_4, \quad \Lambda_5 = \begin{pmatrix} 13 & 9\sqrt{2} \\ 5\sqrt{2} & 7 \end{pmatrix} = M_3 M_1,$$

$$M_1 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix} = B_4, \quad M_3 = \begin{pmatrix} 4\sqrt{2} & 5 \\ 3 & 2\sqrt{2} \end{pmatrix} = \Lambda_1 M_1.$$

Let $(\Lambda, \Lambda'; M, M')$ be a quadruple of matrices in a node of the tree. The children of $(\Lambda, \Lambda'; M, M')$ are defined by using the following algorithm.

Algorithm VS.

- If the node is of type I, then the left child is $(\Lambda, \Lambda(M')^2; M', \Lambda M')$, the center child is $(\Lambda', (M')^2\Lambda'; M', M'\Lambda')$ and the right child is $(\Lambda', \Lambda' M^2;\ M, \Lambda' M)$.
- If the node is of type II, then the left child is $(\Lambda, (M')^2\Lambda; M', M'\Lambda)$, the center child is $(\Lambda', \Lambda'(M')^2; M', \Lambda' M')$ and the right child is $(\Lambda', M^2\Lambda';\ M, M\Lambda')$.

The tree built by Algorithm VS is called the *VS tree*.

The following theorem is proved by induction (see [AR2]).

Theorem 2.3. • *Every matrix which is the first or the second element in any quadruple of the VS tree has a form:*

$$\Lambda_\lambda = \begin{pmatrix} a & \sqrt{2}b \\ \sqrt{2}\lambda & d \end{pmatrix} \in G_4 \text{ and } \operatorname{tr}(\Lambda_\lambda) = 4\lambda \text{ where } \lambda \in \mathcal{N}(\Lambda).$$

- *Every matrix which is the third or the fourth element in any quadruple of the VS tree has a form:*

$$M_m = \begin{pmatrix} \sqrt{2}\alpha & \beta \\ m & \sqrt{2}\delta \end{pmatrix} \in G_4 \text{ and } \operatorname{tr}(M_m) = 2\sqrt{2}m \text{ where } m \in \mathcal{N}(M).$$

Note that (a, b, d) and (α, β, δ) are in $\mathbb{Z}^3$. Note also that the theorem contains the following assertion: if $(\Lambda_{x_1}, \Lambda_{x_2}; M_{y_1}, M_{y_2})$ is a quadruple of matrices in the VS tree, for its left child the $(2,1)$-entry of the second matrix $\Lambda_{x_1} M_{y_2}^2$ (or $M_{y_2}^2 \Lambda_{x_1}$) is $\sqrt{2}(2y_2(4x_1y_2 - y_1) - x_1)$, that of the fourth matrix $\Lambda_{x_1} M_{y_2}$ (or $M_{y_2}\Lambda_{x_1}$) is $4x_1y_2 - y_1$; for its center child the $(2,1)$-entry of $M_{y_2}^2 \Lambda_{x_2}$ (or $\Lambda_{x_2} M_{y_2}^2$) is $\sqrt{2}(2y_2(4x_2y_2 - y_1) - x_2)$, that of $M_{y_2}\Lambda_{x_2}$ (or $\Lambda_{x_2} M_{y_2}$) is $4x_2y_2 - y_1$; for its right child the $(2,1)$-entry of $\Lambda_{x_2} M_{y_1}^2$ (or $M_{y_1}^2 \Lambda_{x_2}$) is $\sqrt{2}(2y_1(4x_2y_1 - y_2) - x_2)$, that of $\Lambda_{x_2} M_{y_1}$ (or $M_{y_1}\Lambda_{x_2}$) is $4x_2y_1 - y_2$.

The following proposition is obtained from Algorithm VS and Proposition 2.1.

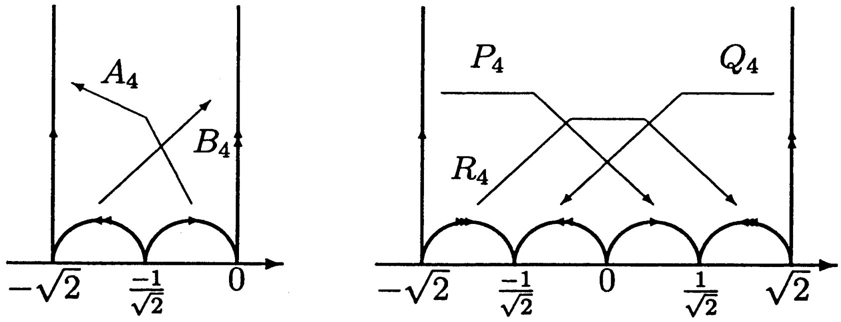

Figure 3: Fundamental domains of Γ_4^o and $\langle P_4, Q_4, R_4\rangle$.

Proposition 2.2. *The axes of the matrices Λ_λ and M_m in the VS tree project to simple closed geodesics on the once punctured torus $\mathbb{T}_4^o$.*

Using this, we proved the following theorem in [AA].

Theorem 2.4. *The VS spectrum is given by the Euclidean heights of the axes of the matrices Λ_λ in the VS tree. These axes project to simple closed geodesics on a particular immersed totally geodesic twice punctured torus in the Borromean rings complement.*

The Borromean rings complement is realized as a quotient space of the upper half-space $\mathbb{H}^3$ by a special subgroup of the Picard group $\mathrm{PSL}(2,\mathbb{Z}[i])$ with index 24. In the proof of this theorem the following fact is crucial: the twice punctured torus in the Borromean rings complement is conformally equivalent to a twice punctured torus $\mathbb{H}^2/\langle P_4, Q_4, R_4\rangle$, denoted by $\mathbb{T}_4^t$, where $P_4 = B_4 A_4^{-1}$, $Q_4 = A_4^{-1}B_4^{-1}$, $R_4 = B_4^2$:

$$P_4 = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 3 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{pmatrix}.$$

Moreover, $\mathbb{T}_4^t$ is a double covering of the once punctured torus $\mathbb{T}_4^o$. (See Figure 3.)

3 Palindromes

Let $w = a_0 \dots a_n$ be a finite word on an alphabet $\mathcal{A}$. We write the reversed word $a_n \dots a_0$ by w^*. The word w is called a *palindrome* if $w = w^*$. Let ℓ be a letter of $\mathcal{A}$. The notation $\#_\ell(w)$ stands for the number of occurrences of ℓ in the word w.

3.1 Cohn-Markoff case

Algorithm CM allows us to write a CM-matrix N_k as a finite word on the alphabet $\{N_1, N_2\}$. We call this word the (N_1, N_2)-*word* of N_k. For the sake of simplicity, in this subsection, we usually use the letters 1 and 2 instead of N_1 and N_2.

Theorem 3.1. *For $k \in K$, $k \geq 5$ we have $N_k = 2\sigma 1$ where σ is a palindrome on $\{1, 2\}$.*

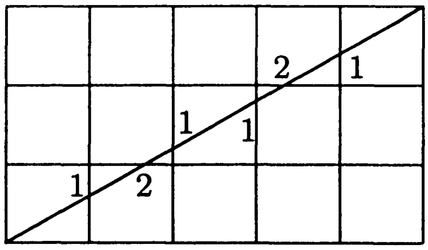

Figure 4: The cutting sequence of $(5, 3)$: 121121

The theorem is proved inductively by using Algorithm CM (see [AR2]). We show below how to find the (N_1, N_2)-word of N_k without knowing the (N_1, N_2)-words of the matrices in the node of its parent.

Let (u, v) be a pair of mutually prime positive integers. Take a rectangle whose horizontal length is u and vertical length is v, and tile this rectangle by unit squares. Consider the diagonal of this rectangle from the left-bottom corner to the top-right corner. Going up along the diagonal, we write 1 each time we cross a vertical line, and 2 each time we cross a horizontal line. The *cutting sequence* $C(u, v)$ of the pair (u, v) is the sequence of 1s and 2s obtained in this way. (For example, see Figure 4.) Note that the assumption that u and v are mutually prime ensures that the diagonal never passes through a point of integer coordinates. Note also that $C(u, v)$ is always a palindrome.

We define the *Frobenius coordinates* of N_k by the pair

$$(\#_{N_1}(N_k), \#_{N_2}(N_k)).$$

Let us consider a Markoff triple (p, q, r). Theorem 2.1 allows us to find a triple of CM-matrices (N_p, N_q, N_r). Then we can take the triple of the Frobenius coordinates of (N_p, N_q, N_r), denoted by $((p_1, p_2), (q_1, q_2), (r_1, r_2))$. (For example, $((1, 0), (0, 1), (1, 1))$ for $(1, 2, 5)$.) We obtain the following relations from Algorithm CM and the definition of the Frobenius coordinates:

$$(p_1, p_2) + (q_1, q_2) = (r_1, r_2) \quad \text{and} \quad p_1q_2 - p_2q_1 = \pm 1.$$

Using these, we can prove the following theorem.

Theorem 3.2. *Let $2\sigma 1$ be the (N_1, N_2)-word of any CM-matrix N_k apart from N_1 and N_2. Then, σ is the cutting sequence of the Frobenius coordinates of N_k.*

Now we can regard Algorithm CM as an algorithm which makes essentially the cutting sequence of a pair of mutually prime positive integers.

The (N_1, N_2)-word of a CM matrix N_k, in other words, the cutting sequence of its Frobenius coordinates directly relates to the continued fraction expansion of N_k.

Recall that any fractional linear transformation of PSL$(2,\mathbb{Z})$ can be written as

$$z \mapsto a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n \pm z}}},$$

where $a_i \in \mathbb{Z}$ and $a_i \neq 0$ for $i = 1, \dots, n-1$. Let N be a CM-matrix whose (N_1, N_2)-word is equal to $w = 2w_1 \dots w_{n-1}1$. Since N_1 and N_2 have the following expansions:

$$N_1(z) = 1 + \cfrac{1}{1 + \cfrac{1}{z}} \quad \text{and} \quad N_2(z) = 2 + \cfrac{1}{2 + \cfrac{1}{z}},$$

applying to w the substitutions $1 \mapsto 11$ and $2 \mapsto 22$, we have the word $22w_1w_1 \dots w_{n-1}w_{n-1}11$. This gives the *continued fraction expansion* of N. If $w_1 \dots w_{n-1}$ is a palindrome, then $w_1w_1 \dots w_{n-1}w_{n-1}$ is also a palindrome. It is the fact that A.Markoff originally used to prove Theorem 1.1.

3.2 Vulakh-Schmidt case

By the construction of the VS tree, we know every matrix N in the VS tree is represented as a word on the alphabet $\{A_4, B_4\}$. We call this the (A_4, B_4)*-word* of N. We define the *Euclidean pair* (briefly, *E-pair*) of N as

$$(\#_{B_4}(N), \#_{A_4}(N)).$$

Theorem 2.3 ensures that for each $\lambda \in \mathcal{N}(\Lambda)$ there exists Λ_λ and that for each $m \in \mathcal{N}(M)$ there exists M_m. Let denote the E-pair of Λ_λ by (λ_1, λ_2) and that of M_m by (m_1, m_2). For a VS quadruple, we have a quadruple of E-pairs. (For example, $((1,1),(3,1);(1,0),(2,1))$ for $(1,5;1,3)$.)

Let $((\lambda_1, \lambda_2), (\lambda'_1, \lambda'_2); (m_1, m_2), (m'_1, m'_2))$ be a quadruple of E-pairs for a VS quadruple $(\lambda, \lambda'; m, m')$. We obtain the following relations from Algorithm VS and the definition of E-pairs:

$$(m'_1, m'_2) = (\lambda_1, \lambda_2) + (m_1, m_2),$$
$$(\lambda'_1, \lambda'_2) = (m_1, m_2) + (m'_1, m'_2) = (\lambda_1, \lambda_2) + 2(m_1, m_2).$$

Using these, we plainly get the following proposition.

Proposition 3.1. *The E-pair* (λ_1, λ_2) *of a matrix* Λ_λ *satisfies* $(\lambda_1, \lambda_2) \equiv (1,1)$ *(mod 2). The E-pair* (m_1, m_2) *of a matrix* M_m *satisfies* $(m_1, m_2) \equiv (1,0)$ *or* $(0,1)$ *(mod 2).*

We can interpret geometrically this proposition. We stated in §2 that the closed torus $\mathbb{C}/\langle 1, i\rangle$ is obtained from $\mathbb{T}_4^o$ by the abelianization of Γ_4^o and that the twice punctured torus $\mathbb{T}_4^t$ is a double covering of $\mathbb{T}_4^o$. We now consider the abelianization of $\langle P_4, Q_4, R_4\rangle$ and have the complex plane $\mathbb{C}$ is a universal covering of the abelianized image of $\mathbb{T}_4^t$. (See Figures 3, 5 and 6.)

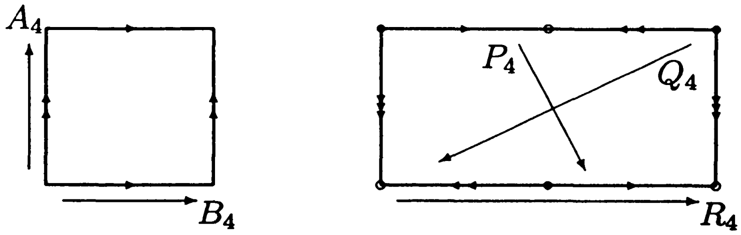

Figure 5: Abelianized images of the fundamental domains in Figure 3.

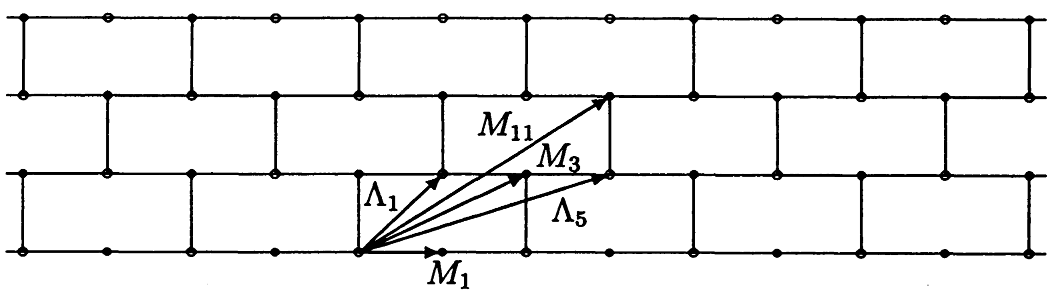

Figure 6: Two colored lattice in the complex plane.

We identify naturally $\mathbb{C}$ with $\mathbb{R}^2$. Let denote $\Omega = \{(k,l) \in \mathbb{Z}^2\}$. Corresponding to the two cusps of $\mathbb{T}_4^t$, the lattice Ω is decomposed into the following two disjoint sub-lattices:

$$\Omega_w = \{(k,l) \in \Omega \mid (k,l) \equiv (0,0) \text{ or } (1,1) \pmod{2}\},$$

$$\Omega_b = \{(k,l) \in \Omega \mid (k,l) \equiv (1,0) \text{ or } (0,1) \pmod{2}\}.$$

In Figure 6, we take the origin $(0,0)$ as a white point, and hence Ω_w is the sub-lattice of white points and Ω_b is the sub-lattice of black points.

As a corollary of Proposition 3.1, we thus have:

Corollary 3.1. *The E-pair of a matrix Λ_λ gives a vector from the origin to a point in Ω_w. The E-pair of a matrix M_m gives a vector from the origin to a point in Ω_b.*

See, for example, Figure 6.

The (N_1, N_2)-word of a CM-matrix was essentially given by the cutting sequence of its Frobenius coordinates. For the (A_4, B_4)-word of a matrix in the VS tree, we can give a similar characterization. For the sake of simplicity, here we use for (A_4, B_4)-words letters f and e instead of A_4 and B_4, respectively, and for making a cutting sequence we code a vertical line by e and a horizontal line by f.

Theorem 3.3. *Let N be a matrix Λ_λ or a matrix M_m in the VS tree. Suppose that N is not $M_1 = e$. Then, the (A_4, B_4)-word of N has a form $ef\sigma$ where σ is the cutting sequence of the E-pair of N.*

We can also regard Algorithm VS as an algorithm which makes essentially the cutting sequence of a pair $(u,v) \in (\mathbb{Z}^+)^2$ where u and v are mutually prime and $u > v$.

Let us consider continued fraction expansions of matrices in the VS tree. Recall that any fractional linear transformation of the Hecke group G_4 can be written as

$$z \longmapsto a_0\sqrt{2} + \cfrac{1}{a_1\sqrt{2} + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1}\sqrt{2} + \cfrac{1}{a_n\sqrt{2} \pm z}}}},$$

where $a_i \in \mathbb{Z}$, not necessarily positive, $a_i \neq 0$ for $i = 1, \dots, n-1$. We call this *$\sqrt{2}$-Rosen continued fraction expansion.*

Making observations of Algorithm VS, we have:

Lemma 3.1. *Let Λ_λ and M_m be matrices of the VS tree. They have the following $\sqrt{2}$-Rosen continued fraction expansions:*

$$\Lambda_\lambda(z) = a_0\sqrt{2} + \cfrac{1}{a_1\sqrt{2} + \cfrac{1}{\ddots + \cfrac{1}{a_k\sqrt{2} + z}}}, M_m(z) = b_0\sqrt{2} + \cfrac{1}{b_1\sqrt{2} + \cfrac{1}{\ddots + \cfrac{1}{b_l\sqrt{2} - z}}},$$

where $a_i, b_i \in \{-2,-1,0,1,2\}$.

Let denote simply these expansions by $\Lambda_\lambda = a_0a_1\dots a_k+$ and $M_m = b_0b_1\dots b_l-$. Moreover, in what follows, we use $\bar{1}$ and $\bar{2}$ instead of -1 and -2. For example, $\Lambda_1 = 110+$, $\Lambda_5 = 111\bar{2}1+$, $M_1 = 1\bar{1}-$ and $M_3 = 111\bar{1}-$. Note that $110+$ is the only word in which the letter 0 occurs.

Unfortunately, $\sqrt{2}$-Rosen continued fraction expansions are not unique. For example, we have the following relations: $2\bar{2} = 11\bar{1}\bar{1}$, $\bar{2}2 = \bar{1}\bar{1}11$, $11\bar{2} = 2\bar{1}\bar{1}$, $\bar{1}\bar{1}2 = \bar{2}11$.

Let σ be a word on $\{1,\bar{1},2,\bar{2}\}$ and let $\bar{\sigma}$ denote the word obtained from σ by exchanging 1 with $\bar{1}$ and 2 with $\bar{2}$. Note that this operation means changing the sign of a word, which is caused by a matrix M_m (see Lemma 3.1).

We thus know there is no simple relation between the (A_4, B_4)-word of a matrix in the VS tree and its $\sqrt{2}$-Rosen continued fraction expansion. However, we can prove the following theorems (see [AR2]).

Theorem 3.4. *Each matrix M_m in the VS tree has a $\sqrt{2}$-Rosen continued fraction expansion $1\sigma\bar{1}-$ where σ is a palindrome on the alphabet $\{1,\bar{1},2\}$.*

Theorem 3.5. *Each matrix Λ_λ in the VS tree has a $\sqrt{2}$-Rosen continued fraction expansion $1\sigma 1+$ where σ is an anti-palindrome, that is, σ has a form $\xi\eta\bar{\xi}^*$ where ξ is a word on $\{1,\bar{1},2,\bar{2}\}$ and $\eta = 2\bar{1}\bar{1}$ or $\bar{1}\bar{1}2$.*

In the proof of these theorems, we make use of cutting sequences. We defined in [AR2] an algorithm by which we obtain the palindrome of Theorem 3.4 from the cutting sequence of the E-pair of M_m and the anti-palindrome of Theorem 3.5 from that of the E-pair of Λ_λ.

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