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# Weighted Green functions of polynomial skew products on $\mathbb{C}^2$

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This report is a resume of [9]. We consider the dynamics of polynomial skew products on $\mathbb{C}^2$. We investigate the existence of the Green and fiberwise Green functions of the maps, which induces generalized Green functions that are well-behaved on $\mathbb{C}^2$, and we give an example of the Green functions which are not defined on some curves in $\mathbb{C}^2$. The generalized Green functions relate to the dynamics of the extensions of the maps to holomorphic or rational maps on weighted projective spaces.

## 1 Introduction

We consider the dynamics of a polynomial skew product on $\mathbb{C}^2$ of the form $f(z,w) = (p(z), q(z,w))$, where $p$ and $q$ are polynomials such that $p(z) = z^{d_1} + O(z^{d_1-1})$ and $q(z,w) = q_z(w) = w^{d_2} + O_z(w^{d_2-1})$. We assume that $d_1 \geq 2$ and $d_2 \geq 2$. Let $d = \max\{d_1, d_2\}$, which coincides with the dynamical degree of $f$. We denote by $f^n$ the $n$-th iterate of $f$. The dynamics of $f$ consists of the dynamics on the base space and the dynamics on the fibers: $f^n(z,w) = (p^n(z), Q_z^n(w))$, where $Q_z^n = q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z$. Hence useful tools for the study of the dynamics of $f$ are the Green function $G_f$ of $f$, the Green function $G_p$ of $p$, and the fiberwise Green function $G_z$ of $f$:

$$G_f(z,w) = \lim_{n\to\infty} d^{-n} \log^+ |f^n(z,w)|,$$

$$G_p(z) = \lim_{n\to\infty} d_1^{-n} \log^+ |p^n(z)|, \text{ and}$$

$$G_z(w) = \lim_{n\to\infty} d_2^{-n} \log^+ |Q_z^n(w)|,$$

where $\log^+ = \max\{\log, 0\}$ and $|(z,w)| = \max\{|z|, |w|\}$. It is well-known that $G_p$ is defined, continuous, and subharmonic on $\mathbb{C}$.

Known results about the existence of the limits $G_f$ and $G_z$ are as follows. We say that a polynomial map is *regular* if it extends to a holomorphic map on the projective space $\mathbb{P}^2$, or equivalently if it extends to a non-degenerate homogeneous map on $\mathbb{C}^3$. It is well-known that if $f$ is regular, then $G_f$ is well-behaved on $\mathbb{C}^2$; that is, it is defined, continuous, and plurisubharmonic on $\mathbb{C}^2$. Note that $f$ is regular if and only if $d_1 = d_2 = \deg q$, where $\deg q$ denotes the algebraic degree of $q$. Several studies have been made on the dynamics of regular polynomial skew products (e.g. [4], [5], [6] and [1]). However, the existence of $G_z$ is not known even if $f$ is regular. Conversely, the existence of $G_z$ implies that of $G_f$. It is clear that $G_z$ is well-behaved on $K_p \times \mathbb{C}$. Favre and Guedj [2] studied the existence and properties of $G_z$ on $K_p \times \mathbb{C}$ without the assumption about the form of $q$. Using an argument in the proof of [2, Theorem 6.1], we showed the existence of $G_z$ on an open subset of $K_p^c \times \mathbb{C}$ in [8, Lemma 2.3] with the assumption $d_1 \leq d_2$, which is improved in this paper.

The organization of the paper is as follows. A summary of our results appears in Section 2, which includes an introduction of the weighted Green functions. In Section 3, we give an example of the Green functions which are not defined on some curves in $\mathbb{C}^2$ and those of which coincide with the weighted Green functions on $\mathbb{C}^2$. Details for the case $d_1 = d_2$ and for the case $d_1 \neq d_2$ appear in Section 4 and in Section 5, respectively.

## 2 Summary of results

Let $f(z, w) = (p(z), q(z, w))$ be a polynomial skew product as before. To state our results, the following rational number defined by $f$ plays a central role:

$$k = \min \left\{ l \in \mathbb{Q} \;\middle|\; \begin{array}{l} ld \geq n_j + l m_j \text{ for any integers } n_j \text{ and } m_j \text{ s.t.} \\ c_j z^{n_j} w^{m_j} \text{ is a term in } q \text{ for some } c_j \neq 0 \end{array} \right\}$$

if $\deg_z q > 0$ and $k = 1$ if $\deg_z q = 0$, where $\deg_z q$ denotes the algebraic degree of $q$ with respect to $z$. Since $q$ has only finitely many terms, one can take the minimum. Indeed, $k$ is equal to

$$\max \left\{ \frac{n_j}{d - m_j} \;\middle|\; \begin{array}{l} c_j z^{n_j} w^{m_j} \text{ is a term in } q \\ \text{with } c_j \neq 0 \text{ and } m_j < d \end{array} \right\}.$$

By definition, $k \leq \deg_z q$ and $k < \deg q$. Moreover, $d^n \leq \deg(f^n) \leq \max\{1, k\} d^n$ for any integer $n$. Hence the dynamical degree of $f$ is equal to $d$ as described previously. Note that $f$ is regular if and only if $d_1 = d_2$ and $k \leq 1$.

Our results on the non-existence and the existence of several Green functions of $f$ are as follows. In the case $d_1 = d_2$, we give an example of the Green functions which are not defined on some curves in $\mathbb{C}^2$ (Example 3.1). On the other hand, $G_z$ is well-behaved on a region defined by using $k$, and $G_z(w)$ tends to $kG_p(z)$ as $(z, w)$ in the region tends to its boundary (Theorem 4.1). If $d_1 < d_2$, then $G_z$ is well-behaved on $\mathbb{C}^2$ and it coincides with $G_f$ (Theorem 5.1). If $d_1 > d_2$, then $\tilde{G}_z^* \leq kG_p$ on $\mathbb{C}^2$, where $\tilde{G}_z^* = \limsup_{n\to\infty} d^{-n} \log^+ |Q_z^n|$ (Theorem 5.3). From these results we have the following main result:

**Theorem 2.1.** *A generalization of the Green function $G_f^k$ of $f$,*

$$G_f^k(z,w) = \lim_{n\to\infty} d^{-n} \log^+ |f^n(z,w)|_k,$$

*where $|(z,w)|_k = \max\{|z|^k, |w|\}$, is defined, continuous, and plurisubharmonic on $\mathbb{C}^2$.*

In this paper we call this function the *weighted Green function* of $f$. The convergence to $G_f^k$ is uniform if $d_1 \leq d_2$.

Moreover, the weighted Green function of $f$ relates to the dynamics of the extension of $f$ to a rational map on the weighted projective space $\mathbb{P}(r, s, 1)$, where $r$ and $s$ are the denominator and numerator of $k$, respectively. The weighted projective space $\mathbb{P}(r, s, 1)$ is a quotient space of $\mathbb{C}^3 - \{O\}$,

$$\mathbb{P}(r,s,1) = \mathbb{C}^3 - \{O\} \big/ \sim,$$

where $(z, w, t) \sim (\lambda^r z, \lambda^s w, \lambda t)$ for any $\lambda$ in $\mathbb{C} - \{0\}$. We denote a point in the weighted projective space $\mathbb{P}(r, s, 1)$ by weighted homogeneous coordinates $[z : w : t]$. It follows from the definition of $k$ that $f$ extends to a rational map $\tilde{f}$ on $\mathbb{P}(r, s, 1)$, where $\tilde{f}[z : w : t] = [\tilde{p}(z,t) : \tilde{q}(z,w,t) : t^d]$ and

$$\tilde{p}(z,t) = p\left(\frac{z}{t^r}\right)t^{dr} \text{ and } \tilde{q}(z,w,t) = q\left(\frac{z}{t^r}, \frac{w}{t^s}\right)t^{ds}.$$

Indeed, it extends to a *weighted homogeneous map $F$* on $\mathbb{C}^3$,

$$F(z,w,t) = \left(p\left(\frac{z}{t^r}\right)t^{dr}, q\left(\frac{z}{t^r}, \frac{w}{t^s}\right)t^{ds}, t^d\right);$$

that is, $F(\lambda^r z, \lambda^s w, \lambda t) = (\lambda^{dr} Z, \lambda^{ds} W, \lambda^d T)$ for any $\lambda$ in $\mathbb{C}$, where $(Z, W, T) = F(z, w, t)$. It follows from definition that $\tilde{f}$ is holomorphic, or equivalently $F$ is non-degenerate if and only if $d_1 = d_2$. See also [3, Theorem C and Section 5.3] for extensions of polynomial maps on $\mathbb{C}^2$ to holomorphic maps on weighted projective spaces.

We define the Fatou set of $\tilde{f}$ as the maximal open set of $\mathbb{P}(r, s, 1)$ where the family of iterates $\{\tilde{f}^n\}_{n\geq 0}$ is normal. The Julia set of $\tilde{f}$ is defined as the complement of the Fatou set of $\tilde{f}$. In the case $d_1 = d_2$, we show that the Fatou and Julia sets of the extension of $f$ are determined by the weighted Green function of $f$. If $d_1 < d_2$, then it follows rather easily that the dynamics near infinity has similar properties as the case $d_1 = d_2$. On the other hand, the dynamics near infinity seems to be unclear if $d_1 > d_2$.

# 3 Example

Polynomial skew products that are semiconjugate to polynomial products give us examples of the Green functions which are not well-behaved on some curves in $\mathbb{C}^2$ and those which coincide with the weighted Green functions on $\mathbb{C}^2$.

**Example 3.1.** *Let $f(z,w) = (z^2, w^2 + cz^4)$ for any $c \neq 0$. Then $k = 2$, and $f$ is semiconjugate to polynomial product $g(z,w) = (z^2, w^2 + c)$ by $\pi(z,w) = (z, z^2 w) : f \circ \pi = \pi \circ g$. Hence $Q_z^n(w) = z^{2^{n+1}} q_c^n(z^{-2}w)$ for $z \neq 0$, where $q_c(w) = w^2 + c$. Let $\tilde{G}_c(w) = \lim_{n\to\infty} 2^{-n} \log |q_c^n(w)|$. If 0 is a periodic point of period $p > 1$, then $\tilde{G}_c$ is not defined on $\{0, q_c(0), \cdots, q_c^{p-1}(0)\}$. Hence $G_z$ and $G_f$ are not defined on curves $\{(z, q_c^j(0)z^2) : |z| > 1\}_{j=0}^{p-1}$. On the other hand, if $q_c^n(0)$ tends to infinity, then $\tilde{G}_c$ is well-behaved on $\mathbb{C}$. Hence $G_z$ and $G_f$ are well-behaved on $\mathbb{C}^2$ and both coincide with the weighted Green function of $f$.*

More generally, for any polynomial $q(w)$ of degree $d$ and for any positive integer $n$ there exists a polynomial skew product of the form $(z^d, q(z,w))$ that is semiconjugate to polynomial product $(z^d, q(w))$ by projection $\pi(z,w) = (z, z^n w)$. These maps produce examples similar to Example 3.1.

# 4 Details in the case $d_1 = d_2$

In this section, we assume that $d_1 = d_2 = d$. First, let us give an explanation of the rational number $k$. If $f$ is regular, then $w^d$ is a term of highest degree in $q$, which makes an important role on the study of the dynamics of $f$. In general, $w^d$ may not be a term of highest degree in $q$. However, if we define the weight of a monomial $z^n w^m$ as $n + km$, then $w^d$ is a term of highest weight in $q$. Although all arguments in this paper hold with suitable modifications for any rational number $l$ which is larger than $k$, the minimum $k$ seems to be better. As a remark, the minimum $k$ might be related to the eigenvaluation of $f$ in [3].

Let $W_R = \{|w| > R|z|^k, |w| > R^{k+1}\}$ and $A_f = \cup_{n\geq 0} f^{-n}(W_R)$ for large $R > 0$. Then there exists $c > 0$ such that $|q_z(w) - w^d| \leq cR^{-1}$ on $W_R$, and $f$ preserves $W_R$. Hence the limit $G_z$ is well-behaved on $A_f$, and $A_f$ is independent of $R$:

**Theorem 4.1.** *If $d_1 = d_2$, then $G_z$ is defined, continuous, and pluriharmonic on $A_f$. Moreover, $G_z(w)$ tends to $kG_p(z)$ as $(z,w)$ in $A_f$ tends to $\partial A_f$.*

Hence $G_f$ is also well-behaved on $A_f$ if $d_1 = d_2$.

**Remark 4.2.** *Even if $f$ is regular, the theorem above is not trivial. Since $G_f$ is well-behaved on $\mathbb{C}^2$, it is well-known that $G_z$ is defined, continuous, and pluriharmonic on $\{G_f > G_p\}$. On the other hand, it follows from Theorem 4.1 that $G_z$ is defined, continuous, and pluriharmonic on $\{G_f > kG_p\}$. If $k < 1$, then our region is bigger than the well-known region.*

Let us consider the dynamics of $f$ near infinity. We define the weight of a monomial $z^n w^m$ as $n + km$. Let $h$ be the weighted homogeneous part of $q$; that is, $h$ is the polynomial consisting of all terms of highest weight $kd$ in $q$. We assume that $k$ is an integer for a while. Put $w = cz^k$, then $h(z, cz^k) = h(1,c)z^{kd}$. Fix any $c$ and define $h(c) = h(1,c)$. It then follows that $h(c)z^{kd}$ is the homogeneous part of $q(z, cz^k)$ of degree $kd$, where $h(c)z^{kd}$ and $q(z, cz^k)$ are polynomials in $z$. Moreover, $h^n(c)z^{kd^n}$ is the homogeneous part of $Q_z^n(cz^k)$ of degree $kd^n$ because $d_1 = d_2$. If $k$ is not an integer, then $z^k$ is not a well-defined function. However, the polynomial $h$ and the Green function $G_h$ have some symmetries related to the denominator $r$ of $k$ in that case: $h(c)$ can be written as $c^l H(c^r)$ for some integer $l \geq 0$ and for some polynomial $H$, the Julia set $J_h$ is preserved by the rotation $\rho c$, where $\rho$ is a $r$-th root of 1, and $G_h(c) = G_h(z^{-k}w)$ is a well-defined function in $z$ and $w$. Hence we get the following asymptotics of $G_f^k$ near infinity.

**Lemma 4.3.** *If $d_1 = d_2$, then $G_f^k(z, cz^k) = k \log|z| + G_h(c) + o(1)$ as $|z| \to \infty$ for any fixed $c$.*

**Proposition 4.4.** *If $d_1 = d_2$, then $G_f^k(z, w) = \log|(z,w)|_k + \rho_h(z^{-k}w) + o(1)$ as $|(z,w)|_k \to \infty$, where $\rho_h(c) = G_h(c) - \log^+|c|$.*

We next consider the dynamics of the extension of $f$ to a holomorphic map $\tilde{f}$ on $\mathbb{P}(r,s,1)$, where $\tilde{f}[z : w : t] = [\tilde{p}(z,t) : \tilde{q}(z,w,t) : t^d]$. By definition, $\tilde{p}(z,t) = z^d + tu(z,t)$ for some polynomial $u$, and $\tilde{q}(z,w,t) = h(z,w) + tv(z,w,t)$ for some polynomial $v$, where $h$ contains $w^d$. Hence $\tilde{f}$ is holomorphic; that is, it has no indeterminacy points. Note that the point $p_\infty = [0 : 1 : 0]$ is a superattracting fixed point and that $A_f$ coincides with the restriction of the attracting basin $A_{p_\infty}$ to $\mathbb{C}^2$. Let $B_{\tilde{f}} = \mathbb{P}(r,s,1) - K_f \cup A_{p_\infty}$. Using the Kobayashi hyperbolicity, we get the following theorem:

**Theorem 4.5.** *If $d_1 = d_2$, then the Fatou set of $\tilde{f}$ consists of $\mathrm{int}K_f$, $A_{p_\infty}$ and $\mathrm{int}B_{\tilde{f}}$. In other words, the Julia set of $\tilde{f}$ coincides with the closure of the set where $G_f^k$ is not pluriharmonic, where the closure is taken in $\mathbb{P}(r,s,1)$.*

**Remark 4.6.** *We saw that $f$ extends to a non-degenerate weighted homogeneous map $F$ on $\mathbb{C}^3$ if $d_1 = d_2$. From an argument similar to the proof of [7, Theorem 1.5] it follows that the weighted Green function $G_F^k$ of $F$,*

$$G_F^k(z,w,t) = \lim_{n\to\infty} \frac{1}{d^n} \log |F^n(z,w,t)|_k,$$

*where $|(z,w,t)|_k = \max\{|z|^s, |w|^r, |t|^{rs}\}$, is well-behaved on $\mathbb{C}^3 - \{O\}$. From identity $G_F^k(z,w,1) = rG_f^k(z,w)$ it follows that the weighted Green function $G_f^k$ of $f$ is well-behaved on $\mathbb{C}^2$. Moreover, one might be able to prove the later statement in Theorem 4.5 for any holomorphic map on a weighted projective space.*

The dynamics of $\tilde{f}$ on $B_{\tilde{f}}$ is as follows. Any point in $B_{\tilde{f}}$ is attracted to the line at infinity $L_\infty = \{t = 0\}$ under the iterations, since $L_\infty$ is $\tilde{f}$-invariant and attracting. Finally, the dynamics on $L_\infty$, which is induced by the weighted homogeneous part $h$ of $q$, should determine the dynamics on $B_{\tilde{f}}$. We saw that $h(c) := h(1,c)$ can be written as $c^l H(c^r)$ for some integer $l \geq 0$ and some polynomial $H$, which is semiconjugate to $c^l H(c)^r$ by $c^r$. Furthermore, the restriction of $\tilde{f}$ to $L_\infty$ is conjugate to $c^l H(c)^r$.

# 5 Details in the case $d_1 \neq d_2$

## 5.1 Case $d_1 < d_2$

In this subsection, we assume that $d_1 < d_2$. The following theorem follows from the same argument as the proof of Theorem 4.1.

**Theorem 5.1.** *If $d_1 < d_2$, then $G_z$ is defined, continuous, and pluriharmonic on $A_f$. Moreover, $G_z(w)$ tends to $0$ as $(z,w)$ in $A_f$ tends to $\partial A_f$.*

**Corollary 5.2.** *If $d_1 < d_2$, then $G_z$ is defined, continuous, and plurisubharmonic on $\mathbb{C}^2$, and it coincides with $G_f$.*

The dynamics of the extension of $f$ to a rational map $\tilde{f}$ on $\mathbb{P}(r,s,1)$ is as follows. By definition, $\tilde{p}$ is divisible by $t$ and $\tilde{q}(z,w,t) = h(z,w) + tv(z,w,t)$ for some polynomial $v$, where $h$ is the weighted homogeneous part of $q$ containing $w^d$. Note that the point $p_\infty = [0:1:0]$ is a superattracting fixed point and that $A_f$ coincides with the restriction of the attracting basin of the point above to $\mathbb{C}^2$. We denote the indeterminacy set of $\tilde{f}$ by $I$. Then $I = \{[z:w:0] : h(z,w) = 0\}$. Since $\tilde{f}$ maps $L_\infty - I$ to $p_\infty$, the attracting basin $A_{p_\infty}$ of $p_\infty$ includes an open neighborhood of $L_\infty - I$. Let $A_I$ be the attracting basin of $I$ not including $I$; then $\mathbb{P}(r,s,1) - K_f$ consists of $A_{p_\infty}$, $A_I$ and $I$. Consequently, the Fatou set of $\tilde{f}$ consists of int$K_f$, $A_{p_\infty}$ and int$A_I$. The Julia set of $\tilde{f}$ consists of $\partial K_f$, $\partial A_{p_\infty}$ and $\partial A_I$. Note that the restriction of $A_I$ to $\mathbb{C}^2$ is the set where $G_f = 0$ and $G_p > 0$. Hence the Julia set of $\tilde{f}$ includes the closure of the set where $G_f$ is not pluriharmonic, where the closure is taken in $\mathbb{P}(r, s, 1)$.

### 5.2 Case $d_1 > d_2$

The aspect is different if $d_1 > d_2$. Fix positive integers $r'$ and $s'$ such that $\frac{s'}{r'} > k$. Then $[1 : 0 : 0]$ is an attracting fixed point of the extension of $f$ to a rational map on $\mathbb{P}(r', s', 1)$. Hence we have the following statement.

**Theorem 5.3.** *If $d_1 > d_2$, then $\tilde{G}_z^* \leq kG_p$ on $\mathbb{C}^2$, where $\tilde{G}_z^* = \limsup_{n\to\infty} d^{-n} \log^+ |Q_z^n|$. In particular, if $k \leq 1$, then $G_f = G_p$ on $\mathbb{C}^2$.*

The dynamics of the extension $\tilde{f}$ is as follows. By definition, $\tilde{p}(z,t) = z^d + tu(z,t)$ for some polynomial $u$, and $\tilde{q}(z,w,t) = h(z,w) + tv(z,w,t)$ for some polynomial $v$, where $h$ is divisible by $z$. Thus $p_\infty$ is the unique indeterminacy point and the dynamics near infinity is unclear.

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