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Extending local holomorphic conjugacies

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Abstract

Polynomial-like mappings plays an important role in studying complex dynamics in one variable. It has been used to study local connectivity of Julia sets and the Mandelbrot set, rigidity of polynomials, self-similarity of the Mandelbrot set, etc. There is an important equivalence relation called *hybrid equivalence* for polynomial-like mappings. Here, we consider a stronger equivalence relation, i.e., holomorphic equivalence and prove that usually polynomial-like restrictions of rational maps or entire functions are not holomorphically equivalent.

1 Introduction and statement of results

Definition. A map $f : U' \to U$ is called a *polynomial-like mapping* if

1. f is proper and holomorphic;
2. $U' \Subset U$ are topological disks in $\mathbb{C}$.

We always assume the degree of any polynomial-like mapping is greater than one.

The *filled Julia set* of a polynomial-like mapping $f : U' \to U$ is the set

$$K(f) = \{z \in U';\ f^n(z) \in U'\ (\forall n > 0)\}.$$

The simplest example of polynomial-like mappings is a restriction of any polynomial; For a polynomial P of degree $d \geq 2$. Let $U = \{|z| < R\}$ for sufficiently large $R > 0$ and $U' = P^{-1}(U)$. Then $P : U' \to U$ is a polynomial-like mappings of degree d.

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A fundamental theorem on polynomial-like mappings is the *straightening theorem* by Douady and Hubbard [DH] as follows. We say two polynomial-like mappings $f : U' \to U$ and $g : V' \to V$ of the same degree are *hybrid equivalent* if there is a quasiconformal conjugacy $\varphi : U'' \to V''$ between some neighborhoods U'' and V'' of $K(f)$ and $K(g)$ respectively such that $\bar{\partial}\varphi \equiv 0$ almost everywhere in $K(f)$.

Theorem (Straightening theorem). *Any hybrid equivalence class of polynomial-like mappings of degree $d \geq 2$ contains a polynomial of degree d.*

More precisely, for any polynomial-like mapping $f : U' \to U$ of degree d, there exists a polynomial P of degree d hybrid equivalent to f.

Here, we consider stronger equivalence relation, that is *holomorphic equivalence.* Namely, we require the conjugacy φ above is holomorphic. Then we can distinguish most of rational maps or entire maps having polynomial-like restrictions.

Main Theorem ([I1]). *For $i = 1, 2$, let f_i be a rational map or an entire map. Assume that there exist polynomial-like restrictions $f_1 : U_1' \to U_1$ and $f_2 : U_2' \to U_2$ of degree not less than two which are analytically conjugate. Then there exist rational or entire maps g, φ_1 and φ_2 such that*

$$f_1 \circ \varphi_1 = \varphi_1 \circ g, \qquad f_2 \circ \varphi_2 = \varphi_2 \circ g \tag{1}$$

and g has a polynomial-like restriction $g : V' \to V$ analytically conjugate to $f_1 : U_1' \to U_1$ by φ_1.

Furthermore,

- *if both of the degrees $d_1 = \deg f_1$ and $d_2 = \deg f_2$ are finite, then g, φ_1, φ_2 are also of finite degrees. In particular, we have $d_1 = d_2$.*
- *If f_1 is a polynomial and f_2 is a rational map, then f_2 is a polynomial by taking a Möbius conjugate and we can take g, φ_1 and φ_2 to be polynomials.*

We say a map g is *semiconjugate to* another map f if there exists a (not necessarily injective) map φ such that $\varphi \circ g = f \circ \varphi$. The conclusion of the theorem says that there exists a (rational or entire) map g which is globally semiconjugate to both f_1 and f_2.

It seems such a (non-trivial) global semiconjugacy is rare. For example, if f, g and φ have finite degrees, then by taking the degree of the equation $\varphi \circ g = f \circ \varphi$, we have $\deg f = \deg g$. Therefore, we have the following:

Corollary 1. *If the degrees of rational maps f_1, f_2 are different, then they do not have polynomial-like restrictions $f_i : U_i' \to U_i$ $(i = 1, 2)$ which are holomorphically conjugate.*

This corollary is one of the essential steps in proving discontinuity of straightening maps [I2].

It is also known that there is no global semiconjugacy from a transcendental map to a polynomial [I1]. However, (non-)existences of the following semiconjugacies are still open:

Conjecture. *1. There is no global semiconjugacy from a polynomial to a transcendental entire map.*

2. There is no global semiconjugacy from a transcendental entire map to a rational map.

Here we give some examples of global semiconjugacy (i.e., $f \circ \varphi = \varphi \circ g$):

Example. 1. $f = h_1 \circ h_2$, $g = h_2 \circ h_1$ and $\varphi = h_1$.

2. $f = g = T_a$ and $\varphi = T_b$ where T_d is the Chebyshev polynomial of degree d (i.e., $T_d(\cos z) = \cos dz$).

3. $f(z) = z^c h(z^b)$, $g(z) = z^c (h(z))^c$ and $\varphi(z) = z^b$.

4. A linear map $g(z) = \lambda z$, $|\lambda| > 1$ can be semiconjugate to power map, Chebyshev map and Lattès map if λ satisfies a certain condition. Note that the power map $z \mapsto z^d$ for $|d| \geq 2$ is semiconjugate to the Chebyshev map T_d, and the Chebyshev map is again semiconjugate to some Lattès map. However this is not the case of the above theorem because there is no polynomial-like restriction preserved by those semiconjugacies.

In the case of polynomials (i.e., the case f, g, φ are all polynomials), any semiconjugate maps are essentially obtained by some combinations of the above 1, 2 and 3 [I1]. This is proved by applying Ritt's theorems on decomposing polynomials in terms of composition [R] and a theorem by Engstrom [En] which is a stronger version of Ritt's second theorem.

Shishikura suggested the following application to the author. Consider a polynomial P having a renormalization hybrid equivalent to itself. More precisely, assume there exists a polynomial-like mapping $P^n : U' \to U$ hybrid equivalent to

P itself. This implies that for any $k \geq 0$, there exists a polynomial-like mapping $f_k := P^{n^k} : U_k' \to U_k$ hybrid equivalent to P. After rescaling, f_k might converge to some polynomial-like map f_0.

In other words, consider the hybrid class F of polynomial-like mappings containing P. Then any maps $f \in F$ has a polynomial-like restriction $\mathcal{R}(f) = f^n : U_f' \to U_f$, which is again in F. After a proper rescaling and giving some topology in F, it is known that the *renormalization operator* $\mathcal{R} : F \to F$ is contraction for some cases (see [L] and [M]). For such a case, $\mathcal{R}^n(f)$ converges to the unique fixed point $f_0 \in F$ satisfying the equation

$$f_0^n(\lambda z) = \frac{1}{\lambda} f_0(z).$$

From this equation, it is easy to see that f_0 is not a rational map. By the main theorem, we can prove that any renormalization fixed point is not a rational map under much weaker assumption:

Corollary 2. *The renormalization fixed point is not a rational map. More precisely, if $f : U' \to U$ is a polynomial-like mapping such that there is a polynomial-like restriction $f^n : V' \to V$ $(n > 1)$ holomorphically conjugate to f, then f is not a rational map.*

Proof. Consider a rational map f such that $f^n : V' \to V$ is a polynomial-like mapping for some $n > 1$, V' and V. Since the degrees of f and f^n are different, $f^n : V' \to V$ is not holomorphically conjugate to f itself by Corollary 1. □

2 Idea of proof

The main idea of the proof of Main Theorem is the "pushing-forward" of the holomorphic conjugacy.

Let $\varphi : U_1 \to U_2$ be a holomorphic conjugacy between $f_1 : U_1' \to U_1$ and $f_2 : U_2' \to U_2$. Let

$$\Gamma_0 = \{(z, \varphi(z)); z \in U_1\} \subset \mathbb{C} \times \mathbb{C}$$

be the graph of φ. Consider the product dynamics $F(x, y) = (f_1(x), f_2(y))$ of f_1 and f_2 and let

$$\Gamma_n = F^n(\Gamma_0).$$

Since F is a proper map, Γ_n is a local analytic set.

Corollary 1. $\Gamma_n \subset \Gamma_{n+1}$.

Proof. Let $(f_1^n(z), f_2^n(\varphi(z))) \in \Gamma_n$. Since $z \in U_1$, there exists some $w \in U'$ such that $f_1^n(w) = z$. Therefore,

$$(f_1^n(z), f_2^n(\varphi(z))) = (f_1^{n+1}(w), f_2^n(\varphi(f_1(w)))) = (f_1^{n+1}(w), f_2^{n+1}(\varphi(w))).$$

Since $U' \subset U$, this proves that $(f_1^n(z), f_2^n(\varphi(z))) \in \Gamma_{n+1}$. $\square$

Therefore, we have an increase sequence of local analytic sets

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n \subset \Gamma_{n+1} \subset \cdots . \tag{2}$$

Let $\pi_n : X_n \to \Gamma_n$ be a resolution of singularities, i.e., X_n is a Riemann surface and π_n is a holomorphic map which is biholomorphic outside a discrete set. By (2) and the property $F(\Gamma_n) = \Gamma_{n+1}$, it follows that there is an injective map $\iota_n : X_n \hookrightarrow X_{n+1}$ and a proper map $g_n : X_n \to X_{n+1}$ for each $n \geq 0$ such that

$$\pi_{n+1} \circ \iota_n = \pi_n, \quad \pi_{n+1} \circ g_n = F \circ \pi_n.$$

Then the following diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook, "\iota_0"] \arrow[dr, "g_0"] & X_1 \arrow[r, hook, "\iota_1"] \arrow[dr, "g_1"] & \cdots \arrow[r, hook, "\iota_{n-1}"] \arrow[dr, "g_{n-1}"] & X_n \arrow[r, hook, "\iota_n"] \arrow[dr, "g_n"] & X_{n+1} \arrow[r, hook, "\iota_{n+1}"] \arrow[dr, "g_{n+1}"] & \cdots \\
X_0 \arrow[r, hook, "\iota_0"] & X_1 \arrow[r, hook, "\iota_1"] & \cdots \arrow[r, hook, "\iota_{n-1}"] & X_n \arrow[r, hook, "\iota_n"] & X_{n+1} \arrow[r, hook, "\iota_{n+1}"] & \cdots
\end{tikzcd}$$

commutes.

Now take the direct limit of X_n:

$$X = \varinjlim X_n.$$

Then X is a complex manifold. Furthermore, π_n and g_n induce holomorphic maps

$$\pi : X \to \Gamma = \bigcup_{n \geq 0} \Gamma_n, \quad g : X \to X$$

such that $\pi \circ g = F \circ \pi$. Let $\varphi_i = p_i \circ \pi$ where p_i is the projection to the i-th coordinate. Then we have $\varphi_i \circ g = f_i \circ \varphi_i$ for $i = 1, 2$. Moreover on X_0, the map $\varphi_i : X_0 \to U_i$ is biholomorphic. Therefore, X has a polynomial-like restriction $g : V' \to V$ holomorphically conjugate to $f_i : U_i' \to U_i$, where $V = X_0$ and $V' = (\varphi_1|_{X_0})^{-1}(U_1')$.

We need to classify $g : X \to X$. To do this end, we apply the uniformization theorem to divide into several cases, and we need to study all the possibilities. For example, we have the following:

- Since g has a polynomial-like restriction, there is a repelling periodic point of g. By the Schwarz lemma, it follows that X is not hyperbolic. Therefore, by taking the universal covering, we may assume that X is either the complex plane $\mathbb{C}$ or the Riemann sphere $\hat{\mathbb{C}}$.

- If both f_1 and f_2 are rational, then we have

$$\deg g \leq \deg F = \deg f_1 \cdot \deg f_2 < +\infty.$$

  Hence it follows that g is not transcendental.

In this way, we can exclude most of the possibilities and prove the theorem.

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