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Component-wise accumulation sets for Axiom A polynomial skew products

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1 Introduction

In this note, we consider Axiom A regular polynomial skew products on $\mathbb{C}^2$. It is of the form : $f(z,w)=(p(z),q(z,w))$, where $p(z)=z^d+\cdots$ and $q_z(w)=q(z,w)=w^d+\cdots$ are polynomials of degree $d\geq 2$. Then its k-th iterate is expressed by :

$$f^k(z,w)=(p^k(z),q_{p^{k-1}(z)}\circ\cdots\circ q_z(w))=:(p^k(z),Q_z^k(w)).$$

Hence it preserves the family of fibers $\{z\}\times\mathbb{C}$ and this makes it possible to study its dynamics more precisely. Let K be the set of points with bounded orbits and put $K_z:=\{w\in\mathbb{C};(z,w)\in K\}$ and $K_{J_p}:=K\cap(J_p\times\mathbb{C})$. The *fiber Julia set* J_z is the boundary of K_z.

Let Ω be the set of *non-wandering points* for f. Then f is said to be *Axiom A* if Ω is compact, hyperbolic and periodic points are dense in Ω. For polynomial skew products, Jonsson [J2] has shown that f is Axiom A if and only if the following three conditions are satisfied :

(a) p is hyperbolic,
(b) f is vertically expanding over J_p,
(c) f is vertically expanding over $A_p:=\{$attracting periodic points of $p\}$.

Here f is *vertically expanding over* $Z\subset\mathbb{C}$ with $p(Z)\subset Z$ if there exist $\lambda>1$ and $C>0$ such that $|(Q_z^k)'(w)|\geq C\lambda^k$ holds for any $z\in Z, w\in J_z$ and $k\geq 0$.

We are interested in the dynamics of f on $J_p\times\mathbb{C}$ because the dynamics outside $J_p\times\mathbb{C}$ is fairly simple. Consider the critical set

$$C_{J_p}=\{(z,w)\in J_p\times\mathbb{C};q_z'(w)=0\}$$

over the *base Julia set* J_p. Let μ be the ergodic measure of maximal entropy for f (see Fornaess and Sibony [FS1]). Its support J_2 is called the *second Julia set* of f. Let $PC_{J_p}:=\overline{\cup_{n\geq 1}f^n(C_{J_p})}$ be the *postcritical set* of C_{J_p}. Jonsson [J2] has shown that

(d) $J_2 = \overline{\cup_{z\in J_p}\{z\} \times J_z}$ (Corollary 4.4),
(e) the condition (b) $\Longleftrightarrow PC_{J_p} \cap J_2 = \emptyset$ (Theorem 3.1),
(f) J_2 is the closure of the set of repelling periodic points of f (Corollary 4.7).

By the Birkhoff ergodic theorem, μ-a.e. x has a dense orbit in J_2. Especially, $J_2 = supp\,\mu$ is transitive. Hence J_2 coincides with the *basic set* of unstable dimension two. See also [FS2].

For any subset X in $\mathbb{C}^2$, its accumulation set is defined by

$$A(X) = \cap_{N\geq 0}\overline{\cup_{n\geq N} f^n(X)}.$$

DeMarco & Hruska [DH1] defined the *pointwise* and *component-wise* accumulation sets of C_{J_p} respectively by

$$A_{pt}(C_{J_p}) = \overline{\cup_{x\in C_{J_p}} A(x)} \quad \text{and} \quad A_{cc}(C_{J_p}) = \overline{\cup_{C\in \mathcal{C}(C_{J_p})} A(C)},$$

where $\mathcal{C}(C_{J_p})$ denotes the collection of connected components of C_{J_p}. It follows from the definition that

$$A_{pt}(C_{J_p}) \subset A_{cc}(C_{J_p}) \subset A(C_{J_p}).$$

It also follows that $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p})$ if J_p is a Cantor set and $A_{cc}(C_{J_p}) = A(C_{J_p})$ if J_p is connected.

Let Λ be the closure of the set of saddle periodic points in $J_p \times \mathbb{C}$. It decomposes into a disjoint union of *saddle basic sets* : $\Lambda = \sqcup_{i=1}^m \Lambda_i$. Put $\Lambda_z = \{w \in \mathbb{C}; (z,w) \in \Lambda\}$. The *stable* and *unstable manifolds* of Λ are respectively defined by

$$\begin{aligned} W^s(\Lambda) &= \{y \in \mathbb{C}^2; f^k(y) \to \Lambda\}, \\ W^u(\Lambda) &= \{y \in \mathbb{C}^2; \exists \text{ backward orbit } \hat{y} = (y_{-k}) \text{ tending to } \Lambda\}. \end{aligned}$$

Theorem A. ([DH1])

$$A_{pt}(C_{J_p}) = \Lambda, \quad A(C_{J_p}) = W^u(\Lambda) \cap (J_p \times \mathbb{C}).$$

Theorem B. ([DH1, DH2])

$$A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \implies \forall C \in \mathcal{C}(C_{J_p}), C \cap K = \emptyset \ \textit{or}\ C \subset K. \quad (1)$$

$$A_{pt}(C_{J_p}) = A(C_{J_p}) \iff \textit{the map } z \mapsto \Lambda_z \textit{ is continuous in } J_p. \quad (2)$$

Under the assumption $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda$,

$$A_{pt}(C_{J_p}) = A(C_{J_p}) \iff \textit{the map } z \mapsto K_z \textit{ is continuous in } J_p. \quad (3)$$

Note that

$$W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \Longleftrightarrow W^u(\Lambda_i) \cap W^s(\Lambda_j) = \emptyset \text{ for any } 1 \le i \ne j \le m. \quad (4)$$

Sumi [S] gives an example of Axiom A polynomial skew product which does not satisfy the condition in (4). It is also (incorrectly) described as Example 5.10 in [DH1].

We define a relation $\succ$ among saddle basic sets by $\Lambda_i \succ \Lambda_j$ if $(W^u(\Lambda_i) \setminus \Lambda_i) \cap (W^s(\Lambda_j) \setminus \Lambda_j) \ne \emptyset$. A *cycle* is a chain of basic sets : $\Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots \succ \Lambda_{i_n} = \Lambda_{i_1}$. For Axiom A open endomorphisms, there is no trivial cycle because $W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$ holds for any i. See [J2], Proposition A.4. Jonsson has also shown that, for Axiom A polynomial skew products on $\mathbb{C}^2$, the non-wandering set Ω coincides with the *chain recurrent set* $\mathcal{R}$. This leads to the following lemma, which we use later.

Lemma 1. ([J2], Corollary 8.14) *Axiom A polynomial skew products on* $\mathbb{C}^2$ *have no cycles.*

Put

$$C_0 := C_{J_p} \setminus K, \quad C_i := C_{J_p} \cap W^s(\Lambda_i) \ (1 \le i \le m).$$

We will try to give characterizations of the equalities $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ and $A_{pt}(C_{J_p}) = A(C_{J_p})$ in terms of C_i.

Lemma 2. $C_{J_p} = \sqcup_{i=0}^m C_i$.

proof. By Proposition 3.1 in Jonsson [J1], $\hat{\Omega}$ has local product structure for open Axiom A endomorphisms. Theorem A implies $A(x) \subset \Lambda$ for any $x \in C_{J_p}$. If $A(x) = \emptyset$, then $x \in C_0$. Otherwise there exist an n and $y \in \Lambda$ such that $f^n(x) \in W^s_{loc}(y)$. Hence $A(x) \subset \Lambda_i$ if $y \in \Lambda_i$. Thus we conclude $C_{J_p} = \sqcup_{i=0}^m C_i$. □

If we put $\Lambda_0 = \emptyset$, we have $A(C_i) \supset A_{pt}(C_i) = \Lambda_i$ for any $i \ge 0$.

Theorem 1.

$$A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \Longleftrightarrow \forall C \in \mathcal{C}(C_{J_p}),\ 0 \le \exists i \le m \ \textit{such that}\ C \subset C_i. \quad (5)$$

In terms of C_i, the condition in (1) is expressed by

$$\forall C \in \mathcal{C}(C_{J_p}), \quad C \subset C_0 \text{ or } C \subset \cup_{i=1}^m C_i.$$

Hence, if $m = 1$, that is, Λ itself is a basic set, then the condition in (5) coincides with that in (1). In general, the condition in (5) is stronger than that in (1).

We have another characterization of $A_{pt}(C_{J_p}) = A(C_{J_p})$ in terms of C_i.

Theorem 2. *For any $i \geq 0$, we have*

$$A(C_i) = \Lambda_i \iff C_i \text{ is closed }. \tag{6}$$

Consequently we have

$$A_{pt}(C_{J_p}) = A(C_{J_p}) \iff C_i \text{ is closed for any } i \geq 0.$$

As for the condition in (3), we have

Theorem 3. *The following three conditions are equivalent to each other.*
(a) C_0 *is closed,*
(b) $A(C_{J_p}) = W^u(\Lambda) \cap W^s(\Lambda)$,
(c) the map $z \mapsto K_z$ *is continuous in* J_p.

Note that Theorem 3 reproves the equivalence (3) in Theorem B. We also note that $A_{pt}(C_{J_p}) = A(C_{J_p})$ is equivalent to

$$W^u(\Lambda) \cap (J_p \times \mathbb{C}) = W^u(\Lambda) \cap W^s(\Lambda) = \Lambda.$$

Corollary 1. *Suppose* C_0 *is closed. Then,*

$$W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff C_i \text{ is closed for any } i \geq 1.$$

We do not know whether the assumption that C_0 is closed can be removed or not. The $(\Rightarrow)$ part holds without this assumption.

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2 Proofs of Theorems

First we prove Theorem 1. Note that $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p})$ if and only if $A(C) \subset \Lambda$ for any $C \in \mathcal{C}(C_{J_p})$.
$(\Rightarrow)$ Suppose $C \in \mathcal{C}(C_{J_p})$ intersects at least two of C_i. By Theorem B, (1), we may assume $C \subset \cup_{i=1}^m C_i$. Then, by Lemma 2, we have $C = \sqcup_{i=1}^m (C \cap C_i)$. If all $C \cap C_i$ are closed, it contradicts the connectivity of C. Thus at least one of them is not closed. We may assume that there exists a sequence $x_n \in C \cap C_i$ tending to $x_0 \in C \cap C_j$ for some $i \neq j$. Take a small open neighborhood U_k of Λ_k for $1 \leq k \leq m$ so that $f(U_k) \cap U_\ell = \emptyset$ for $k \neq \ell$. Since $x_0 \in C_j$, there exists a $K > 1$ such that $f^k(x_0) \in U_j$ for $k \geq K$. Then $f^K(x_n) \in U_j$ for large n. Since $x_n \in C_i$, the orbit of x_n eventually leaves U_j. Hence put

$k_n := min\{k \geq K; f^k(x_n) \notin U_j\}$. We will show $k_n \to \infty$. Otherwise, taking a subsequence, we may assume $\{k_n\}$ is bounded. Then there is a subsequence n_ℓ such that $k_{n_\ell} = L$ for all ℓ. That is, $f^L(x_{n_\ell}) \notin U_j$. Taking $\ell \to \infty$, we have $f^L(x_0) \notin U_j$, which contradicts $L \geq K$. Now let y be an accumulation point of the sequence $\{f^{k_n}(x_n)\}$. From the definition of U_k, we have $y \notin \cup U_k$, hence $y \notin \Lambda$. Since $y \in A(C)$, this implies $A_{cc}(C_{J_p})$ contains a point y outside $\Lambda = A_{pt}(C_{J_p})$. Thus we conclude $A_{pt}(C_{J_p}) \neq A_{cc}(C_{J_p})$.

Moreover we can prove $y \in W^u(\Lambda_j)$. In fact, by taking subsequences if necessary, put $y_{-\ell} = \lim_{n\to\infty} f^{k_n - \ell}(x_n)$. Then $\{y_{-\ell}; \ell \geq 0\}$ forms a backward orbit of y in $\overline{U_j}$. By the local product structure of $\hat{\Omega}$, we conclude $y_{-\ell} \to \Lambda_j$, hence $y \in W^u(\Lambda_j)$.

($\Leftarrow$) We have only to show that $A(C) \subset \Lambda_i$ if $C \subset C_i$. If $C \subset C_0$, then $A(C) = \emptyset$ since C is compact. Suppose $C \subset C_i$ and there exists $x \in A(C) \setminus \Lambda_i$ for $i \geq 1$. By taking U_i small, there exists a neighborhood V of x such that $V \cap U_i = \emptyset$. Since $x \in \overline{\cup_{k\geq N} f^k(C)}$ for any $N \geq 0$, there exist $m_n \nearrow \infty$ and $x_n \in C$ such that $f^{m_n}(x_n) \in V$, i.e. $f^{m_n}(x_n) \notin U_i$ for any n. Since C is closed, we may assume x_n tends to some $x_0 \in C \subset C_i$. As above, if we put $k_n := min\{k \geq K; f^k(x_n) \notin U_i\}$, we have an accumulation point y of $\{f^{k_n}(x_n)\}$ outside Λ. By the above remark, $y \in W^u(\Lambda_i)\setminus\Lambda_i$. We have $y \notin W^s(\Lambda_i)$ because $W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$. Since $y \in A(C)$, $y \in K_{J_p} \setminus J_2 = W^s(\Lambda)$. Thus y must belong to $W^s(\Lambda_{i_1})$ for some $i_1 \neq i$. That is, we have a sequence $\{f^{k_n}(x_n)\}$ in $W^s(\Lambda_i)$ tending to $y \in W^u(\Lambda_i) \cap W^s(\Lambda_{i_1})$, hence we have an order $\Lambda_i \succ \Lambda_{i_1}$.

By successively applying this argument, we have a chain of saddle basic sets :

$$\Lambda_i \succ \Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots, \quad i \neq i_1 \neq i_2 \neq \cdots.$$

Since there exist only finitely many basic sets, we must have a cycle of them, which contradicts Lemma 1. This completes the proof of Theorem 1. □

We will prove Theorem 2. By the same argument as above, we have

Lemma 3. *Let $i, j \geq 1$. If $\overline{C_i} \cap C_j \neq \emptyset$, then $A(C_i) \cap (W^u(\Lambda_j) \setminus \Lambda) \neq \emptyset$. If C_i is closed, then $A(C_i) = \Lambda_i$.*

Note that $A_{pt}(C_{J_p}) = A(C_{J_p})$ if and only if $A(C_i) \subset \Lambda$ for any i. We have only to show (6).

($\Rightarrow$) If C_i for some i is not closed, then there exists a $j \neq i$ such that $\overline{C_i} \cap C_j \neq \emptyset$. If $i \geq 1$, then $j \geq 1$ and by Lemma 3, $A(C_i)$ contains a point outside Λ. Suppose C_0 is not closed. Then there exists a sequence $x_n \in C_0$ tending to a point $x_0 \in C_i$ for some $i \geq 1$. For a fixed large $R > 0$, put $k_n := min\{k \in \mathbb{N}; ||f^k(x_n)|| > R\}$. It is easy to see $k_n \to \infty$. (Otherwise,

$||f^L(x_0)|| \geq R$ for some $L \geq K$, which contradicts $x_0 \in C_i$.) Note that $\{f^{k_n}(x_n)\}$ is bounded. Thus, if we take any one of its accumulation points y, then $y \in A(C_0) \setminus K_{J_p}$, hence $A(C_0)$ intersects $W^u(\Lambda) \setminus K_{J_p}$.

($\Leftarrow$) By Lemma 3, it follows that, for $i \geq 1$, $A(C_i) = \Lambda_i$ if C_i is closed. If C_0 is closed, it is compact, hence $A(C_0) = \emptyset$. This completes the proof of Theorem 2. □

Now we prove Theorem 3.

(a) $\Rightarrow$ (b) By Theorem 2, $A(C_0) = \emptyset$ if C_0 is closed. Then

$$A(C_{J_p}) = \cup_{i=1}^m A(C_i) \subset K_{J_p} \cap (W^u(\Lambda) \cap (J_p \times \mathbb{C})) = W^u(\Lambda) \cap W^s(\Lambda).$$

(b) $\Rightarrow$ (a) As is shown in the proof of Theorem 2, if C_0 is not closed, then $A(C_0)$ intersects $W^u(\Lambda) \setminus K_{J_p}$. Thus $A(C_{J_p}) \neq W^u(\Lambda) \cap W^s(\Lambda)$.

(c) $\Rightarrow$ (a) Suppose C_0 is not closed. Then there exists a sequence $x_n = (z_n, c_n) \in C_0$ tending to a point $x_0 = (z_0, c_0) \in C_i$ for some $i \geq 1$. Then there exists $\delta > 0$ such that $\mathbb{D}(c_0, \delta) \subset int\, K_{z_0}$ since $c_0 \in int\, K_{z_0}$. Note that the map $z \mapsto J_z$ is continuous in J_p. Hence, if z is close to z_0, we have either $\mathbb{D}(c_0, \delta) \subset int\, K_z$ or $\mathbb{D}(c_0, \delta) \cap K_z = \emptyset$. Since for large n, $c_n \in \mathbb{D}(c_0, \delta)$ is outside K_{z_n}, we conclude that $\mathbb{D}(c_0, \delta) \cap K_{z_n} = \emptyset$ for large n. This implies the discontinuity of the map $z \mapsto K_z$ at $z = z_0$.

(a) $\Rightarrow$ (c) We use the argument in Lemma 3.7 of [J2]. Note that $z \mapsto K_z$ is upper semi-continuous in J_p. Hence if $z \mapsto K_z$ is discontinuous at $z = z_0$, then it is not lower semi-continuous there. Thus there exist $w_0 \in int\, K_{z_0}$ and $\delta > 0$ such that $D(w_0, \delta) \cap K_z = \emptyset$ for $z \neq z_0$ close to z_0. Put $(z_k, w_k) = f^k(z_0, w_0)$. By Corollary 3.6 in [J2] (see also Theorem 3.3 and Lemma 3.2 in Comerford [C]), there exist k and a critical point c_k of q_{z_k} in the connected component U_{w_k} of $int\, K_{z_k}$ containing w_k such that, for any $\epsilon > 0$, there exists an n so that $|w_n - Q_{z_k}^{n-k}(c_k)| < \epsilon$. Since C_0 is closed, the set $\cup_{i=1}^m C_i \ni (z_k, c_k)$ is away from C_0. Thus the critical point c'_k of $q_{p^k(z)}$ close to c_k for z sufficiently close to z_0 also belongs to $int\, K_{p^k(z)}$. For this n, take z sufficiently close to z_0 so that $|Q_z^n(w_0) - w_n| < \epsilon$ and that $|Q_{p^k(z)}^{n-k}(c'_k) - Q_{z_k}^{n-k}(c_k)| < \epsilon$. Thus we have

$$\begin{aligned} |Q_z^n(w_0) - Q_{p^k(z)}^{n-k}(c'_k)| &\leq |Q_z^n(w_0) - w_n| + |w_n - Q_{z_k}^{n-k}(c_k)| \\ &\quad + |Q_{z_k}^{n-k}(c_k) - Q_{p^k(z)}^{n-k}(c'_k)| \\ &< 3\epsilon. \end{aligned}$$

Since $Q_z^n(w_0) \notin K_{p^n(z)}$ and $Q_{p^k(z)}^{n-k}(c_k') \in int\, K_{p^n(z)}$, this implies the distance of the postcritical set from J_2 is less than 3ϵ. Since we can take ϵ arbitrarily small, this contradicts the fact that f is Axiom A. This completes the proof of Theorem 3. □

Remark 1. [DH1, DH2] *has proved (c) ⇒ (b).*

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