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Cooperation principle in random complex dynamics and singular functions on the complex plane *

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Abstract

We investigate the random dynamics of rational maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of semigroups of rational maps on $\hat{\mathbb{C}}$. We see that the both fields are related to each other very deeply. We show that regarding random complex dynamics of polynomials, in most cases, the chaos of the averaged system disappears, due to the cooperation of the generators. We investigate the iteration and spectral properties of transition operators. We show that under certain conditions, in the limit stage, "singular functions on the complex plane" appear. In particular, we consider the functions T which represent the probability of tending to infinity with respect to the random dynamics of polynomials. Under certain conditions these functions T are complex analogues of the devil's staircase and Lebesgue's singular functions. More precisely, we show that these functions T are continuous on $\hat{\mathbb{C}}$ and vary only on the Julia sets of associated semigroups. Furthermore, by using ergodic theory and potential theory, we investigate the non-differentiability and regularity of these functions. We also investigate stability and bifurcation of random complex dynamics. We show that stable systems are open and dense in the space of random dynamics of polynomials. We find many phenomena which can hold in the random complex dynamics and the dynamics of semigroups of rational maps, but cannot hold in the usual iteration dynamics of a single holomorphic map. We carry out a systematic study of these phenomena and their mechanisms.

1 Introduction

This is a research announcement article. Many results of this article has been written in [41], and the detail of some new results of this article will be written in [42].

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In this paper, we investigate the random dynamics of rational maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of rational semigroups (i.e., semigroups of non-constant rational maps where the semigroup operation is functional composition) on $\hat{\mathbb{C}}$. We see that the both fields are related to each other very deeply. In fact, we develop both theories simultaneously.

One motivation for research in complex dynamical systems is to describe some mathematical models on ethology. For example, the behavior of the population of a certain species can be described by the dynamical system associated with iteration of a polynomial $f(z) = az(1-z)$ such that f preserves the unit interval and the postcritical set in the plane is bounded (cf. [7]). However, when there is a change in the natural environment, some species have several strategies to survive in nature. From this point of view, it is very natural and important not only to consider the dynamics of iteration, where the same survival strategy (i.e., function) is repeatedly applied, but also to consider random dynamics, where a new strategy might be applied at each time step. The first study of random complex dynamics was given by J. E. Fornaess and N. Sibony ([9]). For research on random complex dynamics of quadratic polynomials, see [2, 3, 4, 5, 6, 10]. For research on random dynamics of polynomials (of general degrees) with bounded planar postcritical set, see the author's works [36, 35, 37, 38, 39, 40].

The first study of dynamics of rational semigroups was conducted by A. Hinkkanen and G. J. Martin ([13]), who were interested in the role of the dynamics of polynomial semigroups (i.e., semigroups of non-constant polynomial maps) while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group ([11]), who studied such semigroups from the perspective of random dynamical systems. Since the Julia set $J(G)$ of a finitely generated rational semigroup $G = \langle h_1, \dots, h_m \rangle$ has "backward self-similarity," i.e., $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$ (see [27, Lemma 1.1.4]), the study of the dynamics of rational semigroups can be regarded as the study of "backward iterated function systems," and also as a generalization of the study of self-similar sets in fractal geometry.

For recent work on the dynamics of rational semigroups, see the author's papers [27]–[40] and [26, 43, 44, 45, 46].

In order to consider the random dynamics of a family of polynomials on $\hat{\mathbb{C}}$, let $T_\infty(z)$ be the probability of tending to $\infty \in \hat{\mathbb{C}}$ starting with the initial value $z \in \hat{\mathbb{C}}$. In this paper, we see that under certain conditions, the function $T_\infty : \hat{\mathbb{C}} \to [0,1]$ is continuous on $\hat{\mathbb{C}}$ and has some singular properties (for instance, varies only inside a thin fractal set, the so-called Julia set of a polynomial semigroup), and this function is a complex analogue of the devil's staircase (Cantor function) or Lebesgue's singular functions (see Example 4.2, Figures 2, 3, and 4). Before going into detail, let us recall the definition of the devil's staircase (Cantor function) and Lebesgue's singular functions. Note that the following definitions look a little bit different from those in [47], but it turns out that they are equivalent to those in [47].

Definition 1.1 ([47]). Let $\varphi : \mathbb{R} \to [0,1]$ be the unique bounded function which satisfies the following functional equation:

$$\frac{1}{2}\varphi(3x) + \frac{1}{2}\varphi(3x-2) \equiv \varphi(x), \ \varphi|_{(-\infty,0]} \equiv 0, \ \varphi|_{[1,+\infty)} \equiv 1. \tag{1}$$

The function $\varphi|_{[0,1]} : [0,1] \to [0,1]$ is called the **devil's staircase (or Cantor function)**.

Remark 1.2. The above $\varphi : \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R}$ and varies precisely on the Cantor middle third set. Moreover, it is monotone (see Figure 1).

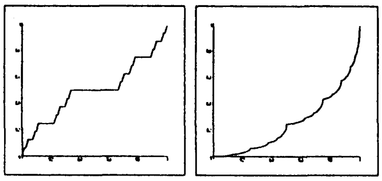
Definition 1.3 ([47]). Let $0 < a < 1$ be a constant. We denote by $\psi_a : \mathbb{R} \to [0,1]$ the unique bounded function which satisfies the following functional equation:

$$a\psi_a(2x) + (1-a)\psi_a(2x-1) \equiv \psi_a(x), \ \psi_a|_{(-\infty,0]} \equiv 0, \ \psi_a|_{[1,+\infty)} \equiv 1. \tag{2}$$

For each $a \in (0,1)$ with $a \neq 1/2$, the function $L_a := \psi_a|_{[0,1]} : [0,1] \to [0,1]$ is called **Lebesgue's singular function** with respect to the parameter a.

Remark 1.4. The function $\psi_a : \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R}$, monotone on $\mathbb{R}$, and strictly monotone on $[0,1]$. Moreover, if $a \neq 1/2$, then for almost every $x \in [0,1]$ with respect to the one-dimensional Lebesgue measure, the derivative of ψ_a at x is equal to zero (see Figure 1).

Figure 1: (From left to right) The graphs of the devil's staircase and Lebesgue's singular function.



These singular functions defined on $[0,1]$ can be redefined by using random dynamical systems on $\mathbb{R}$ as follows. Let $f_1(x) := 3x, f_2(x) := 3(x-1)+1$ $(x \in \mathbb{R})$ and we consider the random dynamical system (random walk) on $\mathbb{R}$ such that at every step we choose f_1 with probability $1/2$ and f_2 with probability $1/2$. We set $\hat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We denote by $T_{+\infty}(x)$ the probability of tending to $+\infty \in \hat{\mathbb{R}}$ starting with the initial value $x \in \mathbb{R}$. Then, we can see that the function $T_{+\infty}|_{[0,1]} : [0,1] \to [0,1]$ is equal to the devil's staircase.

Similarly, let $g_1(x) := 2x, g_2(x) := 2(x-1)+1$ $(x \in \mathbb{R})$ and let $0 < a < 1$ be a constant. We consider the random dynamical system on $\mathbb{R}$ such that at every step we choose the map g_1 with probability a and the map g_2 with probability $1-a$. Let $T_{+\infty,a}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$. Then, we can see that the function $T_{+\infty,a}|_{[0,1]} : [0,1] \to [0,1]$ is equal to Lebesgue's singular function L_a with respect to the parameter a.

We remark that in most of the literature, the theory of random dynamical systems has not been used directly to investigate these singular functions on the interval, although some researchers have used it implicitly.

One of the main purposes of this paper is to consider the complex analogue of the above story. In order to do that, we have to investigate the independent and identically-distributed (abbreviated by i.i.d.) random dynamics of rational maps and the dynamics of semigroups of rational maps on $\hat{\mathbb{C}}$ simultaneously. We develop both the theory of random dynamics of rational maps and that of the dynamics of semigroups of rational maps. The author thinks this is the best strategy since when we want to investigate one of them, we need to investigate another.

To introduce the main idea of this paper, we let G be a rational semigroup and denote by $F(G)$ the Fatou set of G, which is defined to be the maximal open subset of $\hat{\mathbb{C}}$ where

G is equicontinuous with respect to the spherical distance on $\hat{\mathbb{C}}$. We call $J(G) := \hat{\mathbb{C}} \setminus F(G)$ the Julia set of G. The Julia set is backward invariant under each element $h \in G$, but might not be forward invariant. This is a difficulty of the theory of rational semigroups. Nevertheless, we "utilize" this as follows. The key to investigating random complex dynamics is to consider the following **kernel Julia set** of G, which is defined by $J_{\ker}(G) = \bigcap_{g \in G} g^{-1}(J(G))$. This is the largest forward invariant subset of $J(G)$ under the action of G. Note that if G is a group or if G is a commutative semigroup, then $J_{\ker}(G) = J(G)$. However, for a general rational semigroup G generated by a family of rational maps h with $\deg(h) \geq 2$, it may happen that $\emptyset = J_{\ker}(G) \neq J(G)$ (see subsection 3.5, section 4).

Let Rat be the space of all non-constant rational maps on the Riemann sphere $\hat{\mathbb{C}}$, endowed with the distance κ which is defined by $\kappa(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\hat{\mathbb{C}}$. Let Rat_+ be the space of all rational maps g with $\deg(g) \geq 2$. Let $\mathcal{P}$ be the space of all polynomial maps g with $\deg(g) \geq 2$. Let τ be a Borel probability measure on Rat with compact support. We consider the i.i.d. random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \text{Rat}$ according to τ. Thus this determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and each Borel measurable subset A of $\hat{\mathbb{C}}$, the transition probability $p(x, A)$ of the Markov process is defined as $p(x, A) = \tau(\{g \in \text{Rat} \mid g(x) \in A\})$. Let G_τ be the rational semigroup generated by the support of τ. Let $C(\hat{\mathbb{C}})$ be the space of all complex-valued continuous functions on $\hat{\mathbb{C}}$ endowed with the supremum norm. Let M_τ be the operator on $C(\hat{\mathbb{C}})$ defined by $M_\tau(\varphi)(z) = \int \varphi(g(z)) d\tau(g)$. This M_τ is called the transition operator of the Markov process induced by τ. For a topological space X, let $\mathfrak{M}_1(X)$ be the space of all Borel probability measures on X endowed with the topology induced by the weak convergence (thus $\mu_n \to \mu$ in $\mathfrak{M}_1(X)$ if and only if $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{R}$). Note that if X is a compact metric space, then $\mathfrak{M}_1(X)$ is compact and metrizable. For each $\tau \in \mathfrak{M}_1(X)$, we denote by $\operatorname{supp} \tau$ the topological support of τ. Let $\mathfrak{M}_{1,c}(X)$ be the space of all Borel probability measures τ on X such that $\operatorname{supp} \tau$ is compact. Let $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the dual of M_τ. This M_τ^* can be regarded as the "averaged map" on the extension $\mathfrak{M}_1(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}}$ (see Remark 2.21). We define the "Julia set" $J_{meas}(\tau)$ of the dynamics of M_τ^* as the set of all elements $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ satisfying that for each neighborhood B of μ, $\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n \in \mathbb{N}}$ is not equicontinuous on B (see Definition 2.17). For each sequence $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\text{Rat})^{\mathbb{N}}$, we denote by J_γ the set of non-equicontinuity of the sequence $\{\gamma_n \circ \cdots \circ \gamma_1\}_{n \in \mathbb{N}}$ with respect to the spherical distance on $\hat{\mathbb{C}}$. This J_γ is called the Julia set of γ. Let $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau \in \mathfrak{M}_1((\text{Rat})^{\mathbb{N}})$.

We prove the following theorem.

Theorem 1.5 ([41], Cooperation Principle I, see Theorem 3.14). *Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$. Then $J_{meas}(\tau) = \emptyset$. Moreover, for $\tilde{\tau}$-a.e. $\gamma \in (\text{Rat})^{\mathbb{N}}$, the 2-dimensional Lebesgue measure of J_γ is equal to zero.*

This theorem means that if all the maps in the support of τ cooperate, the set of sensitive initial values of the averaged system disappears. Note that for any $h \in \text{Rat}_+$, $J_{meas}(\delta_h) \neq \emptyset$. Thus the above result deals with a phenomenon which can hold in the random complex dynamics but cannot hold in the usual iteration dynamics of a single rational map h with $\deg(h) \geq 2$.

From the above result and some further detailed arguments, we prove the following theorem. To state the theorem, for a $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we denote by U_τ the space of all finite linear combinations of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, where an eigenvector is said to be unitary if the absolute value of the corresponding eigenvalue is equal to one. Moreover, we set $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0\}$. Under the above notations, we have the following.

Theorem 1.6 ([41], Cooperation Principle II: Disappearance of Chaos, see Theorem 3.15).

Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then we have all of the following statements.

(1) *There exists a direct decomposition $C(\hat{\mathbb{C}}) = U_\tau \oplus \mathcal{B}_{0,\tau}$. Moreover, $\dim_{\mathbb{C}} U_\tau < \infty$ and $\mathcal{B}_{0,\tau}$ is a closed subspace of $C(\hat{\mathbb{C}})$. Moreover, there exists a non-empty M_τ^*-invariant compact subset A of $\mathfrak{M}_1(\hat{\mathbb{C}})$ with finite topological dimension such that for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$, $d((M_\tau^*)^n(\mu), A) \to 0$ in $\mathfrak{M}_1(\hat{\mathbb{C}})$ as $n \to \infty$. Furthermore, each element of U_τ is locally constant on $F(G_\tau)$. Therefore each element of U_τ is a continuous function on $\hat{\mathbb{C}}$ which varies only on the Julia set $J(G_\tau)$.*

(2) *For each $z \in \hat{\mathbb{C}}$, there exists a Borel subset $\mathcal{A}_z$ of $(\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{A}_z) = 1$ with the following property.*

 - *For each $\gamma = (\gamma_1, \gamma_2, \dots) \in \mathcal{A}_z$, there exists a number $\delta = \delta(z, \gamma) > 0$ such that diam$(\gamma_n \cdots \gamma_1(B(z,\delta))) \to 0$ as $n \to \infty$, where diam denotes the diameter with respect to the spherical distance on $\hat{\mathbb{C}}$, and $B(z,\delta)$ denotes the ball with center z and radius δ.*

(3) *There exists at least one and at most finitely many minimal sets for $(G_\tau, \hat{\mathbb{C}})$, where we say that a non-empty compact subset L of $\hat{\mathbb{C}}$ is a minimal set for $(G_\tau, \hat{\mathbb{C}})$ if L is minimal in $\{C \subset \hat{\mathbb{C}} \mid \emptyset \neq C \text{ is compact}, \forall g \in G_\tau, g(C) \subset C\}$ with respect to inclusion.*

(4) *Let S_τ be the union of minimal sets for $(G_\tau, \hat{\mathbb{C}})$. Then for each $z \in \hat{\mathbb{C}}$ there exists a Borel subset $\mathcal{C}_z$ of $(\mathrm{Rat})^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{C}_z) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \dots) \in \mathcal{C}_z$, $d(\gamma_n \cdots \gamma_1(z), S_\tau) \to 0$ as $n \to \infty$.*

This theorem means that if all the maps in the support of τ cooperate, the chaos of the averaged system disappears. Theorem 1.6 describes new phenomena which can hold in random complex dynamics but cannot hold in the usual iteration dynamics of a single $h \in \mathrm{Rat}_+$. For example, for any $h \in \mathrm{Rat}_+$, if we take a point $z \in J(h)$, where $J(h)$ denotes the Julia set of the semigroup generated by h, then for any ball B with $B \cap J(h) \neq \emptyset$, $h^n(B)$ expands as $n \to \infty$, and we have infinitely many minimal sets (periodic cycles) of h.

In Theorem 3.15, we completely investigate the structure of U_τ and the set of unitary eigenvalues of M_τ (Theorem 3.15). Using the above result, we show that if $\dim_{\mathbb{C}} U_\tau > 1$ and $\mathrm{int}(J(G_\tau)) = \emptyset$ where $\mathrm{int}(\cdot)$ denotes the set of interior points, then $F(G_\tau)$ has infinitely many connected components (Theorem 3.15-20). Thus the random complex dynamics can be applied to the theory of dynamics of rational semigroups. The key to proving

Theorem 1.6 (Theorem 3.15) is to show that for almost every $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\text{Rat})^{\mathbb{N}}$ with respect to $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau$ and for each compact set Q contained in a connected component U of $F(G_\tau)$, $\text{diam}\gamma_n \circ \cdots \circ \gamma_1(Q) \to 0$ as $n \to \infty$. This is shown by using careful arguments on the hyperbolic metric of each connected component of $F(G_\tau)$. Combining this with the decomposition theorem on "almost periodic operators" on Banach spaces from [19], we prove Theorem 1.6 (Theorem 3.15).

Considering these results, we have the following natural question: "When is the kernel Julia set empty?" Since the kernel Julia set of G is forward invariant under G, Montel's theorem implies that if τ is a Borel probability measure on $\mathcal{P}$ with compact support, and if the support of τ contains an admissible subset of $\mathcal{P}$ (see Definition 3.54), then $J_{\text{ker}}(G_\tau) = \emptyset$ (Lemma 3.56). In particular, if the support of τ contains an interior point with respect to the topology of $\mathcal{P}$, then $J_{\text{ker}}(G_\tau) = \emptyset$ (Lemma 3.52). From this result, it follows that for any Borel probability measure τ on $\mathcal{P}$ with compact support, there exists a Borel probability measure ρ with finite support, such that ρ is arbitrarily close to τ, such that the support of ρ is arbitrarily close to the support of τ , and such that $J_{\text{ker}}(G_\rho) = \emptyset$ (Proposition 3.57). The above results mean that in a certain sense, $J_{\text{ker}}(G_\tau) = \emptyset$ for most Borel probability measures τ on $\mathcal{P}$. Summarizing these results we can state the following. In order to state the result, let $\mathcal{O}$ be the topology of $\mathfrak{M}_{1,c}(\text{Rat})$ such that $\tau_n \to \tau$ in $(\mathfrak{M}_{1,c}(\text{Rat}), \mathcal{O})$ if and only if (a) $\int \varphi d\tau_n \to \int \varphi d\tau$ for each bounded continuous function φ on Rat, and (b) $\text{supp}\,\tau_n \to \text{supp}\,\tau$ with respect to the Hausdorff metric.

Theorem 1.7 ([41], Cooperation Principle III, see Lemmas 3.52, 3.56, Proposition 3.57). *Let $A := \{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid J_{\text{ker}}(G_\tau) = \emptyset\}$ and $B := \{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid J_{\text{ker}}(G_\tau) = \emptyset, \sharp\text{supp}\,\tau < \infty\}$. Then we have all of the following.*

(1) *A and B are dense in $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$.*

(2) *If the interior of the support of τ is not empty with respect to the topology of $\mathcal{P}$, then $\tau \in A$.*

(3) *For each $\tau \in A$, the chaos of the averaged system of the Markov process induced by τ disappears (more precisely, all the statements in Theorems 1.5, 1.6 hold).*

In the subsequent paper [42], we investigate more detail on the above result.

We remark that in 1983, by numerical experiments, K. Matsumoto and I. Tsuda ([21]) observed that if we add some uniform noise to the dynamical system associated with iteration of a chaotic map on the unit interval $[0,1]$, then under certain conditions, the quantities which represent chaos (e.g., entropy, Lyapunov exponent, etc.) decrease. More precisely, they observed that the entropy decreases and the Lyapunov exponent turns negative. They called this phenomenon "noise-induced order", and many physicists have investigated it by numerical experiments, although there has been only a few mathematical supports for it.

Moreover, in this paper, we introduce "mean stable" rational semigroups in subsection 3.6. If G is mean stable, then $J_{\text{ker}}(G) = \emptyset$ and a small perturbation H of G is still mean stable. We show that if Γ is a compact subset of Rat_+ and if the semigroup G generated by Γ is semi-hyperbolic (see Definition 2.12) and $J_{\text{ker}}(G) = \emptyset$, then there exists a neighborhood $\mathcal{V}$ of Γ in the space of non-empty compact subset of Rat such that for each $\Gamma' \in \mathcal{V}$, the semigroup G' generated by Γ' is mean stable, and $J_{\text{ker}}(G') = \emptyset$. Regarding the stability and mean stability, we present the following.

Theorem 1.8 (Cooperation Principle IV, Theorems 3.101, 3.106).

(1) *The set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau$ *is mean stable*$\}$ *is open and dense in* $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$. *Moreover, the set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid J_{\ker}(G_\tau) = \emptyset, J(G_\tau) \neq \emptyset\}$ *contains* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau$ *is mean stable*$\}$.

(2) *The set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau$ *is mean stable,* $\sharp\Gamma_\tau < \infty\}$ *is dense in* $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$.

(3) *Let* $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$ *be mean stable. Then there exists a neighborhood* Ω *of* τ *in* $(\mathfrak{M}_{1,c}(\mathrm{Rat}_+), \mathcal{O})$ *such that* $\nu \mapsto U_\nu$ *is continuous on* Ω *and the cardinality of the set of all minimal sets for* $(G_\nu, \hat{\mathbb{C}})$ *is constant on* Ω.

By using the above results, we investigate the random dynamics of polynomials. Let τ be a Borel probability measure on $\mathcal{P}$ with compact support. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and the smallest filled-in Julia set $\hat{K}(G_\tau)$ (see Definition 3.19) of G_τ is not empty. Then we show that the function $T_{\infty,\tau}$ of probability of tending to $\infty \in \hat{\mathbb{C}}$ belongs to U_τ and is not constant (Theorem 3.22). Thus $T_{\infty,\tau}$ is non-constant and continuous on $\hat{\mathbb{C}}$ and varies only on $J(G_\tau)$. Moreover, the function $T_{\infty,\tau}$ is characterized as the unique Borel measurable bounded function $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ which satisfies $M_\tau(\varphi) = \varphi$, $\varphi|_{F_\infty(G_\tau)} \equiv 1$, and $\varphi|_{\hat{K}(G_\tau)} \equiv 0$, where $F_\infty(G_\tau)$ denotes the connected component of the Fatou set $F(G_\tau)$ of G_τ containing ∞ (Proposition 3.26). From these results, we can show that $T_{\infty,\tau}$ has a kind of "monotonicity," and applying it, we get information regarding the structure of the Julia set $J(G_\tau)$ of G_τ (Theorem 3.31). We call the function $T_{\infty,\tau}$ a **devil's coliseum,** especially when $\mathrm{int}(J(G_\tau)) = \emptyset$ (see Example 4.2, Figures 2, 3, and 4). Note that for any $h \in \mathcal{P}$, T_{∞,δ_h} is not continuous at any point of $J(h) \neq \emptyset$. Thus the above results deal with a phenomenon which can hold in the random complex dynamics, but cannot hold in the usual iteration dynamics of a single polynomial.

It is a natural question to ask about the regularity of non-constant $\varphi \in U_\tau$ (e.g., $\varphi = T_{\infty,\tau}$) on the Julia set $J(G_\tau)$. For a rational semigroup G, we set $P(G) := \overline{\bigcup_{h\in G}\{\text{all critical values of } h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}$, where the closure is taken in $\hat{\mathbb{C}}$, and we say that G is **hyperbolic** if $P(G) \subset F(G)$. If G is generated by $\{h_1, \dots, h_m\}$ as a semigroup, we write $G = \langle h_1, \dots, h_m\rangle$. We prove the following theorem.

Theorem 1.9 ([41], see Theorem 3.82 and Theorem 3.84). *Let* $m \geq 2$ *and let* $(h_1, \dots, h_m) \in \mathcal{P}^m$. *Let* $G = \langle h_1, \dots, h_m\rangle$. *Let* $0 < p_1, p_2, \dots, p_m < 1$ *with* $\sum_{i=1}^m p_i = 1$. *Let* $\tau = \sum_{i=1}^m p_i\delta_{h_i}$. *Suppose that* $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ *for each* (i,j) *with* $i \neq j$ *and suppose also that* G *is hyperbolic. Then we have all of the following statements.*

(1) $J_{\ker}(G_\tau) = \emptyset$, *int*$(J(G_\tau)) = \emptyset$, *and* $\dim_H(J(G)) < 2$, *where* $\dim_H$ *denotes the Hausdorff dimension with respect to the spherical distance on* $\hat{\mathbb{C}}$.

(2) *Suppose further that at least one of the following conditions* (a)(b)(c) *holds.*

   (a) $\sum_{j=1}^m p_j \log(p_j \deg(h_j)) > 0$.

   (b) $P(G) \setminus \{\infty\}$ *is bounded in* $\mathbb{C}$.

   (c) $m = 2$.

Then there exists a non-atomic "invariant measure" λ on $J(G)$ with $\mathrm{supp}\,\lambda = J(G)$ and an uncountable dense subset A of $J(G)$ with $\lambda(A) = 1$ and $\dim_H(A) > 0$, such that for every $z \in A$ and for each non-constant $\varphi \in U_\tau$, the pointwise Hölder exponent of φ at z, which is defined to be

$$\inf\{\alpha \in \mathbb{R} \mid \limsup_{y \to z} \frac{|\varphi(y) - \varphi(z)|}{|y - z|^\alpha} = \infty\},$$

is strictly less than 1 *and φ is not differentiable at z (Theorem 3.82).*

(3) *In* (2) *above, the pointwise Hölder exponent of φ at z can be represented in terms of $p_j, \log(\deg(h_j))$ and the integral of the sum of the values of the Green's function of the basin of ∞ for the sequence $\gamma = (\gamma_1, \gamma_2, \dots) \in \{h_1, \dots, h_m\}^{\mathbb{N}}$ at the finite critical points of γ_1 (Theorem 3.82).*

(4) *Under the assumption of (2), for almost every point $z \in J(G)$ with respect to the δ-dimensional Hausdorff measure H^δ where $\delta = \dim_H(J(G))$, the pointwise Hölder exponent of a non-constant $\varphi \in U_\tau$ at z can be represented in terms of the p_j and the derivatives of h_j (Theorem 3.84).*

Combining Theorems 1.5, 1.6, 1.9, it follows that under the assumptions of Theorem 1.9, the chaos of the averaged system disappears in the C^0 "sense", but it remains in the C^1 "sense". From Theorem 1.9, we also obtain that if p_1 is small enough, then for almost every $z \in J(G)$ with respect to H^δ and for each $\varphi \in U_\tau$, φ is differentiable at z and the derivative of φ at z is equal to zero, even though a non-constant $\varphi \in U_\tau$ is not differentiable at any point of an uncountable dense subset of $J(G)$ (Remark 3.86). To prove these results, we use Birkhoff's ergodic theorem, potential theory, the Koebe distortion theorem and thermodynamic formalisms in ergodic theory. We can construct many examples of $(h_1, \dots, h_m) \in \mathcal{P}^m$ such that $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$, where $G = \langle h_1, \dots, h_m \rangle$, G is hyperbolic, $\hat{K}(G) \neq \emptyset$, and U_τ possesses non-constant elements (e.g., $T_{\infty,\tau}$) for any $\tau = \sum_{i=1}^m p_i \delta_{h_i}$ (see Proposition 4.1, Example 4.2, Proposition 4.3, Proposition 4.4, and Remark 4.6).

As pointed out in the previous paragraphs, we find many new phenomena which can hold in random complex dynamics and the dynamics of rational semigroups, but cannot hold in the usual iteration dynamics of a single rational map. These new phenomena and their mechanisms are systematically investigated.

In the proofs of all results, we employ the skew product map associated with the support of τ (Definition 3.46), and some detailed observations concerning the skew product are required. It is a new idea to use the kernel Julia set of the associated semigroup to investigate random complex dynamics. Moreover, it is both natural and new to combine the theory of random complex dynamics and the theory of rational semigroups. Without considering the Julia sets of rational semigroups, we are unable to discern the singular properties of the non-constant finite linear combinations φ (e.g., $\varphi = T_{\infty,\tau}$, a devil's coliseum) of the unitary eigenvectors of M_τ.

In section 2, we give some fundamental notations and definitions. In section 3, we present the main results of this article. The results of subsections 3.1–3.8 have been written in [41]. For the proofs of the results of subsections 3.1–3.8, see [41]. The proofs of the results of subsections 3.9–3.10 will be written in [42]. In section 4, we give many examples to which the main results are applicable.

In the subsequent paper [42], we investigate the stability and bifurcation of M_τ.

2 Preliminaries

In this section, we give some basic definitions and notations on the dynamics of semigroups of holomorphic maps and the i.i.d. random dynamics of holomorphic maps.

Notation: Let (X, d) be a metric space, A a subset of X, and $r > 0$. We set $B(A, r) := \{z \in X \mid d(z, A) < r\}$. Moreover, for a subset C of $\mathbb{C}$, we set $D(C, r) := \{z \in \mathbb{C} \mid \inf_{a \in C} |z - a| < r\}$. Moreover, for any topological space Y and for any subset A of Y, we denote by $\mathrm{int}(A)$ the set of all interior points of A.

Definition 2.1. Let Y be a metric space. We set $\mathrm{CM}(Y) := \{f : Y \to Y \mid f \text{ is continuous}\}$ endowed with the compact-open topology. Moreover, we set $\mathrm{OCM}(Y) := \{f \in \mathrm{CM}(Y) \mid f \text{ is an open map}\}$ endowed with the relative topology from $\mathrm{CM}(Y)$. Furthermore, we set $C(Y) := \{\varphi : Y \to \mathbb{C} \mid \varphi \text{ is continuous }\}$. When Y is compact, we endow $C(Y)$ with the supremum norm $\|\cdot\|_\infty$. Moreover, for a subset $\mathcal{F}$ of $C(Y)$, we set $\mathcal{F}_{nc} := \{\varphi \in \mathcal{F} \mid \varphi \text{ is not constant}\}$.

Definition 2.2. Let Y be a complex manifold. We set $\mathrm{HM}(Y) := \{f : Y \to Y \mid f \text{ is holomorphic}\}$ endowed with the compact open topology. Moreover, we set $\mathrm{NHM}(Y) := \{f \in \mathrm{HM}(Y) \mid f \text{ is not constant}\}$ endowed with the compact open topology.

Remark 2.3. $\mathrm{CM}(Y)$, $\mathrm{OCM}(Y)$, $\mathrm{HM}(Y)$, and $\mathrm{NHM}(Y)$ are semigroups with the semigroup operation being functional composition.

Definition 2.4. A **rational semigroup** is a semigroup generated by a family of non-constant rational maps on the Riemann sphere $\hat{\mathbb{C}}$ with the semigroup operation being functional composition([13, 11]). A **polynomial semigroup** is a semigroup generated by a family of non-constant polynomial maps. We set $\mathrm{Rat} := \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-constant rational map}\}$ endowed with the distance κ which is defined by $\kappa(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\hat{\mathbb{C}}$. Moreover, we set $\mathrm{Rat}_+ := \{h \in \mathrm{Rat} \mid \deg(h) \geq 2\}$ endowed with the relative topology from Rat. Furthermore, we set $\mathcal{P} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g \text{ is a polynomial}, \deg(g) \geq 2\}$ endowed with the relative topology from Rat.

Definition 2.5. Let Y be a compact metric space and let G be a subsemigroup of $\mathrm{CM}(Y)$. The **Fatou set** of G is defined to be $F(G) :=$
$\{z \in Y \mid \exists \text{ neighborhood } U \text{ of } z \text{ s.t. } \{g|_U : U \to Y\}_{g \in G} \text{ is equicontinuous on } U\}$. (For the definition of equicontinuity, see [1].) The **Julia set** of G is defined to be $J(G) := Y \setminus F(G)$. If G is generated by $\{g_i\}_i$, then we write $G = \langle g_1, g_2, \ldots \rangle$. If G is generated by a subset Γ of $\mathrm{CM}(Y)$, then we write $G = \langle \Gamma \rangle$. For finitely many elements $g_1, \ldots, g_m \in \mathrm{CM}(Y)$, we set $F(g_1, \ldots, g_m) := F(\langle g_1, \ldots, g_m \rangle)$ and $J(g_1, \ldots, g_m) := J(\langle g_1, \ldots, g_m \rangle)$. For a subset A of Y, we set $G(A) := \bigcup_{g \in G} g(A)$ and $G^{-1}(A) := \bigcup_{g \in G} g^{-1}(A)$. We set $G^* := G \cup \{\mathrm{Id}\}$, where Id denotes the identity map.

By using the method in [13, 11], it is easy to see that the following lemma holds.

Lemma 2.6. *Let Y be a compact metric space and let G be a subsemigroup of* $\mathrm{OCM}(Y)$. *Then for each $h \in G$, $h(F(G)) \subset F(G)$ and $h^{-1}(J(G)) \subset J(G)$. Note that the equality does not hold in general.*

The following is the key to investigating random complex dynamics.

Definition 2.7. Let Y be a compact metric space and let G be a subsemigroup of $\mathrm{CM}(Y)$. We set $J_{\mathrm{ker}}(G) := \bigcap_{g \in G} g^{-1}(J(G))$. This is called the **kernel Julia set** of G.

Remark 2.8. Let Y be a compact metric space and let G be a subsemigroup of $\mathrm{CM}(Y)$. (1) $J_{\mathrm{ker}}(G)$ is a compact subset of $J(G)$. (2) For each $h \in G$, $h(J_{\mathrm{ker}}(G)) \subset J_{\mathrm{ker}}(G)$. (3) If G is a rational semigroup and if $F(G) \neq \emptyset$, then $\mathrm{int}(J_{\mathrm{ker}}(G)) = \emptyset$. (4) If G is generated by a single map or if G is a group, then $J_{\mathrm{ker}}(G) = J(G)$. However, for a general rational semigroup G, it may happen that $\emptyset = J_{\mathrm{ker}}(G) \neq J(G)$ (see subsection 3.5 and section 4).

The following **postcritical set** is important when we investigate the dynamics of rational semigroups.

Definition 2.9. For a rational semigroup G, let $P(G) := \overline{\bigcup_{g \in G}\{\text{all critical values of } g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}$ where the closure is taken in $\hat{\mathbb{C}}$. This is called the **postcritical set** of G.

Remark 2.10. If $\Gamma \subset \mathrm{Rat}$ and $G = \langle \Gamma \rangle$, then $P(G) = \overline{G^{*}(\bigcup_{h \in \Gamma}\{\text{all critical values of } h\})}$. From this one may know the figure of $P(G)$, in the finitely generated case, using a computer.

Definition 2.11. Let G be a rational semigroup. Let N be a positive integer. We denote by $SH_N(G)$ the set of points $z \in \hat{\mathbb{C}}$ satisfying that there exists a positive number δ such that for each $g \in G$, $\deg(g : V \to B(z, \delta)) \leq N$, for each connected component V of $g^{-1}(B(z, \delta))$. Moreover, we set $UH(G) := \hat{\mathbb{C}} \setminus \bigcup_{N \in \mathbb{N}} SH_N(G)$.

Definition 2.12. Let G be a rational semigroup. We say that G is **hyperbolic** if $P(G) \subset F(G)$. We say that G is **semi-hyperbolic** if $UH(G) \subset F(G)$.

Remark 2.13. We have $UH(G) \subset P(G)$. If G is hyperbolic, then G is semi-hyperbolic.

It is sometimes important to investigate the dynamics of sequences of maps.

Definition 2.14. Let Y be a compact metric space. For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathrm{CM}(Y))^{\mathbb{N}}$ and each $m, n \in \mathbb{N}$ with $m \geq n$, we set $\gamma_{m,n} = \gamma_m \circ \cdots \circ \gamma_n$ and we set

$$F_\gamma := \{z \in Y \mid \exists \text{ neighborhood } U \text{ of } z \text{ s.t. } \{\gamma_{n,1}\}_{n \in \mathbb{N}} \text{ is equicontinuous on } U\}$$

and $J_\gamma := Y \setminus F_\gamma$. The set F_γ is called the **Fatou set** of the sequence γ and the set J_γ is called the **Julia set** of the sequence γ.

Remark 2.15. Let $Y = \hat{\mathbb{C}}$ and let $\gamma \in (\mathrm{Rat}_+)^{\mathbb{N}}$. Then by [1, Theorem 2.8.2], $J_\gamma \neq \emptyset$. Moreover, if Γ is a non-empty compact subset of Rat_+ and $\gamma \in \Gamma^{\mathbb{N}}$, then by [30], J_γ is a perfect set and J_γ has uncountably many points.

We now give some notations on random dynamics.

Definition 2.16. For a topological space Y, we denote by $\mathfrak{M}_1(Y)$ the space of all Borel probability measures on Y endowed with the topology such that $\mu_n \to \mu$ in $\mathfrak{M}_1(Y)$ if and only if for each bounded continuous function $\varphi : Y \to \mathbb{C}$, $\int \varphi \, d\mu_n \to \int \varphi \, d\mu$. Note that if Y is a compact metric space, then $\mathfrak{M}_1(Y)$ is a compact metric space with the metric $d_0(\mu_1, \mu_2) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\int \phi_j d\mu_1 - \int \phi_j d\mu_2|}{1+|\int \phi_j d\mu_1 - \int \phi_j d\mu_2|}$, where $\{\phi_j\}_{j\in\mathbb{N}}$ is a dense subset of $C(Y)$. Moreover, for each $\tau \in \mathfrak{M}_1(Y)$, we set $\operatorname{supp}\tau := \{z \in Y \mid \forall \text{ neighborhood } U \text{ of } z, \ \tau(U) > 0\}$. Note that $\operatorname{supp}\tau$ is a closed subset of Y. Furthermore, we set $\mathfrak{M}_{1,c}(Y) := \{\tau \in \mathfrak{M}_1(Y) \mid \operatorname{supp}\tau \text{ is compact}\}$.

For a complex Banach space $\mathcal{B}$, we denote by $\mathcal{B}^*$ the space of all continuous complex linear functionals $\rho : \mathcal{B} \to \mathbb{C}$, endowed with the weak* topology.

For any $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$, we will consider the i.i.d. random dynamics on Y such that at every step we choose a map $g \in \mathrm{CM}(Y)$ according to τ (thus this determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space Y such that for each $x \in Y$ and each Borel measurable subset A of Y, the transition probability $p(x, A)$ of the Markov process is defined as $p(x, A) = \tau(\{g \in \mathrm{CM}(Y) \mid g(x) \in A\})$).

Definition 2.17. Let Y be a compact metric space. Let $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$.

1. We set $\Gamma_\tau := \operatorname{supp}\tau$ (thus Γ_τ is a closed subset of $\mathrm{CM}(Y)$). Moreover, we set $X_\tau := (\Gamma_\tau)^{\mathbb{N}}$ $(= \{\gamma = (\gamma_1, \gamma_2, \ldots) \mid \gamma_j \in \Gamma_\tau\})$ endowed with the product topology. Furthermore, we set $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau$. This is the unique Borel probability measure on X_τ such that for each cylinder set $A = A_1 \times \cdots \times A_n \times \Gamma_\tau \times \Gamma_\tau \times \cdots$ in X_τ, $\tilde{\tau}(A) = \prod_{j=1}^{n} \tau(A_j)$. We denote by G_τ the subsemigroup of $\mathrm{CM}(Y)$ generated by the subset Γ_τ of $\mathrm{CM}(Y)$.

2. Let M_τ be the operator on $C(Y)$ defined by $M_\tau(\varphi)(z) := \int_{\Gamma_\tau} \varphi(g(z)) \, d\tau(g)$. M_τ is called the **transition operator** of the Markov process induced by τ. Moreover, let $M_\tau^* : C(Y)^* \to C(Y)^*$ be the dual of M_τ, which is defined as $M_\tau^*(\mu)(\varphi) = \mu(M_\tau(\varphi))$ for each $\mu \in C(Y)^*$ and each $\varphi \in C(Y)$. Remark: we have $M_\tau^*(\mathfrak{M}_1(Y)) \subset \mathfrak{M}_1(Y)$ and for each $\mu \in \mathfrak{M}_1(Y)$ and each open subset V of Y, we have $M_\tau^*(\mu)(V) = \int_{\Gamma_\tau} \mu(g^{-1}(V)) \, d\tau(g)$.

3. We denote by $F_{meas}(\tau)$ the set of $\mu \in \mathfrak{M}_1(Y)$ satisfying that there exists a neighborhood B of μ in $\mathfrak{M}_1(Y)$ such that the sequence $\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(Y)\}_{n\in\mathbb{N}}$ is equicontinuous on B. We set $J_{meas}(\tau) := \mathfrak{M}_1(Y) \setminus F_{meas}(\tau)$.

4. We denote by $F_{meas}^0(\tau)$ the set of $\mu \in \mathfrak{M}_1(Y)$ satisfying that the sequence$\{(M_\tau^*)^n : \mathfrak{M}_1(Y) \to \mathfrak{M}_1(Y)\}_{n\in\mathbb{N}}$ is equicontinuous at the one point μ. We set $J_{meas}^0(\tau) := \mathfrak{M}_1(Y) \setminus F_{meas}^0(\tau)$.

Remark 2.18. We have $F_{meas}(\tau) \subset F_{meas}^0(\tau)$ and $J_{meas}^0(\tau) \subset J_{meas}(\tau)$.

Remark 2.19. Let Γ be a closed subset of Rat. Then there exists a $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ such that $\Gamma_\tau = \Gamma$. By using this fact, we sometimes apply the results on random complex dynamics to the study of the dynamics of rational semigroups.

Definition 2.20. Let Y be a compact metric space. Let $\Phi : Y \to \mathfrak{M}_1(Y)$ be the topological embedding defined by: $\Phi(z) := \delta_z$, where δ_z denotes the Dirac measure at z. Using this topological embedding $\Phi : Y \to \mathfrak{M}_1(Y)$, we regard Y as a compact subset of $\mathfrak{M}_1(Y)$.

Remark 2.21. If $h \in \mathrm{CM}(Y)$ and $\tau = \delta_h$, then we have $M_\tau^* \circ \Phi = \Phi \circ h$ on Y. Moreover, for a general $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$, $M_\tau^*(\mu) = \int h_*(\mu) d\tau(h)$ for each $\mu \in \mathfrak{M}_1(Y)$. Therefore, for a general $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$, the map $M_\tau^* : \mathfrak{M}_1(Y) \to \mathfrak{M}_1(Y)$ can be regarded as the "averaged map" on the extension $\mathfrak{M}_1(Y)$ of Y.

Remark 2.22. If $\tau = \delta_h \in \mathfrak{M}_1(\mathrm{Rat}_+)$ with $h \in \mathrm{Rat}_+$, then $J_{meas}(\tau) \neq \emptyset$. In fact, using the embedding $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$, we have $\emptyset \neq \Phi(J(h)) \subset J_{meas}(\tau)$.

The following is an important and interesting object in random dynamics.

Definition 2.23. Let Y be a compact metric space and let A be a subset of Y. Let $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$. For each $z \in Y$, we set $T_{A,\tau}(z) := \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in X_\tau \mid d(\gamma_{n,1}(z), A) \to 0 \text{ as } n \to \infty\})$. This is the probability of tending to A starting with the initial value $z \in Y$. For any $a \in Y$, we set $T_{a,\tau} := T_{\{a\},\tau}$.

3 Results

In this section, we present the main results of this article. The results of subsections 3.1–3.8 have been written in [41]. For the proofs of the results of subsections 3.1–3.8, see [41]. The results and their proofs of subsections 3.9–3.10 will be written in [42].

3.1 General results and properties of M_τ

In this subsection, we present some general results and some results on properties of the iteration of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ and $M_\tau^* : C(\hat{\mathbb{C}})^* \to C(\hat{\mathbb{C}})^*$. We need some notations.

Definition 3.1. Let Y be a n-dimensional smooth manifold. We denote by Leb_n the two-dimensional Lebesgue measure on Y.

Definition 3.2. Let $\mathcal{B}$ be a complex vector space and let $M : \mathcal{B} \to \mathcal{B}$ be an operator. Let $\varphi \in \mathcal{B}$ and $a \in \mathbb{C}$ be such that $\varphi \neq 0, |a| = 1$, and $M(\varphi) = a\varphi$. Then we say that φ is a unitary eigenvector of M with respect to a, and we say that a is a unitary eigenvalue.

Definition 3.3. Let Y be a compact metric space and let $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$. Let K be a non-empty subset of Y such that $G(K) \subset K$. We denote by $\mathcal{U}_{f,\tau}(K)$ the set of all unitary eigenvectors of $M_\tau : C(K) \to C(K)$. Moreover, we denote by $\mathcal{U}_{v,\tau}(K)$ the set of all unitary eigenvalues of $M_\tau : C(K) \to C(K)$. Similarly, we denote by $\mathcal{U}_{f,\tau,*}(K)$ the set of all unitary eigenvectors of $M_\tau^* : C(K)^* \to C(K)^*$, and we denote by $\mathcal{U}_{v,\tau,*}(K)$ the set of all unitary eigenvalues of $M_\tau^* : C(K)^* \to C(K)^*$.

Definition 3.4. Let V be a complex vector space and let A be a subset of V. We set $\mathrm{LS}(A) := \{\sum_{j=1}^m a_j v_j \mid a_1, \ldots, a_m \in \mathbb{C}, v_1, \ldots, v_m \in A, m \in \mathbb{N}\}$.

Definition 3.5. Let Y be a topological space and let V be a subset of Y. We denote by $C_V(Y)$ the space of all $\varphi \in C(Y)$ such that for each connected component U of V, there exists a constant $c_U \in \mathbb{C}$ with $\varphi|_U \equiv c_U$.

Remark 3.6. $C_V(Y)$ is a linear subspace of $C(Y)$. Moreover, if Y is compact, metrizable, and locally connected and V is an open subset of Y, then $C_V(Y)$ is a closed subspace of $C(Y)$. Furthermore, if Y is compact, metrizable, and locally connected, $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$, and G_τ is a subsemigroup of $\mathrm{OCM}(Y)$, then $M_\tau(C_{F(G_\tau)}(Y)) \subset C_{F(G_\tau)}(Y)$.

Definition 3.7. For a topological space Y, we denote by $\mathrm{Cpt}(Y)$ the space of all nonempty compact subsets of Y. If Y is a metric space, we endow $\mathrm{Cpt}(Y)$ with the Hausdorff metric.

Definition 3.8. Let Y be a metric space and let G be a subsemigroup of $\mathrm{CM}(Y)$. Let $K \in \mathrm{Cpt}(Y)$. We say that K is a minimal set for (G, Y) if K is minimal among the space $\{L \in \mathrm{Cpt}(Y) \mid G(L) \subset L\}$ with respect to inclusion. Moreover, we set $\mathrm{Min}(G, Y) := \{K \in \mathrm{Cpt}(Y) \mid K \text{ is minimal for } (G, Y)\}$.

Remark 3.9. Let Y be a metric space and let G be a subsemigroup of $\mathrm{CM}(Y)$. By Zorn's lemma, it is easy to see that if $K_1 \in \mathrm{Cpt}(Y)$ and $G(K_1) \subset K_1$, then there exists a $K \in \mathrm{Min}(G, Y)$ with $K \subset K_1$. Moreover, it is easy to see that for each $K \in \mathrm{Min}(G, Y)$ and each $z \in K$, $\overline{G(z)} = K$. In particular, if $K_1, K_2 \in \mathrm{Min}(G, Y)$ with $K_1 \neq K_2$, then $K_1 \cap K_2 = \emptyset$. Moreover, by the formula $\overline{G(z)} = K$, we obtain that for each $K \in \mathrm{Min}(G, Y)$, either (1) $\sharp K < \infty$ or (2) K is perfect and $\sharp K > \aleph_0$. Furthermore, it is easy to see that if $\Gamma \in \mathrm{Cpt}(\mathrm{CM}(Y)), G = \langle \Gamma \rangle$, and $K \in \mathrm{Min}(G, Y)$, then $K = \bigcup_{h \in \Gamma} h(K)$.

Definition 3.10. Let Y be a compact metric space. Let $\rho \in C(Y)^*$. We denote by $a(\rho)$ the set of points $z \in Y$ which satisfies that there exists a neighborhood U of z in Y such that for each $\varphi \in C(Y)$ with $\mathrm{supp}\,\varphi \subset U$, $\rho(\varphi) = 0$. We set $\mathrm{supp}\,\rho := Y \setminus a(\rho)$.

Definition 3.11. Let $\{\varphi_n : U \to \hat{\mathbb{C}}\}_{n=1}^{\infty}$ be a sequence of holomorphic maps on an open set U of $\hat{\mathbb{C}}$. Let $\varphi : U \to \hat{\mathbb{C}}$ be a holomorphic map. We say that φ is a limit function of $\{\varphi_n\}_{n=1}^{\infty}$ if there exists a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ in $\mathbb{N}$ such that $\varphi_{n_j} \to \varphi$ as $j \to \infty$ locally uniformly on U.

Definition 3.12. For a topological space Z, we denote by $\mathrm{Con}(Z)$ the set of all connected components of Z.

Definition 3.13. Let G be a rational semigroup. We set $J_{res}(G) := \{z \in J(G) \mid \forall U \in \mathrm{Con}(F(G)), z \notin \partial U\}$. This is called the **residual Julia set** of G.

We now present the main results.

Theorem 3.14 (Cooperation Principle I). *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{NHM}(\mathbb{CP}^n))$, where $\mathbb{CP}^n$ denotes the n-dimensional complex projective space. Suppose that $J_{\ker}(G_\tau) = \emptyset$. Then, $F_{meas}(\tau) = \mathfrak{M}_1(\mathbb{CP}^n)$, and for $\tilde{\tau}$-a.e. $\gamma \in (\mathrm{NHM}(\mathbb{CP}^n))^{\mathbb{N}}$, $\mathrm{Leb}_{2n}(J_\gamma) = 0$.*

Theorem 3.15 (Cooperation Principle II: Disappearance of Chaos). *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ and let $S_\tau := \bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L$. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then, all of the following statements 1,...,21 hold.*

1. *Let $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}$. Then, $\mathcal{B}_{0,\tau}$ is a closed subspace of $C(\hat{\mathbb{C}})$ and there exists a direct sum decomposition $C(\hat{\mathbb{C}}) = \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \oplus \mathcal{B}_{0,\tau}$. Moreover, $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \subset C_{F(G_\tau)}(\hat{\mathbb{C}})$ and $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) < \infty$.*

2. *Let $q := \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))$. Let $\{\varphi_j\}_{j=1}^q$ be a basis of $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$ such that for each $j = 1, \ldots, q$, there exists an $\alpha_j \in \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ with $M_\tau(\varphi_j) = \alpha_j \varphi_j$. Then, there exists a unique family $\{\rho_j : C(\hat{\mathbb{C}}) \to \mathbb{C}\}_{j=1}^q$ of complex linear functionals such that for each $\varphi \in C(\hat{\mathbb{C}})$, $\|M_\tau^n(\varphi - \sum_{j=1}^q \rho_j(\varphi)\varphi_j)\|_\infty \to 0$ as $n \to \infty$. Moreover, $\{\rho_j\}_{j=1}^q$ satisfies all of the following.*

(a) *For each* $j = 1, \dots, q$, $\rho_j : C(\hat{\mathbb{C}}) \to \mathbb{C}$ *is continuous.*

(b) *For each* $j = 1, \dots, q$, $M_\tau^*(\rho_j) = \alpha_j \rho_j$.

(c) *For each* (i, j), $\rho_i(\varphi_j) = \delta_{ij}$. *Moreover,* $\{\rho_j\}_{j=1}^q$ *is a basis of* $\mathrm{LS}(\mathcal{U}_{f,\tau,*}(\hat{\mathbb{C}}))$.

(d) *For each* $j = 1, \dots, q$, $\operatorname{supp} \rho_j \subset S_\tau$.

3. *We have* $\sharp J(G_\tau) \geq 3$. *In particular, for each* $U \in \mathrm{Con}(F(G_\tau))$, *we can take the hyperbolic metric on* U.

4. *There exists a Borel measurable subset* $\mathcal{A}$ *of* $(\mathrm{Rat})^{\mathbb{N}}$ *with* $\tilde{\tau}(\mathcal{A}) = 1$ *such that*

   (a) *for each* $\gamma \in \mathcal{A}$ *and for each* $U \in \mathrm{Con}(F(G_\tau))$, *each limit function of* $\{\gamma_{n,1}|_U\}_{n=1}^\infty$ *is constant, and*

   (b) *for each* $\gamma \in \mathcal{A}$ *and for each* $Q \in \mathrm{Cpt}(F(G_\tau))$, $\sup_{a \in Q} \|\gamma_{n,1}'(a)\|_h \to 0$ *as* $n \to \infty$, *where* $\|\gamma_{n,1}'(a)\|_h$ *denotes the norm of the derivative of* $\gamma_{n,1}$ *at a measured from the hyperbolic metric on the element* $U_0 \in \mathrm{Con}(F(G_\tau))$ *with* $a \in U_0$ *to that on the element* $U_n \in \mathrm{Con}(F(G_\tau))$ *with* $\gamma_{n,1}(a) \in U_n$.

5. *For each* $z \in \hat{\mathbb{C}}$, *there exists a Borel subset* $\mathcal{A}_z$ *of* $(\mathrm{Rat})^{\mathbb{N}}$ *with* $\tilde{\tau}(\mathcal{A}_z) = 1$ *with the following property.*

   - *For each* $\gamma = (\gamma_1, \gamma_2, \dots) \in \mathcal{A}_z$, *there exists a number* $\delta = \delta(z, \gamma) > 0$ *such that* $diam(\gamma_n \cdots \gamma_1(B(z, \delta))) \to 0$ *as* $n \to \infty$, *where diam denotes the diameter with respect to the spherical distance on* $\hat{\mathbb{C}}$, *and* $B(z, \delta)$ *denotes the ball with center* z *and radius* δ.

6. $\sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) < \infty$.

7. *Let* $W := \bigcup_{A \in \mathrm{Con}(F(G_\tau)), A \cap S_\tau \neq \emptyset} A$. *Then* S_τ *is compact. Moreover, for each* $z \in \hat{\mathbb{C}}$ *there exists a Borel measurable subset* $\mathcal{C}_z$ *of* $(\mathrm{Rat})^{\mathbb{N}}$ *with* $\tilde{\tau}(\mathcal{C}_z) = 1$ *such that for each* $\gamma \in \mathcal{C}_z$, *there exists an* $n \in \mathbb{N}$ *with* $\gamma_{n,1}(z) \in W$ *and* $d(\gamma_{m,1}(z), S_\tau) \to 0$ *as* $m \to \infty$.

8. *Let* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *and* $r_L := \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$. *Then,* $\mathcal{U}_{v,\tau}(L)$ *is a finite subgroup of* S^1 *with* $\sharp \mathcal{U}_{v,\tau}(L) = r_L$. *Moreover, there exists an* $a_L \in S^1$ *and a family* $\{\psi_{L,j}\}_{j=1}^{r_L}$ *in* $\mathcal{U}_{f,\tau}(L)$ *such that*

   (a) $a_L^{r_L} = 1$, $\mathcal{U}_{v,\tau}(L) = \{a_L^j\}_{j=1}^{r_L}$,

   (b) $M_\tau(\psi_{L,j}) = a_L^j \psi_{L,j}$ *for each* $j = 1, \dots, r_L$,

   (c) $\psi_{L,j} = (\psi_{L,1})^j$ *for each* $j = 1, \dots, r_L$, *and*

   (d) $\{\psi_{L,j}\}_{j=1}^{r_L}$ *is a basis of* $\mathrm{LS}(\mathcal{U}_{f,\tau}(L))$.

9. *Let* $\Psi_{S_\tau} : \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \to C(S_\tau)$ *be the map defined by* $\varphi \mapsto \varphi|_{S_\tau}$. *Then,* $\Psi_{S_\tau}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) = \mathrm{LS}(\mathcal{U}_{f,\tau}(S_\tau))$ *and* $\Psi_{S_\tau} : \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \to \mathrm{LS}(\mathcal{U}_{f,\tau}(S_\tau))$ *is a linear isomorphism. Furthermore,* $\Psi_{S_\tau} \circ M_\tau = M_\tau \circ \Psi_{S_\tau}$ *on* $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$.

10. $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}}) = \mathcal{U}_{v,\tau}(S_\tau) = \bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \mathcal{U}_{v,\tau}(L) = \bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \{a_L^j\}_{j=1}^{r_L}$ *and* $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) = \sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} r_L$.

11. $\mathcal{U}_{v,\tau,*}(\hat{\mathbb{C}}) = \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$, $\mathcal{U}_{v,\tau,*}(S_\tau) = \mathcal{U}_{v,\tau}(S_\tau)$, *and* $\mathcal{U}_{v,\tau,*}(L) = \mathcal{U}_{v,\tau}(L)$ *for each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$.

12. *Let* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$. *Let* $\Lambda_{r_L} := \{g_1 \circ \cdots \circ g_{r_L} \mid \forall j, g_j \in \Gamma_\tau\}$. *Moreover, let* $G_\tau^{r_L} := \langle \Lambda_{r_L} \rangle$. *Then,* $r_L = \sharp \mathrm{Min}(G_\tau^{r_L}, L)$.

13. *There exists a basis* $\{\varphi_{L,i} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \ldots, r_L\}$ *of* $\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$ *and a basis* $\{\rho_{L,i} \mid L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}}), i = 1, \ldots, r_L\}$ *of* $\mathrm{LS}(\mathcal{U}_{f,\tau,*}(\hat{\mathbb{C}}))$ *such that for each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *and for each* $i = 1, \ldots, r_L$, *we have all of the following.*
    (a) $M_\tau(\varphi_{L,i}) = a_L^i \varphi_{L,i}$.
    (b) $|\varphi_{L,i}||_L \equiv 1$.
    (c) $\varphi_{L,i}|_{L'} \equiv 0$ *for any* $L' \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ *with* $L' \neq L$.
    (d) $\varphi_{L,i}|_L = (\varphi_{L,1}|_L)^i$.
    (e) $\mathrm{supp}\, \rho_{L,i} = L$.
    (f) $\rho_{L,i}(\varphi_{L,j}) = \delta_{ij}$ *for each* $j = 1, \ldots, r_L$.

14. *For each* $\nu \in \mathfrak{M}_1(\hat{\mathbb{C}})$, $d_0((M_\tau^*)^n(\nu), \mathrm{LS}(\mathcal{U}_{f,\tau,*}(\hat{\mathbb{C}})) \cap \mathfrak{M}_1(\hat{\mathbb{C}})) \to 0$ *as* $n \to \infty$. *Moreover,* $\dim_T(\mathrm{LS}(\mathcal{U}_{f,\tau,*}(\hat{\mathbb{C}})) \cap \mathfrak{M}_1(\hat{\mathbb{C}})) \le 2 \dim_{\mathbb{C}} \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) < \infty$, *where* $\dim_T$ *denotes the topological dimension.*

15. *For each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, $T_{L,\tau} : \hat{\mathbb{C}} \to [0,1]$ *is continuous and* $M_\tau(T_{L,\tau}) = T_{L,\tau}$. *Moreover,* $\sum_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} T_{L,\tau}(z) = 1$ *for each* $z \in \hat{\mathbb{C}}$.

16. *If* $\sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) \ge 2$, *then* (a) *for each* $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$, $T_{L,\tau}(J(G_\tau)) = [0,1]$, *and* (b) $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) > 1$.

17. $S_\tau = \overline{\{z \in F(G) \cap S_\tau \mid \exists g \in G_\tau \text{ s.t. } g(z) = z, |m(g,z)| < 1\}}$, *where the closure is taken in* $\hat{\mathbb{C}}$, *and* $m(g,z)$ *denotes the multiplier ([1]) of* g *at the fixed point* z.

18. *If* $\Gamma_\tau \cap \mathrm{Rat}_+ \neq \emptyset$, *then* $S_\tau = \overline{\{z \in F(G) \cap S_\tau \mid \exists g \in G_\tau \cap \mathrm{Rat}_+ \text{ s.t. } g(z) = z, |m(g,z)| < 1\}} \subset UH(G_\tau) \subset P(G_\tau)$.

19. *If* $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) > 1$, *then for any* $\varphi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))_{nc}$ *there exists an uncountable subset* A *of* $\mathbb{C}$ *such that for each* $t \in A$, $\emptyset \neq \varphi^{-1}(\{t\}) \cap J(G_\tau) \subset J_{res}(G_\tau)$.

20. *If* $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) > 1$ *and* $\mathrm{int}(J(G_\tau)) = \emptyset$, *then* $\sharp \mathrm{Con}(F(G_\tau)) = \infty$.

21. *Suppose that* $G_\tau \cap \mathrm{Aut}(\hat{\mathbb{C}}) \neq \emptyset$, *where* $\mathrm{Aut}(\hat{\mathbb{C}})$ *denotes the set of all holomorphic automorphisms on* $\hat{\mathbb{C}}$. *If there exists a loxodromic or parabolic element of* $G_\tau \cap \mathrm{Aut}(\hat{\mathbb{C}})$, *then* $\sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) = 1$ *and* $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) = 1$.

Remark 3.16. Let G be a rational semigroup with $G \cap \mathrm{Rat}_+ \neq \emptyset$. Then by [1, Theorem 4.2.4], $\sharp(J(G)) \ge 3$.

Remark 3.17. Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be such that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. The union S_τ of minimal sets for $(G_\tau, \hat{\mathbb{C}})$ may meet $J(G_\tau)$. See Example 4.7.

Remark 3.18. Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ be such that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then $\dim_{\mathbb{C}}(\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) > 1$ if and only if $(\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc} \neq \emptyset$.

Definition 3.19. Let G be a polynomial semigroup. We set
$\hat{K}(G) := \{z \in \mathbb{C} \mid \{g(z) \mid g \in G\}$ is bounded in $\mathbb{C}\}$. $\hat{K}(G)$ is called the **smallest filled-in Julia set** of G. For any $h \in \mathcal{P}$, we set $K(h) := \hat{K}(\langle h \rangle)$. This is called the filled-in Julia set of h.

Remark 3.20. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ be such that $J_{\ker}(G_\tau) = \emptyset$ and $\hat{K}(G_\tau) \neq \emptyset$. Then $\sharp\text{Min}(G_\tau, \hat{\mathbb{C}}) \geq 2$. Thus by Theorem 3.15-16, $\dim_{\mathbb{C}}(\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) > 1$.

Remark 3.21. There exist many examples of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ such that $J_{\ker}(G_\tau) = \emptyset, \hat{K}(G_\tau) \neq \emptyset$ and $\text{int}(J(G_\tau)) = \emptyset$ (see Proposition 4.1, Proposition 4.3, Proposition 4.4, Theorem 3.82, and [28, Theorem 2.3]).

3.2 Properties on $T_{\infty,\tau}$

In this subsection, we present some results on properties of $T_{\infty,\tau}$ for a $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Moreover, we present some results on the structure of $J(G_\tau)$ for a $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $J_{\ker}(G_\tau) = \emptyset$.

By Theorem 3.14 or Theorem 3.15, we obtain the following result.

Theorem 3.22. *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$. Then, the function $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ is continuous on the whole $\hat{\mathbb{C}}$, and $M_\tau(T_{\infty,\tau}) = T_{\infty,\tau}$.*

Remark 3.23. Let $h \in \mathcal{P}$ and let $\tau := \delta_h$. Then, $T_{\infty,\tau}(\hat{\mathbb{C}}) = \{0,1\}$ and $T_{\infty,\tau}$ is not continuous at every point in $J(h) \neq \emptyset$.

On the one hand, we have the following, due to Vitali's theorem.

Lemma 3.24. *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Then, for each connected component U of $F(G_\tau)$, there exists a constant $C_U \in [0,1]$ such that $T_{\infty,\tau}|_U \equiv C_U$.*

Definition 3.25. Let G be a polynomial semigroup. If $\infty \in F(G)$, then we denote by $F_\infty(G)$ the connected component of $F(G)$ containing ∞. (Note that if G is generated by a compact subset of $\mathcal{P}$, then $\infty \in F(G)$.)

We give a characterization of $T_{\infty,\tau}$.

Proposition 3.26. *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $\hat{K}(G_\tau) \neq \emptyset$. Then, there exists a unique bounded Borel measurable function $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ such that $\varphi = M_\tau(\varphi)$, $\varphi|_{F_\infty(G_\tau)} \equiv 1$ and $\varphi|_{\hat{K}(G_\tau)} \equiv 0$. Moreover, $\varphi = T_{\infty,\tau}$.*

Remark 3.27. Combining Theorem 3.22 and Lemma 3.24, it follows that under the assumptions of Theorem 3.22, if $T_{\infty,\tau} \not\equiv 1$, then the function $T_{\infty,\tau}$ is continuous on $\hat{\mathbb{C}}$ and varies only on the Julia set $J(G_\tau)$ of G_τ. In this case, the function $T_{\infty,\tau}$ is called the **devil's coliseum** (see Figures 3, 4). This is a complex analogue of the devil's staircase or Lebesgue's singular functions. We will see the monotonicity of this function $T_{\infty,\tau}$ in Theorem 3.31.

In order to present the result on the monotonicity of the function $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$, the level set of $T_{\infty,\tau}|_{J(G_\tau)}$ and the structure of the Julia set $J(G_\tau)$, we need the following notations.

Definition 3.28. Let $K_1, K_2 \in \mathrm{Cpt}(\hat{\mathbb{C}})$.

1. "$K_1 <_s K_2$" indicates that K_1 is included in the union of all bounded components of $\mathbb{C} \setminus K_2$.

2. "$K_1 \leq_s K_2$" indicates that $K_1 <_s K_2$ or $K_1 = K_2$.

Remark 3.29. This "$\leq_s$" is a partial order in $\mathrm{Cpt}(\hat{\mathbb{C}})$. This "$\leq_s$" is called the **surrounding order.**

We present a necessary and sufficient condition for $T_{\infty,\tau}$ to be the constant function 1.

Lemma 3.30. *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Then, the following* (1), (2), *and* (3) *are equivalent.* (1) $T_{\infty,\tau} \equiv 1$. (2) $T_{\infty,\tau}|_{J(G_\tau)} \equiv 1$. (3) $\hat{K}(G_\tau) = \emptyset$.

By Theorem 3.22 and Lemma 3.24, we obtain the following result.

Theorem 3.31 (Monotonicity of $T_{\infty,\tau}$ and the structure of $J(G_\tau)$). *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that $J_{\mathrm{ker}}(G_\tau) = \emptyset$ and $\hat{K}(G_\tau) \neq \emptyset$. Then, we have all of the following.*

1. $\mathrm{int}(\hat{K}(G_\tau)) \neq \emptyset$.

2. $T_{\infty,\tau}(J(G_\tau)) = [0,1]$.

3. *For each $t_1, t_2 \in [0,1]$ with $0 \leq t_1 < t_2 \leq 1$, we have $T_{\infty,\tau}^{-1}(\{t_1\}) <_s T_{\infty,\tau}^{-1}(\{t_2\}) \cap J(G_\tau)$.*

4. *For each $t \in (0,1)$, we have $\hat{K}(G_\tau) <_s T_{\infty,\tau}^{-1}(\{t\}) \cap J(G_\tau) <_s \overline{F_\infty(G_\tau)}$.*

5. *There exists an uncountable dense subset A of $[0,1]$ with $\sharp([0,1] \setminus A) \leq \aleph_0$ such that for each $t \in A$, we have $\emptyset \neq T_{\infty,\tau}^{-1}(\{t\}) \cap J(G_\tau) \subset J_{res}(G_\tau)$.*

Remark 3.32. If G is generated by a single map $h \in \mathcal{P}$, then $\partial \hat{K}(G) = \partial F_\infty(G) = J(G)$ and so $\hat{K}(G)$ and $\overline{F_\infty(G)}$ cannot be separated. However, under the assumptions of Theorem 3.31, the theorem implies that $\hat{K}(G_\tau)$ and $\overline{F_\infty(G_\tau)}$ are separated by the uncountably many level sets $\{T_{\infty,\tau}|_{J(G_\tau)}^{-1}(\{t\})\}_{t\in(0,1)}$, and that these level sets are totally ordered with respect to the surrounding order, respecting the usual order in $(0,1)$. Note that there are many $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ such that $J_{\mathrm{ker}}(G_\tau) = \emptyset$ and $\hat{K}(G_\tau) \neq \emptyset$. See section 4.

Remark 3.33. For each $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$, there exists a $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ such that $\Gamma_\tau = \Gamma$. Thus, Theorem 3.31 tells us the information of the Julia set of a polynomial semigroup G generated by a compact subset Γ of $\mathcal{P}$ such that $J_{\mathrm{ker}}(G) = \emptyset$ and $\hat{K}(G) \neq \emptyset$.

Applying Theorem 3.22 and Lemma 3.24, we obtain the following result.

Theorem 3.34. *Let Γ be a non-empty compact subset of $\mathcal{P}$ and let $G = \langle \Gamma \rangle$. Suppose that $\hat{K}(G) \neq \emptyset$ and $J_{\mathrm{ker}}(G) = \emptyset$. Then, at least one of the following statements* (a) *and* (b) *holds.*

(a) $\mathrm{int}(J(G)) \neq \emptyset$. (b) $\sharp\{U \in \mathrm{Con}(F(G)) \mid U \neq F_\infty(G)$ *and* $U \not\subset \mathrm{int}(\hat{K}(G))\} = \infty$.

Remark 3.35. There exist finitely generated polynomial semigroups G in $\mathcal{P}$ such that $\mathrm{int}(J(G)) \neq \emptyset$ and $J(G) \neq \hat{\mathbb{C}}$ (see [14], Example 4.11).

3.3 Planar postcritical set and the condition that $\hat{K}(G_\tau) = \emptyset$

In this subsection, we present some results which are deduced from the condition that the planar postcritical set is unbounded. Moreover, we present some results which are deduced from the condition that $\hat{K}(G_\tau) = \emptyset$.

Definition 3.36. For a polynomial semigroup G, we set $P^*(G) := P(G) \setminus \{\infty\}$. This is called the **planar postcritical set** of the polynomial semigroup G.

Definition 3.37. Let Y be a complete metric space. We say that a subset A of Y is residual if A contains a countable intersection of open dense subsets of Y. Note that by Baire's category theorem, a residual subset A of Y is dense in Y.

The following theorem generalizes [2, Theorem 1.5] and [4, Theorem 2.3].

Theorem 3.38. *Let $\Gamma \in \text{Cpt}(\mathcal{P})$ and let $G = \langle \Gamma \rangle$. Suppose that $P^*(G)$ is not bounded in $\mathbb{C}$. Then, there exists a residual subset $\mathcal{U}$ of $\Gamma^{\mathbb{N}}$ such that for each $\tau \in \mathfrak{M}_1(\mathcal{P})$ with $\Gamma_\tau = \Gamma$, we have $\tilde{\tau}(\mathcal{U}) = 1$, and such that for each $\gamma \in \mathcal{U}$, the Julia set J_γ of γ has uncountably many connected components.*

Question 3.39. What happens if $\hat{K}(G_\tau) = \emptyset$ (i.e., if $T_{\infty,\tau} \equiv 1$) ?

Definition 3.40. Let $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{P}^{\mathbb{N}}$. We set $K_\gamma := \{z \in \mathbb{C} \mid \{\gamma_{n,1}(z)\}_{n \in \mathbb{N}} \text{ is bounded in } \mathbb{C}\}$. Moreover, we set $A_{\infty,\gamma} := \{z \in \hat{\mathbb{C}} \mid \gamma_{n,1}(z) \to \infty\}$.

Theorem 3.41. *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that $\hat{K}(G_\tau) = \emptyset$. Then, we have all of the following statements 1,...,4.*

1. *$J_{\text{ker}}(G_\tau) = \emptyset$.*
2. *$F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$ and $(M_\tau^*)^n(\nu) \to \delta_\infty$ as $n \to \infty$ uniformly on $\nu \in \mathfrak{M}_1(\hat{\mathbb{C}})$.*
3. *$T_{\infty,\tau} \equiv 1$ on $\hat{\mathbb{C}}$.*
4. *For $\tilde{\tau}$-a.e. $\gamma \in \mathcal{P}^{\mathbb{N}}$,* (a) $\text{Leb}_2(K_\gamma) = 0$, (b) *$K_\gamma = J_\gamma$, and* (c) *$K_\gamma = J_\gamma$ has uncountably many connected components.*

Remark 3.42. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that $J_{\text{ker}}(G_\tau) = \emptyset$. From Theorem 3.22 and Theorem 3.41, it follows that $\hat{K}(G_\tau) \neq \emptyset$ if and only if $(\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc} \neq \emptyset$.

Example 3.43. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ and suppose that there exist two elements $h_1, h_2 \in \Gamma_\tau$ such that $K(h_1) \cap K(h_2) = \emptyset$. Then $\hat{K}(G_\tau) = \emptyset$. For more examples of τ with $\hat{K}(G_\tau) = \emptyset$, see Example 3.59.

3.4 Conditions to be $\text{Leb}_2(J_\gamma) = 0$ for $\tilde{\tau}$-a.e. γ (even if $J_{\text{ker}}(G_\tau) \neq \emptyset$)

In this subsection, we present some sufficient conditions to be $\text{Leb}_2(J_\gamma) = 0$ for $\tilde{\tau}$-a.e. γ. More precisely, we show that even if $J_{\text{ker}}(G_\tau) \neq \emptyset$, under certain conditions, for $\tilde{\tau}$-a.e. γ, for Leb_2-a.e. $z \in \hat{\mathbb{C}}$, there exists a number $n_0 \in \mathbb{N}$ such that for each n with $n \geq n_0$, $\gamma_{n,1}(z) \in F(G_\tau)$. We also define other kinds of Julia sets of M_τ^*.

Definition 3.44. Let Y be a compact metric space. Let $\tau \in \mathfrak{M}_1(\mathrm{CM}(Y))$. Regarding Y as a compact subset of $\mathfrak{M}_1(Y)$ as in Definition 2.20, we use the following notation.

1. We denote by $F_{pt}(\tau)$ the set of $z \in Y$ satisfying that there exists a neighborhood B of z in Y such that the sequence$\{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(Y)\}_{n\in\mathbb{N}}$ is equicontinuous on B. We set $J_{pt}(\tau) := Y \setminus F_{pt}(\tau)$.

2. Similarly, we denote by $F_{pt}^0(\tau)$ the set of $z \in Y$ such that the sequence $\{(M_\tau^*)^n|_Y : Y \to \mathfrak{M}_1(Y)\}_{n\in\mathbb{N}}$ is equicontinuous at the one point $z \in Y$. We set $J_{pt}^0(\tau) := Y \setminus F_{pt}^0(\tau)$.

Remark 3.45. We have $F_{pt}(\tau) \subset F_{pt}^0(\tau)$ and $J_{pt}^0(\tau) \subset J_{pt}(\tau) \cap J_{meas}^0(\tau)$.

We also need the following notations on the skew products. In fact, we heavily use the idea and the notations of the dynamics of skew products, to prove many results of this paper.

Definition 3.46. Let Y be a compact metric space and let Γ be a non-empty compact subset of $\mathrm{CM}(Y)$. We define a map $f : \Gamma^{\mathbb{N}} \times Y \to \Gamma^{\mathbb{N}} \times Y$ as follows: For a point $(\gamma, y) \in \Gamma^{\mathbb{N}} \times Y$ where $\gamma = (\gamma_1, \gamma_2, \ldots)$, we set $f(\gamma, y) := (\sigma(\gamma), \gamma_1(y))$, where $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is the shift map, that is, $\sigma(\gamma_1, \gamma_2, \ldots) = (\gamma_2, \gamma_3, \ldots)$. The map $f : \Gamma^{\mathbb{N}} \times Y \to \Gamma^{\mathbb{N}} \times Y$ is called the **skew product associated with the generator system** Γ. Moreover, we use the following notation.

1. Let $\pi : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}}$ and $\pi_Y : \Gamma^{\mathbb{N}} \times Y \to Y$ be the canonical projections. For each $\gamma \in \Gamma^{\mathbb{N}}$ and $n \in \mathbb{N}$, we set $f_\gamma^n := f^n|_{\pi^{-1}\{\gamma\}} : \pi^{-1}\{\gamma\} \to \pi^{-1}\{\sigma^n(\gamma)\}$. Moreover, we set $f_{\gamma,n} := \gamma_n \circ \cdots \circ \gamma_1$.

2. For each $\gamma \in \Gamma^{\mathbb{N}}$, we set $J^\gamma := \{\gamma\} \times J_\gamma$ $(\subset \Gamma^{\mathbb{N}} \times Y)$. Moreover, we set $\tilde{J}(f) := \overline{\bigcup_{\gamma\in\Gamma^{\mathbb{N}}} J^\gamma}$, where the closure is taken in the product space $\Gamma^{\mathbb{N}} \times Y$. Furthermore, we set $\tilde{F}(f) := (\Gamma^{\mathbb{N}} \times Y) \setminus \tilde{J}(f)$.

3. For each $\gamma \in \Gamma^{\mathbb{N}}$, we set $\hat{J}^{\gamma,\Gamma} := \pi^{-1}\{\gamma\} \cap \tilde{J}(f)$, $\hat{F}^{\gamma,\Gamma} := \pi^{-1}(\{\gamma\}) \setminus \hat{J}^{\gamma,\Gamma}$, $\hat{J}_{\gamma,\Gamma} := \pi_Y(\hat{J}^{\gamma,\Gamma})$, and $\hat{F}_{\gamma,\Gamma} := Y \setminus \hat{J}_{\gamma,\Gamma}$. Note that $J_\gamma \subset \hat{J}_{\gamma,\Gamma}$.

4. When $\Gamma \subset \mathrm{Rat}$, for each $z = (\gamma, y) \in \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$, we set $f'(z) := (\gamma_1)'(y)$.

Remark 3.47. Under the above notation, let $G = \langle \Gamma \rangle$. Then $\pi_Y(\tilde{J}(f)) \subset J(G)$ and $\pi \circ f = \sigma \circ \pi$ on $\Gamma^{\mathbb{N}} \times Y$. Moreover, for each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_1(J_\gamma) \subset J_{\sigma(\gamma)}$, $\gamma_1(\hat{J}_{\gamma,\Gamma}) \subset \hat{J}_{\sigma(\gamma),\Gamma}$, and $f(\tilde{J}(f)) \subset \tilde{J}(f)$. Furthermore, if $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$, then for each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_1(J_\gamma) = J_{\sigma(\gamma)}$, $\gamma_1^{-1}(J_{\sigma(\gamma)}) = J_\gamma$, $\gamma_1(\hat{J}_{\gamma,\Gamma}) = \hat{J}_{\sigma(\gamma),\Gamma}$, $\gamma_1^{-1}(\hat{J}_{\sigma(\gamma),\Gamma}) = \hat{J}_{\gamma,\Gamma}$, $f(\tilde{J}(f)) = \tilde{J}(f) = f^{-1}(\tilde{J}(f))$, and $f(\tilde{F}(f)) = \tilde{F}(f) = f^{-1}(\tilde{F}(f))$ (see [30, Lemma 2.4]).

We now present the results. Even if $J_{\ker}(G_\tau) \neq \emptyset$, we have the following.

Theorem 3.48. *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that $J_{\ker}(G_\tau)$ is included in the unbounded component of $\mathbb{C} \setminus (UH(G_\tau) \cap J(G_\tau))$. Then, we have the following.*

1. *For $\tilde{\tau}$-a.e. $\gamma \in X_\tau$, $\mathrm{Leb}_2(J_\gamma) = \mathrm{Leb}_2(\hat{J}_{\gamma,\Gamma_\tau}) = 0$.*

2. *For* Leb_2*-a.e.* $y \in \hat{\mathbb{C}}$, *for* $\tilde{\tau}$*-a.e.* $\gamma \in X_\tau$, *there exists an* $n = n(y,\gamma) \in \mathbb{N}$ *such that* $\gamma_{n,1}(y) \in F(G_\tau)$.

3. $\mathrm{Leb}_2(J^0_{pt}(\tau)) = 0$.

4. *For* Leb_2*-a.e. point* $y \in \hat{\mathbb{C}}$, $T_{\infty,\tau}$ *is continuous at* y.

Remark 3.49. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. If $J_{\ker}(G_\tau)$ is included in the unbounded component of $\mathbb{C} \setminus (P(G_\tau) \cap J(G_\tau))$, then $J_{\ker}(G_\tau)$ is included in the unbounded component of $\mathbb{C} \setminus (UH(G_\tau) \cap J(G_\tau))$ (see Remark 2.13).

Remark 3.50. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that for each $h \in \Gamma_\tau$, h is a real polynomial and each critical value of h in $\mathbb{C}$ belongs to $\mathbb{R}$. Suppose also that for each $z \in P(G_\tau) \cap J(G_\tau)$, there exists an element $g_z \in G_\tau$ such that $g_z(z) \in F(G_\tau)$. Then $J_{\ker}(G_\tau)$ is included in the unbounded component of $\mathbb{C} \setminus (UH(G_\tau) \cap J(G_\tau))$.

3.5 Conditions to be $J_{\ker}(G) = \emptyset$

In this subsection, we present some sufficient conditions to be $J_{\ker}(G) = \emptyset$.
The following is a natural question.

Question 3.51. When do we have that $J_{\ker}(G) = \emptyset$?

We give several answers to this question.

Lemma 3.52. *Let* Γ *be a subset of* Rat *such that the interior of* Γ *with respect to the topology of* Rat *is not empty. Let* $G = \langle \Gamma \rangle$. *Suppose that* $F(G) \neq \emptyset$. *Then,* $J_{\ker}(G) = \emptyset$.

Definition 3.53. Let Λ be a finite dimensional complex manifold and let $\{g_\lambda\}_{\lambda\in\Lambda}$ be a family of rational maps on $\hat{\mathbb{C}}$. We say that $\{g_\lambda\}_{\lambda\in\Lambda}$ is a holomorphic family of rational maps if the map $(z,\lambda) \in \hat{\mathbb{C}} \times \Lambda \mapsto g_\lambda(z) \in \hat{\mathbb{C}}$ is holomorphic on $\hat{\mathbb{C}} \times \Lambda$. We say that $\{g_\lambda\}_{\lambda\in\Lambda}$ is a holomorphic family of polynomials if $\{g_\lambda\}_{\lambda\in\Lambda}$ is a holomorphic family of rational maps and each g_λ is a polynomial.

Definition 3.54. Let $\mathcal{Y}$ be a subset of $\mathcal{P}$.

1. We say that $\mathcal{Y}$ is *admissible* if for each $z_0 \in \mathbb{C}$ there exists a holomorphic family of polynomials $\{g_\lambda\}_{\lambda\in\Lambda}$ such that $\{g_\lambda \mid \lambda \in \Lambda\} \subset \mathcal{Y}$ and the map $\lambda \mapsto g_\lambda(z_0)$ is nonconstant in Λ.

2. We say that $\mathcal{Y}$ is *strongly admissible* if for each $(z_0, h_0) \in \mathbb{C} \times \mathcal{Y}$ there exists a holomorphic family $\{g_\lambda\}_{\lambda\in\Lambda}$ of polynomials and a point $\lambda_0 \in \Lambda$ such that $\{g_\lambda \mid \lambda \in \Lambda\} \subset \mathcal{Y}$, $g_{\lambda_0} = h_0$, and the map $\lambda \mapsto g_\lambda(z_0) \in \mathbb{C}$ is nonconstant in any neighborhood of λ_0 in Λ.

Example 3.55.

1. Let $\mathcal{Y}$ be a strongly admissible subset of $\mathcal{P}$. Let $\mathcal{Y}$ be endowed with the relative topology from $\mathcal{P}$. If Γ is a non-empty open subset of $\mathcal{Y}$, then Γ is strongly admissible. If Γ' is a subset of $\mathcal{Y}$ such that the interior of Γ' in $\mathcal{Y}$ is not empty, then Γ' is admissible.

2. $\mathcal{P}$ is strongly admissible. If Γ is a subset of $\mathcal{P}$ such that the interior of Γ in $\mathcal{P}$ is not empty, then Γ is admissible.

3. For a fixed $h_0 \in \mathcal{P}, \mathcal{Y} := \{h_0 + c \mid c \in \mathbb{C}\}$ is a strongly admissible closed subset of $\mathcal{P}$. If Γ is a subset of $\mathcal{Y}$ such that the interior of Γ in $\mathcal{Y}$ is not empty, then Γ is admissible.

Lemma 3.56. *Let Γ be a relative compact admissible subset of $\mathcal{P}$. Let $G = \langle \Gamma \rangle$. Then, $J_{\ker}(G) = \emptyset$.*

Proposition 3.57. *Let $\mathcal{Y}$ be a closed subset of an open subset of $\mathcal{P}$. Suppose that $\mathcal{Y}$ is strongly admissible. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{Y})$. Let V_1 be any neighborhood of τ in $\mathfrak{M}_1(\mathcal{Y})$ and V_2 any neighborhood of Γ_τ in* $\text{Cpt}(\hat{\mathbb{C}})$. *Then, there exists an element $\rho \in \mathfrak{M}_1(\mathcal{Y})$ such that $\rho \in V_1$, $\Gamma_\rho \in V_2$, $\sharp \Gamma_\rho < \infty$, and $J_{\ker}(G_\rho) = \emptyset$.*

Remark 3.58 (Cooperation Principle III). By Lemma 3.56, Proposition 3.57, Theorems 3.14, 3.15, we can state that for most $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$, the chaos of the averaged system of the Markov process induced by τ disappears. In the subsequent paper [42], we investigate the further detail regarding this result.

Example 3.59. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ be such that Γ_τ is admissible. Suppose that there exists an element $h \in \Gamma_\tau$ with $\text{int}(K(h)) = \emptyset$. Then $\hat{K}(G_\tau) = \emptyset$ and the statements in Theorem 3.41 hold. For, if $\hat{K}(G_\tau) \neq \emptyset$, then since Γ_τ is admissible and since $G_\tau(\hat{K}(G_\tau)) \subset \hat{K}(G_\tau)$, we have $\text{int}(\hat{K}(G_\tau)) \neq \emptyset$. However, since $\text{int}(K(h)) = \emptyset$, this is a contradiction. Thus $\hat{K}(G_\tau) = \emptyset$.

From the above argument, we obtain many examples of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ such that $\hat{K}(G_\tau) = \emptyset$. For example, if $h(z) = z^2 + c$ belongs to the boundary of the Mandelbrot set and Γ_τ contains a neighborhood of h in the c-plane, then from the above argument, $\hat{K}(G_\tau) = \emptyset$ and the statements in Theorem 3.41 hold. Thus the above argument generalizes [4, Theorem 2.2] and a statement in [2, Theorem 2.4].

3.6 Mean stability

In this subsection, we introduce mean stable rational semigroups, and we present some results on mean stability.

Definition 3.60. Let Y be a compact metric space and let $\Gamma \in \text{Cpt}(\text{CM}(Y))$. Let $G = \langle \Gamma \rangle$. We say that G is **mean stable** if there exist non-empty open subsets U, V of $F(G)$ and a number $n \in \mathbb{N}$ such that all of the following hold.

(1) $\overline{V} \subset U$ and $\overline{U} \subset F(G)$.

(2) For each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_{n,1}(\overline{U}) \subset V$.

(3) For each point $z \in Y$, there exists an element $g \in G$ such that $g(z) \in U$.

Note that this definition does not depend on the choice of a compact set Γ which generates G. Moreover, for a $\Gamma \in \text{Cpt}(\text{CM}(Y))$, we say that Γ is mean stable if $\langle \Gamma \rangle$ is mean stable. Furthermore, for a $\tau \in \mathfrak{M}_{1,c}(\text{CM}(Y))$, we say that τ is mean stable if G_τ is mean stable.

Remark 3.61. It is easy to see that if G is mean stable, then $J_{\ker}(G) = \emptyset$.

By Montel's theorem, it is easy to see that the following lemma holds.

Lemma 3.62. *Let* $\Gamma \in \mathrm{Cpt(Rat)}$ *be mean stable. Suppose* $\sharp(\hat{\mathbb{C}} \setminus V) \geq 3$, *where* V *is the open set coming from Definition 3.60. Then there exists a neighborhood* $\mathcal{U}$ *of* Γ *in* $\mathrm{Cpt(Rat)}$ *with respect to the Hausdorff metric such that each* $\Gamma' \in \mathcal{U}$ *is mean stable.*

Proposition 3.63. *Let* $\Gamma \in \mathrm{Cpt(Rat_+)}$. *Suppose that* $J_{\ker}(\langle \Gamma \rangle) = \emptyset$ *and* $\langle \Gamma \rangle$ *is semi-hyperbolic. Then there exists an open neighborhood* U *of* Γ *in* $\mathrm{Cpt(Rat)}$ *such that for each* $\Gamma' \in U$, Γ' *is mean stable and* $J_{\ker}(\langle \Gamma' \rangle) = \emptyset$.

Remark 3.64. Let $\Gamma \in \mathrm{Cpt(Rat_+)}$. Suppose that $J_{\ker}(\langle \Gamma \rangle) = \emptyset$ and $\langle \Gamma \rangle$ is semi-hyperbolic. Then for a small perturbation Γ' of Γ, Γ' is mean stable, which is the consequence of Proposition 3.63, but $\langle \Gamma' \rangle$ may not be semi-hyperbolic. See Proposition 4.1-(c).

Proposition 3.65. *Let* $\tau \in \mathfrak{M}_{1,c}$ *be mean stable. Suppose that* $J(G_\tau) \neq \emptyset$. *Let* V *be the set coming from Definition 3.60. Let* $S_\tau := \bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L$. *Then we have all of the following.*

1. $S_\tau \subset \overline{G_\tau^*(\overline{V})} \subset F(G_\tau)$.
2. *Let* $W := \bigcup_{A \in \mathrm{Con}(F(G_\tau)), A \cap S_\tau \neq \emptyset} A$. *Let* $\mathcal{U}_W := \{\varphi \in C_W(W) \mid \exists a \in S^1, M_\tau(\varphi) = a\varphi, \varphi \neq 0\}$ *Moreover, let* $\Psi_W : \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \to C_W(W)$ *be the map defined by* $\varphi \mapsto \varphi|_W$. *Then* $\Psi_W(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) = \mathrm{LS}(\mathcal{U}_W)$ *and* $\Psi_W : \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \to \mathrm{LS}(\mathcal{U}_W)$ *is a linear isomorphism.*
3. *Let* $Z := \bigcup_{A \in \mathrm{Con}(F(G_\tau)), A \cap \overline{G^*(\overline{V})} \neq \emptyset} A$. *Let* $\mathcal{U}_Z := \{\varphi \in C_Z(Z) \mid \exists a \in S^1, M_\tau(\varphi) = a\varphi, \varphi \neq 0\}$ *Moreover, let* $\Psi_Z : \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \to C_Z(Z)$ *be the map defined by* $\varphi \mapsto \varphi|_Z$. *Then* $\Psi_Z(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))) = \mathrm{LS}(\mathcal{U}_Z)$ *and* $\Psi_Z : \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \to \mathrm{LS}(\mathcal{U}_Z)$ *is a linear isomorphism.*

Remark 3.66. Under the assumptions and notation of Proposition 3.65, we have $\dim_{\mathbb{C}} C_W(W) < \infty$ and $\dim_{\mathbb{C}} C_Z(Z) < \infty$. Thus, in order to seek $\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})$ and $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$, it suffices to consider the eigenvectors and eigenvalues of the matrix representation of M_τ on the finite dimensional linear space $C_W(W)$ or $C_Z(Z)$.

Remark 3.67. Let $\Gamma \in \mathrm{Cpt(Rat_+)}$ and let $G = \langle \Gamma \rangle$.

1. Suppose that G is semi-hyperbolic and $J_{\ker}(G) = \emptyset$. Then by Proposition 3.63, G is mean stable. Moreover, the set V in Definition 3.60 can be taken to be a small neighborhood of $A(G)$ in $F(G)$, where
$$A(G) := \overline{G(\{z \in \hat{\mathbb{C}} \mid \exists g \in G \text{ s.t. } g(z) = z, |m(g,z)| < 1\})}.$$
In this case, $\{A \in \mathrm{Con}\ (F(G)) \mid A \cap \overline{G^*(\overline{V})} \neq \emptyset\} = \{A \in \mathrm{Con}(F(G)) \mid A \cap A(G) \neq \emptyset.\}$.
2. Similarly, suppose that G is hyperbolic and $J_{\ker}(G) = \emptyset$. Then by Proposition 3.63, G is mean stable. Moreover, the set V in Definition 3.60 can be taken to be a small neighborhood of $P(G)$ in $F(G)$. In this case, $\{A \in \mathrm{Con}\ (F(G)) \mid A \cap \overline{G^*(\overline{V})} \neq \emptyset\} = \{A \in \mathrm{Con}(F(G)) \mid A \cap P(G) \neq \emptyset.\}$.

3.7 Necessary and Sufficient conditions to be $J_{\ker}(G_\tau) \neq \emptyset$

In this subsection, we present some results on necessary and sufficient conditions to be $J_{\ker}(G_\tau) \neq \emptyset$. The proofs are given in subsection 3.7.

The following is a natural question.

Question 3.68. What happens if $J_{\ker}(G_\tau) \neq \emptyset$?

Definition 3.69. Let Y be a compact metric space with $\dim_H(Y) < \infty$ and let $\tau \in \mathfrak{M}_{1,c}(\mathrm{CM}(Y))$. Since the function $\gamma \mapsto \dim_H(\hat{J}_{\gamma,\Gamma_\tau})$ is Borel measurable and since $(\sigma, \tilde{\tau})$ is ergodic, there exists a number $a \in [0, \infty)$ such that for $\tilde{\tau}$-a.e. $\gamma \in \Gamma_\tau$, $\dim_H(\hat{J}_{\gamma,\Gamma_\tau}) = a$. We set $\mathrm{MHD}(\tau) := a$.

Remark 3.70. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $G = \langle \Gamma \rangle$. Suppose that G is semi-hyperbolic and $F(G) \neq \emptyset$. Then, $\gamma \mapsto J_\gamma$ is continuous on $\Gamma^{\mathbb{N}}$ with respect to the Hausdorff metric (this is non-trivial) and for each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma = \hat{J}_{\gamma,\Gamma}$ (see [30, Theorem 2.14]). Moreover, there exists a constant $0 \leq b < 2$ such that for each $\gamma \in \Gamma^{\mathbb{N}}$, $\dim_H(J_\gamma) \leq b$ (see [33, Theorem 1.16]). Note that if we do not assume semi-hyperbolicity, then $\gamma \mapsto J_\gamma$ is not continuous in general.

Theorem 3.71. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$. Suppose that G_τ is semi-hyperbolic and $F(G_\tau) \neq \emptyset$. Then, we have all of the following.*

1. $\dim_H(J^0_{pt}(\tau)) \leq \mathrm{MHD}(\tau) < 2$.
2. $J_{\ker}(G_\tau) \subset J^0_{pt}(\tau)$.
3. $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$ *if and only if* $J_{\ker}(G_\tau) = \emptyset$. *If* $J_{\ker}(G_\tau) \neq \emptyset$, *then* $J_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$.
4. *If, in addition to the assumption,* $\sharp\Gamma_\tau < \infty$, *then we have the following.*
   (a) $G_\tau^{-1}(J_{\ker}(G_\tau)) \subset J^0_{pt}(\tau)$.
   (b) *Either* $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$ *or* $J_{pt}(\tau) = J(G_\tau)$.

Remark 3.72. Let G be a hyperbolic rational semigroup with $G \cap \mathrm{Rat}_+ \neq \emptyset$. Then, G is semi-hyperbolic and $F(G) \neq \emptyset$.

3.8 Singular properties and regularity of non-constant finite linear combinations of unitary eigenvectors of M_τ

In this subsection, we present some results on singular properties and regularity of non-constant finite linear combinations φ of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$. It turns out that under certain conditions, such φ is non-differentiable at each point of an uncountable dense subset of $J(G_\tau)$ (see Theorem 3.82). Moreover, we investigate the pointwise Hölder exponent of such φ (see Theorem 3.82 and Theorem 3.84).

Lemma 3.73. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let Y be a compact metric space and let $h_1, h_2, \ldots, h_m \in \mathrm{OCM}(Y)$. Let $G = \langle h_1, \ldots, h_m \rangle$. Suppose that for each (i,j) with $i \neq j$, $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$. Then, $J_{\ker}(G) = \emptyset$.*

Definition 3.74. For each $m \in \mathbb{N}$, we set $\mathcal{W}_m := \{(p_1, \dots, p_m) \in (0,1)^m \mid \sum_{j=1}^m p_j = 1\}$.

Lemma 3.75. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat})^m$ and let $G = \langle h_1, \dots, h_m \rangle$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j}$. Suppose that $J(G) \neq \emptyset$ and that $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$. Then $\mathrm{int}(J(G)) = \emptyset$ and for each $\varphi \in (\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc}$,*

$$J(G) = \{z \in \hat{\mathbb{C}} \mid \textit{for any neighborhood } U \textit{ of } z, \varphi|_U \textit{ is not constant}\}.$$

Definition 3.76. Let U be a domain in $\hat{\mathbb{C}}$ and let $g : U \to \hat{\mathbb{C}}$ be a meromorphic function. For each $z \in U$, we denote by $\|g'(z)\|_s$ the norm of the derivative of g at z with respect to the spherical metric.

Definition 3.77. Let $m \in \mathbb{N}$. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat})^m$ be an element such that $h_1, \dots, h_m$ are mutually distinct. We set $\Gamma := \{h_1, \dots, h_m\}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be an f-invariant Borel probability measure. For each $p = (p_1, \dots, p_m) \in \mathcal{W}_m$, we define a function $\tilde{p} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \mathbb{R}$ by $\tilde{p}(\gamma, y) := p_j$ if $\gamma_1 = h_j$ (where $\gamma = (\gamma_1, \gamma_2, \dots)$), and we set

$$u(h, p, \mu) := \frac{-(\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log \tilde{p} \, d\mu)}{\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log \|f'\|_s \, d\mu}$$

(when the integral of the denominator converges).

Definition 3.78. Let $h = (h_1, \dots, h_m) \in \mathcal{P}^m$ be an element such that $h_1, \dots, h_m$ are mutually distinct. We set $\Gamma := \{h_1, \dots, h_m\}$. For any $(\gamma, y) \in \Gamma^{\mathbb{N}} \times \mathbb{C}$, let $G_\gamma(y) := \lim_{n \to \infty} \frac{1}{\deg(\gamma_{n,1})} \log^+ |\gamma_{n,1}(y)|$, where $\log^+ a := \max\{\log a, 0\}$ for each $a > 0$. By the arguments in [25], for each $\gamma \in \Gamma^{\mathbb{N}}$, $G_\gamma(y)$ exists, G_γ is subharmonic on $\mathbb{C}$, and $G_\gamma|_{A_{\infty,\gamma}}$ is equal to the Green's function on $A_{\infty,\gamma}$ with pole at ∞. Moreover, $(\gamma, y) \mapsto G_\gamma(y)$ is continuous on $\Gamma^{\mathbb{N}} \times \mathbb{C}$. Let $\mu_\gamma := dd^c G_\gamma$, where $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$. Note that by the argument in [16, 17], μ_γ is a Borel probability measure on J_γ such that $\mathrm{supp}\, \mu_\gamma = J_\gamma$. Furthermore, for each $\gamma \in \Gamma^{\mathbb{N}}$, let $\Omega(\gamma) = \sum_c G_\gamma(c)$, where c runs over all critical points of γ_1 in $\mathbb{C}$, counting multiplicities.

Remark 3.79. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat}_+)^m$ be an element such that $h_1, \dots, h_m$ are mutually distinct. Let $\Gamma = \{h_1, \dots, h_m\}$ and let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product map associated with Γ. Moreover, let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma)$. Then, there exists a unique f-invariant ergodic Borel probability measure μ on $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ such that $\pi_*(\mu) = \tilde{\tau}$ and $h_\mu(f|\sigma) = \max_{\rho \in \bullet_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}) : f_*(\rho) = \rho, \pi_*(\rho) = \tilde{\tau}} h_\rho(f|\sigma) = \sum_{j=1}^m p_j \log(\deg(h_j))$, where $h_\rho(f|\sigma)$ denotes the relative metric entropy of (f, ρ) with respect to $(\sigma, \tilde{\tau})$, and $\mathfrak{E}_1(\cdot)$ denotes the space of ergodic measures (see [29]). This μ is called the **maximal relative entropy measure** for f with respect to $(\sigma, \tilde{\tau})$.

Definition 3.80. Let V be a non-empty open subset of $\hat{\mathbb{C}}$. Let $\varphi : V \to \mathbb{C}$ be a function and let $y \in V$ be a point. Suppose that φ is bounded around y. Then we set

$$\mathrm{H\ddot{o}l}(\varphi, y) := \inf\{\beta \in \mathbb{R} \mid \limsup_{z \to y} \frac{|\varphi(z) - \varphi(y)|}{d(z,y)^\beta} = \infty\},$$

where d denotes the spherical distance. This is called the **pointwise Hölder exponent of φ at y**.

Remark 3.81. If $\text{Höl}(\varphi, y) < 1$, then φ is non-differentiable at y. If $\text{Höl}(\varphi, y) > 1$, then φ is differentiable at y and the derivative at y is equal to 0.

We now present a result on non-differentiability of non-constant finite linear combinations of unitary eigenvectors of M_τ at almost every point in $J(G_\tau)$ with respect to the projection of maximal relative entropy measure.

Theorem 3.82 (Non-differentiability of $\varphi \in (\mathbf{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc}$ at points in $J(G_\tau)$). *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in (\text{Rat}_+)^m$ and we set $\Gamma := \{h_1, h_2, \dots, h_m\}$. Let $G = \langle h_1, \dots, h_m \rangle$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{P})$. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be the maximal relative entropy measure for $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ with respect to $(\sigma, \tilde{\tau})$. Moreover, let $\lambda := (\pi_{\hat{\mathbb{C}}})_*(\mu) \in \mathfrak{M}_1(\hat{\mathbb{C}})$. Suppose that G is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i, j) with $i \neq j$. Then, we have all of the following.*

1. *$G_\tau = G$ is mean stable and $J_{\text{ker}}(G) = \emptyset$.*
2. *$0 < \dim_H(J(G)) < 2$.*
3. *$\text{supp}\, \lambda = J(G)$.*
4. *For each $z \in J(G)$, $\lambda(\{z\}) = 0$.*
5. *There exists a Borel subset A of $J(G)$ with $\lambda(A) = 1$ such that for each $z_0 \in A$ and each $\varphi \in (\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc}$, $\text{Höl}(\varphi, z_0) = u(h, p, \mu)$.*
6. *If $h = (h_1, \dots, h_m) \in \mathcal{P}^m$, then*
$$u(h, p, \mu) = \frac{-(\sum_{j=1}^m p_j \log p_j)}{\sum_{j=1}^m p_j \log \deg(h_j) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma)\, d\tilde{\tau}(\gamma)}$$
and
$$\begin{aligned} 2 >& \dim_H(\{z \in J(G) \mid \text{ for each } \varphi \in (\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc},\ \text{Höl}(\varphi, z) = u(h, p, \mu)\}) \\ \geq& \frac{\sum_{j=1}^m p_j \log \deg(h_j) - \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log \deg(h_j) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma)\, d\tilde{\tau}(\gamma)} > 0. \end{aligned}$$
7. *Suppose $h = (h_1, \dots, h_m) \in \mathcal{P}^m$. Moreover, suppose that at least one of the following (a), (b), and (c) holds: (a) $\sum_{j=1}^m p_j \log(p_j \deg(h_j)) > 0$. (b) $P^*(G)$ is bounded in $\mathbb{C}$. (c) $m = 2$. Then, $u(h, p, \mu) < 1$ and for each non-empty open subset U of $J(G)$ there exists an uncountable dense subset A_U of U such that for each $z \in A_U$ and each $\varphi \in (\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc}$, φ is non-differentiable at z.*

Remark 3.83. By Theorems 3.15 and 3.82, it follows that under the assumptions of Theorem 3.82, the chaos of the averaged system disappears in the C^0 "sense", but it remains in the C^1 "sense".

We now present a result on the representation of pointwise Hölder exponent of $\varphi \in (\text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc}$ at almost every point in $J(G_\tau)$ with respect to the δ-dimensional Hausdorff measure, where $\delta = \dim_H(J(G_\tau))$.

Theorem 3.84. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in (\mathrm{Rat}_+)^m$ and we set $\Gamma := \{h_1, h_2, \dots, h_m\}$. Let $G = \langle h_1, \dots, h_m \rangle$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathrm{Rat}_+)$. Suppose that G is hyperbolic and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$. Let $\delta := \dim_H J(G)$ and let H^δ be the δ-dimensional Hausdorff measure. Let $\tilde{L} : C(\tilde{J}(f)) \to C(\tilde{J}(f))$ be the operator defined by $\tilde{L}(\varphi)(z) = \sum_{f(w)=z} \varphi(w) \|f'(w)\|_s^{-\delta}$. Moreover, let $L : C(J(G)) \to C(J(G))$ be the operator defined by $L(\varphi)(z) = \sum_{j=1}^m \sum_{h_j(w)=z} \varphi(w) \|h_j'(w)\|_s^{-\delta}$. Then, we have all of the following.*

1. *$G_\tau = G$ is mean stable and $J_{\mathrm{ker}}(G) = \emptyset$.*
2. *There exists a unique element $\tilde{\nu} \in \mathfrak{M}_1(\tilde{J}(f))$ such that $\tilde{L}^*(\tilde{\nu}) = \tilde{\nu}$. Moreover, the limits $\tilde{\alpha} = \lim_{n\to\infty} \tilde{L}^n(1) \in C(\tilde{J}(f))$ and $\alpha = \lim_{n\to\infty} L^n(1) \in C(J(G))$ exist, where 1 denotes the constant function taking its value 1.*
3. *Let $\nu := (\pi_{\hat{\mathbb{C}}})_*(\tilde{\nu}) \in \mathfrak{M}_1(J(G))$. Then $0 < \delta < 2$, $0 < H^\delta(J(G)) < \infty$, and $\nu = \frac{H^\delta}{H^\delta(J(G))}$.*
4. *Let $\tilde{\rho} := \tilde{\alpha}\tilde{\nu} \in \mathfrak{M}_1(\tilde{J}(f))$. Then $\tilde{\rho}$ is f-invariant and ergodic. Moreover, $\min_{z\in J(G)} \alpha(z) > 0$.*
5. *There exists a Borel subset of A of $J(G)$ with $H^\delta(A) = H^\delta(J(G))$ such that for each $z_0 \in A$ and each $\varphi \in (\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc}$,*

$$\mathrm{H\ddot{o}l}(\varphi, z_0) = u(h, p, \tilde{\rho}) = \frac{-\sum_{j=1}^m (\log p_j) \int_{h_j^{-1}(J(G))} \alpha(y)\, dH^\delta(y)}{\sum_{j=1}^m \int_{h_j^{-1}(J(G))} \alpha(y) \log \|h_j'(y)\|_s\, dH^\delta(y)}.$$

Remark 3.85. Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in \mathcal{P}^m$ and let $G = \langle h_1, \dots, h_m \rangle$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j}$. Suppose that $\hat{K}(G) \neq \emptyset$, G is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$. Then, by Lemma 3.73 and Theorem 3.22, $T_{\infty,\tau} \in (\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc}$.

Remark 3.86. Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in \mathcal{P}^m$ and we set $\Gamma := \{h_1, \dots, h_m\}$. Let $G = \langle h_1, \dots, h_m \rangle$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{P})$. Suppose that $\hat{K}(G) \neq \emptyset$, G is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$. Moreover, suppose we have at least one of the following (a),(b),(c): (a) $\sum_{j=1}^m p_j \log(p_j \deg(h_j)) > 0$. (b) $P^*(G)$ is bounded in $\mathbb{C}$. (c) $m = 2$. Then, combining Theorem 3.82, Theorem 3.84, and Remark 3.85, it follows that there exists a number $q > 0$ such that if $p_1 < q$, then we have all of the following.

1. Let μ be the maximal relative entropy measure for f with respect to $(\sigma, \tilde{\tau})$. Let $\lambda = (\pi_{\hat{\mathbb{C}}})_*\mu \in \mathfrak{M}_1(J(G))$. Then for λ-a.e. $z_0 \in J(G)$ and for any $\varphi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))_{nc}$ (e.g., $\varphi = T_{\infty,\tau}$), $\limsup_{n\to\infty} \frac{|\varphi(y)-\varphi(z_0)|}{|y-z_0|} = \infty$ and φ is not differentiable at z_0.
2. Let $\delta = \dim_H(J(G))$ and let H^δ be the δ-dimensional Hausdorff measure. Then $0 < H^\delta(J(G)) < \infty$ and for H^δ-a.e. $z_0 \in J(G)$ and for any $\varphi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$ (e.g., $\varphi = T_{\infty,\tau}$), $\limsup_{n\to\infty} \frac{|\varphi(y)-\varphi(z_0)|}{|y-z_0|} = 0$ and φ is differentiable at z_0.

Combining Theorem 3.15 and Theorem 3.82, we obtain the following result.

Corollary 3.87. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in \mathcal{P}^m$ and we set $\Gamma := \{h_1, \dots, h_m\}$. Let $G = \langle h_1, \dots, h_m \rangle$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := \sum_{j=1}^{m} p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{P})$. Suppose that $\hat{K}(G) \neq \emptyset$, G is hyperbolic, and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i, j) with $i \neq j$. Moreover, suppose we have at least one of the following* (a), (b), (c): (a) $\sum_{j=1}^{m} p_j \log(p_j \deg(h_j)) > 0$. (b) *$P^*(G)$ is bounded in $\mathbb{C}$.* (c) *$m = 2$. Let $\varphi \in C(\hat{\mathbb{C}})$. Then, we have exactly one of the following* (i) *and* (ii).

(i) *There exists a constant function $\zeta \in C(\hat{\mathbb{C}})$ such that $M_\tau^n(\varphi) \to \zeta$ as $n \to \infty$ in $C(\hat{\mathbb{C}})$.*

(ii) *There exists an element $\psi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))_{nc}$ and a number $l \in \mathbb{N}$ such that*

- *$M_\tau^l(\psi) = \psi$,*
- *$\{M_\tau^j(\psi)\}_{j=0}^{l-1} \subset (\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc} \subset C_{F(G)}(\hat{\mathbb{C}})$,*
- *there exists an uncountable dense subset A of $J(G)$ such that for each $z_0 \in A$ and each j, $M_\tau^j(\psi)$ is not differentiable at z_0, and*
- *$M_\tau^{nl+j}(\varphi) \to M_\tau^j(\psi)$ as $n \to \infty$ for each $j = 0, \dots, l-1$.*

We present a result on Hölder continuity of $\varphi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$.

Theorem 3.88. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \dots, h_m) \in \mathrm{Rat}_+^m$ and we set $\Gamma := \{h_1, \dots, h_m\}$. Let $G = \langle h_1, \dots, h_m \rangle$. Let $p = (p_1, \dots, p_m) \in \mathcal{W}_m$ and let $\tau := \sum_{j=1}^{m} p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathrm{Rat}_+)$. Suppose that G is hyperbolic and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i, j) with $i \neq j$. Then, G is mean stable and there exists an $\alpha > 0$ such that for each $\varphi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$, $\varphi : \hat{\mathbb{C}} \to [0,1]$ is α-Hölder continuous on $\hat{\mathbb{C}}$.*

Remark 3.89. In the proof of Theorem 3.82, we use the Birkhoff ergodic theorem and the Koebe distortion theorem, in order to show that for each $\varphi \in (\mathrm{LS}(\mathcal{U}_{f,\tau}))_{nc}$, $\mathrm{H\ddot{o}l}(\varphi, z_0) = u(h, p, \mu)$. Moreover, we apply potential theory in order to calculate $u(h, p, \mu)$ by using p, $\deg(h_j)$, and $\Omega(\gamma)$.

3.9 Stability and bifurcation

In this subsection, we present some results on stability and bifurcation of M_τ or M_τ^*. The proofs of the results will be written in [42].

Definition 3.90. Let $\mathcal{O}$ be the topology of $\mathfrak{M}_{1,c}(\mathrm{Rat})$ such that $\mu_n \to \mu$ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}), \mathcal{O})$ as $n \to \infty$ if and only if (1) $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : \mathrm{Rat} \to \mathbb{C}$, and (2) $\mathrm{supp}\, \mu_n \to \mathrm{supp}\, \mu$ with respect to the Hausdorff metric.

Definition 3.91. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ and let $G = \langle \Gamma \rangle$. We say that $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ is attracting (for $(G, \hat{\mathbb{C}})$) if there exist non-empty open subsets U, V of $F(G)$ and a positive integer n such that all of the following hold.

(1) $L \subset V \subset \overline{V} \subset U \subset \overline{U} \subset F(G)$, $\sharp(\hat{\mathbb{C}} \setminus V) \geq 3$.

(2) For each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_{n,1}(\overline{U}) \subset V$.

Remark 3.92. For each $h \in G \cap \mathrm{Rat}_+$,

$$\sharp\{\text{attracting minimal set for } (G, \hat{\mathbb{C}})\} \leq \{\text{attracting cycles of } h\} < \infty.$$

Lemma 3.93. *Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat})$ and let $G = \langle \Gamma \rangle$. Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$ be attracting for $(G, \hat{\mathbb{C}})$. Let U, V be as in Definition 3.91. Let $W := \bigcup_{A \in \mathrm{Con}(F(G)), A \cap L \neq \emptyset} A$ and let W' be a relative compact open subset of W including L. Then there exists an open neighborhood $\mathcal{U}$ of Γ in $\mathrm{Cpt}(\mathrm{Rat})$ such that the following hold.*

(1) *For each $\Gamma' \in \mathcal{U}$, there exists a unique $L' \in \mathrm{Min}(\langle \Gamma' \rangle, \hat{\mathbb{C}})$ with $L' \subset W'$.*

(2) *The above L' is attracting for $(\langle \Gamma' \rangle, \hat{\mathbb{C}})$.*

Lemma 3.94. *Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $G = \langle \Gamma \rangle$. Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$. Suppose that (1) $L \subset F(G)$ and (2) for each $g \in G$ and for each $U \in \mathrm{Con}(F(G))$ with $U \cap L \neq \emptyset$ and $g(U) \subset U$, U is not a subset of a Siegel disk or a Hermann ring of g. Then, L is attracting for $(G, \hat{\mathbb{C}})$.*

Lemma 3.95. *Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $G = \langle \Gamma \rangle$. Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$. Then exactly one of the following holds.*

(1) *L is attracting.*

(2) *$L \cap J(G) \neq \emptyset$.*

(3) *$L \subset F(G)$ and there exists an element $g \in G$ and an element $U \in \mathrm{Con}(F(G))$ with $L \cap U \neq \emptyset$ such that $g(U) \subset U$ and U is a subset of a Siegel disk or a Hermann ring of g.*

Definition 3.96. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $G = \langle \Gamma \rangle$. Let $L \in \mathrm{Min}(G, \hat{\mathbb{C}})$.

- We say that L is J-touching (for $(G, \hat{\mathbb{C}})$) if $L \cap J(G) \neq \emptyset$.
- We say that L is sub-rotative (for $(G, \hat{\mathbb{C}})$) if (3) in Lemma 3.95 holds.

Definition 3.97. Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$. Suppose L is J-touching or sub-rotative. Moreover, suppose $L \neq \hat{\mathbb{C}}$. Let $g \in \Gamma$. We say that g is a bifurcation element for (Γ, L) if one of the following statements (1)(2) holds.

(1) L is J-touching and there exists a point $z \in L \cap J(\langle \Gamma \rangle)$ such that $g(z) \in J(\langle \Gamma \rangle)$.

(2) L is sub-rotative and there exist an open subset U of $\hat{\mathbb{C}}$ with $U \cap L \neq \emptyset$ and finitely many elements $\gamma_1, \ldots, \gamma_{n-1} \in \Gamma$ such that $g \circ \gamma_{n-1} \cdots \circ \gamma_1(U) \subset U$ and U is a subset of a Siegel disk or a Hermann ring of $g \circ \gamma_{n-1} \cdots \circ \gamma_1$.

Furthermore, we say that an element $g \in \Gamma$ is a bifurcation element for Γ if there exists an $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$ such that g is a bifurcation element for (Γ, L).

Lemma 3.98. *Let $\Gamma \in \mathrm{Cpt}(\mathrm{Rat}_+)$ and let $L \in \mathrm{Min}(\langle \Gamma \rangle, \hat{\mathbb{C}})$. Suppose L is J-touching or sub-rotative. Moreover, suppose $L \neq \hat{\mathbb{C}}$. Let $g \in \Gamma$ be a bifurcation element for (Γ, L). Then, $g \in \partial \Gamma$, where the boundary of Γ is taken in the space Rat_+.*

Theorem 3.99. *Let* $\Gamma \in \text{Cpt}(\text{Rat}_+)$. *Suppose that there exists an attracting* $L \in \text{Min}(\langle\Gamma\rangle, \hat{\mathbb{C}})$. *Let* $\{L_j\}_{j=1}^r$ *be the set of attracting minimal sets for* $(\langle\Gamma\rangle, \hat{\mathbb{C}})$ *such that* $L_i \neq L_j$ *if* $i \neq j$ *(Remark: by Remark 3.92, the set of attracting minimal sets is finite). Let* $\mathcal{U}$ *be a neighborhood of* Γ *in* $\text{Cpt}(\text{Rat}_+)$. *For each* $j = 1, \ldots, r$, *let* $\mathcal{V}_j$ *be a neighborhood of* L_j *with respect to the Hausdorff metric in* $\text{Cpt}(\hat{\mathbb{C}})$. *Then, there exists an element* $\Gamma' \in \mathcal{U}$ *with* $\Gamma' \supset \Gamma$ *such that all of the following hold.*

(1) $\sharp\{L' \in \text{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) \mid L' \textit{ is attracting }\} = r$ *and for each* $j = 1, \ldots, r$, *there exists a unique element* $L'_j \in \text{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}})$ *with* $L'_j \in \mathcal{V}_j$ *such that* L'_j *is attracting for* $(\langle\Gamma'\rangle, \hat{\mathbb{C}})$.

(2) $\langle\Gamma'\rangle$ *is mean stable and* $\sharp\text{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = r$.

Theorem 3.100. *Let* $\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+)$. *Suppose that there exists an attracting* $L \in \text{Min}(G_\tau, \hat{\mathbb{C}})$. *Let* $\{L_j\}_{j=1}^r$ *be the set of attracting minimal sets for* $(G_\tau, \hat{\mathbb{C}})$ *such that* $L_i \neq L_j$ *if* $i \neq j$. *Let* $\mathcal{U}$ *be a neighborhood of* τ *in* $(\mathfrak{M}_{1,c}(\text{Rat}_+), \mathcal{O})$. *For each* $j = 1, \ldots, r$, *let* $\mathcal{V}_j$ *be a neighborhood of* L_j *with respect to the Hausdorff metric in* $\text{Cpt}(\hat{\mathbb{C}})$. *Then, there exists an element* $\rho \in \mathcal{U}$ *with* $\sharp\Gamma_\rho < \infty$ *such that all of the following hold.*

(1) $\sharp\{L' \in \text{Min}(G_\rho, \hat{\mathbb{C}}) \mid L' \textit{ is attracting }\} = r$ *and for each* $j = 1, \ldots, r$, *there exists a unique element* $L'_j \in \text{Min}(G_\rho, \hat{\mathbb{C}})$ *with* $L'_j \in \mathcal{V}_j$ *such that* L'_j *is attracting for* $(G_\rho, \hat{\mathbb{C}})$.

(2) G_ρ *is mean stable and* $\sharp\text{Min}(G_\rho, \hat{\mathbb{C}}) = r$.

Theorem 3.101 (Cooperation Principle IV).

(1) *The set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau \textit{ is mean stable}\}$ *is open and dense in* $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$. *Moreover, the set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid J_{\ker}(G_\tau) = \emptyset, J(G_\tau) \neq \emptyset\}$ *contains* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau \textit{ is mean stable}\}$.

(2) *The set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau \textit{ is mean stable, } \sharp\Gamma_\tau < \infty\}$ *is dense in* $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$.

Proposition 3.102. *Let* $\Gamma \in \text{Cpt}(\text{Rat}_+)$. *Suppose that there exists no attracting minimal set for* $(G, \hat{\mathbb{C}})$. *Let* $\mathcal{U}$ *be a neighborhood of* Γ *in* $\text{Cpt}(\text{Rat}_+)$. *Then, there exists an element* $\Gamma' \in \mathcal{U}$ *with* $\Gamma' \supset \Gamma$ *such that* $\text{Min}(\langle\Gamma'\rangle, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ *and* $J(\langle\Gamma'\rangle) = \hat{\mathbb{C}}$.

Corollary 3.103. *Let* $\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+)$. *Suppose that there exists no attracting minimal set for* $(G_\tau, \hat{\mathbb{C}})$. *Let* $\mathcal{U}$ *be a neighborhood of* τ *in* $(\mathfrak{M}_{1,c}(\text{Rat}_+), \mathcal{O})$. *Then, there exists an element* $\rho \in \mathcal{U}$ *such that* $\text{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}$ *and* $J(G_\rho) = \hat{\mathbb{C}}$.

Corollary 3.104. *The set*

$$\{\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+) \mid \tau \textit{ is mean stable }\} \cup \{\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+) \mid \text{Min}(G_\rho, \hat{\mathbb{C}}) = \{\hat{\mathbb{C}}\}, J(G_\rho) = \hat{\mathbb{C}}\}$$

is dense in $(\mathfrak{M}_{1,c}(\text{Rat}_+), \mathcal{O})$.

Definition 3.105. Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ be such that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then by Theorem 3.15, we have the direct sum decomposition $C(\hat{\mathbb{C}}) = \text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})) \oplus \mathcal{B}_{0,\tau}$. We denote by $\pi_\tau : C(\hat{\mathbb{C}}) \to \text{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$ the canonical projection with respect to this direct sum decomposition.

Theorem 3.106. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$ be mean stable. Then there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}_+), \mathcal{O})$ such that all of the following hold.*

(1) *For each $\nu \in \Omega$, $\sharp\mathrm{Min}(G_\nu, \hat{\mathbb{C}}) = \sharp\mathrm{Min}(G_\tau, \hat{\mathbb{C}})$.*

(2) *For each $\nu \in \Omega$, $\dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))) = \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))$.*

(3) *The map $\nu \mapsto \pi_\nu$ is continuous on Ω. More precisely, for each $\nu \in \Omega$, there exists a basis $\{\varphi_{j,\nu}\}_{j=1}^q$ of $\mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))$, where $q = \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))$, and a finite family $\{\rho_{j,\nu}\}_{j=1}^q$ in $C(\hat{\mathbb{C}})^*$ such that all of the following hold.*

   (a) *For each j, $\nu \mapsto \varphi_{j,\nu} \in C(\hat{\mathbb{C}})$ is continuous on Ω.*

   (b) *For each j, $\nu \mapsto \rho_{j,\nu} \in C(\hat{\mathbb{C}})^*$ is continuous on Ω.*

   (c) *For each (i,j) and each $\nu \in \Omega$, $\rho_{i,\nu}(\varphi_{j,\nu}) = \delta_{ij}$.*

   (d) *For each $\nu \in \Omega$ and each $\varphi \in C(\hat{\mathbb{C}})$, $\pi_\nu(\varphi) = \sum_{j=1}^q \rho_{j,\nu}(\varphi) \cdot \varphi_{j,\mu}$.*

Theorem 3.107. *We consider the following subsets A, B, C, D of $\mathfrak{M}_{1,c}(\mathrm{Rat}_+)$ which are defined as follows.*

(1) *$A := \{\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+) \mid \tau$ is mean stable$\}$.*

(2) *Let B be the set of $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$ satisfying that there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}_+), \mathcal{O})$ such that* (a) *for each $\nu \in \Omega$, $J_{\mathrm{ker}}(G_\nu) = \emptyset$, and* (b) *$\nu \mapsto \sharp\mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ is constant on Ω.*

(3) *Let C be the set of $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$ satisfying that there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}_+), \mathcal{O})$ such that* (a) *for each $\nu \in \Omega$, $F(G_\nu) \neq \emptyset$, and* (b) *$\nu \mapsto \sharp\mathrm{Min}(G_\nu, \hat{\mathbb{C}})$ is constant on Ω.*

(4) *Let D be the set of $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat}_+)$ satisfying that there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\mathrm{Rat}_+), \mathcal{O})$ such that* (a) *for each $\nu \in \Omega$, $J_{\mathrm{ker}}(G_\nu) = \emptyset$, and* (b) *$\nu \mapsto \mathrm{LS}(\mathcal{U}_{f,\nu}(\hat{\mathbb{C}}))$ is continuous on Ω (i.e., statements* (2) *and* (3) *in* Theorem 3.106 *hold).*

Then, $A = B = C = D$.

Proposition 3.108. *Let $\mathcal{Y}$ be a strongly admissible closed subset of $\mathcal{P}$. For each $t \in [0,1]$, let μ_t be an element of $\mathfrak{M}_{1,c}(\mathcal{Y})$. Suppose that all of the following conditions (1)–(5) hold.*

(1) *$t \mapsto \mu_t \in (\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ is continuous on $[0,1]$.*

(2) *For each $t \in [0,1]$, there exists an open subset V_t of $\mathcal{Y}$ such that $\Gamma_{\mu_t} = \overline{V_t}$ where the closure is taken in the space $\mathcal{Y}$.*

(3) *If $t_1, t_2 \in [0,1]$ and $t_1 < t_2$, then $\Gamma_{\mu_{t_1}} \subset \mathrm{int}\Gamma_{\mu_{t_2}}$ with respect to the topology of $\mathcal{Y}$.*

(4) *μ_0 and μ_1 are mean stable.*

(5) *$\sharp(\mathrm{Min}(\langle\Gamma_{\mu_0}\rangle, \hat{\mathbb{C}})) \neq \sharp(\mathrm{Min}(\langle\Gamma_{\mu_1}\rangle, \hat{\mathbb{C}}))$.*

Then,

$$1 \leq \sharp\{t \in [0,1] \mid \textit{there exists a bifurcation element } g \in \Gamma_{\mu_t} \textit{ for } \Gamma_{\mu_t}\} < \infty.$$

Example 3.109. Let c be a point in the Mandelbrot set $\mathcal{M}$ such that $z \mapsto z^2 + c$ is hyperbolic. Let $r_0 > 0$ be a small number. Let $r_1 > 0$ be such that $D(c, r_1) \cap (\mathbb{C} \setminus \mathcal{M}) \neq \emptyset$. For each $t \in [0,1]$, let $\mu_t \in \mathfrak{M}_1(\overline{D(c,(1-t)r_0 + tr_1)})$ be the normalized 2-dimensional Lebesgue measure on $\overline{D(c,(1-t)r_0 + tr_1)}$. Then $\{\mu_t\}_{t \in [0,1]}$ satisfies the conditions (1)–(5) in Proposition 3.108 (for example, $\sharp\mathrm{Min}(\langle \Gamma_{\mu_0} \rangle, \hat{\mathbb{C}}) > \sharp\mathrm{Min}(\langle \Gamma_{\mu_0} \rangle, \hat{\mathbb{C}}) = 1$). Thus

$$1 \leq \sharp\{t \in [0,1] \mid \text{there exists a bifurcation element } g \in \Gamma_{\mu_t} \text{ for } \Gamma_{\mu_t}\} < \infty.$$

3.10 Spectral properties of M_τ and stability

In this subsection, we present some results on spectral properties of M_τ acting on the space of Hölder continuous functions on $\hat{\mathbb{C}}$ and the stability. The proofs of the results will be written in [42].

Theorem 3.110. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$. Then, there exists an $\alpha > 0$ such that each $\varphi \in \mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))$ is α-Hölder continuous on $\hat{\mathbb{C}}$.*

Notation: For each $\alpha \in (0,1)$, let $H_\alpha(\hat{\mathbb{C}})$ be the Banach space of all complex-valued α-Hölder continuous functions on $\hat{\mathbb{C}}$ endowed with the α-Hölder norm $\|\cdot\|_\alpha$.

If $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is mean stable and $J(G_\tau) \neq \emptyset$, then by Proposition 3.65, we have $\bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L \subset F(G_\tau)$. From this point of view, we consider the situation that $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ satisfies $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and $\bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L \subset F(G_\tau)$. Under this situation, we have several very strong results.

Theorem 3.111. *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$. Suppose that $J_{\ker}(G_\tau) = \emptyset$, $J(G_\tau) \neq \emptyset$, and $\bigcup_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} L \subset F(G_\tau)$. Let $r := \prod_{L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})} \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$. Then, there exists a constant $\alpha \in (0,1)$, a constant $\lambda \in (0,1)$, and a constant $C > 0$ such that for each $\varphi \in H_\alpha(\hat{\mathbb{C}})$, we have all of the following.*

(1) $\|M_\tau^{nr}(\varphi) - \pi_\tau(\varphi)\|_\alpha \leq C\lambda^n \|\varphi - \pi_\tau(\varphi)\|_\alpha$ *for each* $n \in \mathbb{N}$.

(2) $\|M_\tau^{n}(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n \|\varphi - \pi_\tau(\varphi)\|_\alpha$ *for each* $n \in \mathbb{N}$.

(3) $\|M_\tau^{n}(\varphi - \pi_\tau(\varphi))\|_\alpha \leq C\lambda^n \|\varphi\|_\alpha$ *for each* $n \in \mathbb{N}$.

(4) $\|\pi_\tau(\varphi)\|_\alpha \leq C\|\varphi\|_\alpha$.

We now consider the spectrum $\mathrm{Spec}(M_\tau)$ of $M_\tau : H_\alpha(\hat{\mathbb{C}}) \to H_\alpha(\hat{\mathbb{C}})$. From Theorem 3.111, we can show that the distance between $\mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ and $\mathrm{Spec}(M_\tau) \setminus \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$ is positive.

Theorem 3.112. *Under the assumptions of Theorem 3.111, we have all of the following.*

(1) $\mathrm{Spec}(M_\tau) \subset \{z \in \mathbb{C} \mid |z| \leq \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}})$, *where* $\lambda \in (0,1)$ *denotes the constant in Theorem 3.111.*

(2) *Let* $\zeta \in \mathbb{C} \setminus (\{z \in \mathbb{C} \mid |z| \le \lambda\} \cup \mathcal{U}_{v,\tau}(\hat{\mathbb{C}}))$. *Then,* $(\zeta I - M_\tau)^{-1} : H_\alpha(\hat{\mathbb{C}}) \to H_\alpha(\hat{\mathbb{C}})$ *is equal to*

$$(\zeta I - M_\tau)|_{\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}}))}^{-1} \circ \pi_\tau + \sum_{n=0}^{\infty} \frac{M_\tau^n}{\zeta^{n+1}}(I - \pi_\tau),$$

where I *denotes the identity on* $H_\alpha(\hat{\mathbb{C}})$.

Remark 3.113. Let $h_1, \ldots, h_m \in \mathrm{Rat}$. Let $G = \langle h_1, \ldots, h_m \rangle$. Suppose that $J_{\ker}(G) = \emptyset$, $J(G) \neq \emptyset$ and $\cup_{L \in \mathrm{Min}(G,\hat{\mathbb{C}})} L \subset F(G)$. Let $\mathcal{W}_m := \{(a_1, \ldots, a_m) \in (0,1)^m \mid \sum_{j=1}^m a_j = 1\}$. For each $a = (a_1, \ldots, a_m) \in \mathcal{W}_m$, let $\tau_a := \sum_{j=1}^m a_j \delta_{h_j}$. Then by Theorem 3.112 and [18], it follows that for each $b \in \mathcal{W}_m$, there exists an $\alpha \in (0,1)$ such that $a \mapsto (\pi_{\tau_a} : H_\alpha(\hat{\mathbb{C}}) \to H_\alpha(\hat{\mathbb{C}}))$ is real-analytic on an open neighborhood of b in $\mathcal{W}_m$.

4 Examples

We give some examples to which we can apply Theorem 3.14, Theorem 3.15, Theorem 3.22, Proposition 3.26, Theorem 3.31, Theorem 3.34, Proposition 3.63, Theorem 3.82, Theorem 3.84, Corollary 3.87, and Theorem 3.88.

Proposition 4.1. *Let* $f_1 \in \mathcal{P}$. *Suppose that* $\mathrm{int}(K(f_1))$ *is not empty. Let* $b \in \mathrm{int}(K(f_1))$ *be a point. Let* d *be a positive integer such that* $d \ge 2$. *Suppose that* $(\deg(f_1), d) \neq (2,2)$. *Then, there exists a number* $c > 0$ *such that for each* $\lambda \in \{\lambda \in \mathbb{C} : 0 < |\lambda| < c\}$, *setting* $f_\lambda = (f_{\lambda,1}, f_{\lambda,2}) = (f_1, \lambda(z-b)^d + b)$ *and* $G_\lambda := \langle f_1, f_{\lambda,2} \rangle$, *we have all of the following.*

(a) f_λ *satisfies the open set condition with an open subset* U_λ *of* $\hat{\mathbb{C}}$ *(i.e.,* $f_{\lambda,1}^{-1}(U_\lambda) \cup f_{\lambda,2}^{-1}(U_\lambda) \subset U_\lambda$ *and* $f_{\lambda,1}^{-1}(U_\lambda) \cap f_{\lambda,2}^{-1}(U_\lambda) = \emptyset$*),* $f_{\lambda,1}^{-1}(J(G_\lambda)) \cap f_{\lambda,2}^{-1}(J(G_\lambda)) = \emptyset$, $\mathrm{int}(J(G_\lambda)) = \emptyset$, $J_{\ker}(G_\lambda) = \emptyset$, $G_\lambda(K(f_1)) \subset K(f_1) \subset \mathrm{int}(K(f_{\lambda,2}))$ *and* $\emptyset \neq K(f_1) \subset \hat{K}(G_\lambda)$.

(b) *If* $K(f_1)$ *is connected, then* $P^*(G_\lambda)$ *is bounded in* $\mathbb{C}$.

(c) *If* f_1 *is semi-hyperbolic (resp. hyperbolic) and* $K(f_1)$ *is connected, then* G_λ *is semi-hyperbolic (resp. hyperbolic),* $J(G_\lambda)$ *is porous (for the definition of porosity, see [33]), and* $\dim_H(J(G_\lambda)) < 2$.

Example 4.2 (Devil's coliseum). Let $g_1(z) := z^2 - 1$, $g_2(z) := z^2/4$, $h_1 := g_1^2$, and $h_2 := g_2^2$. Let $G = \langle h_1, h_2 \rangle$ and $\tau := \sum_{i=1}^2 \frac{1}{2}\delta_{h_i}$. Then it is easy to see that setting $A := K(h_2) \setminus D(0, 0.4)$, we have $\overline{D(0,0.4)} \subset \mathrm{int}(K(h_1))$, $h_2(K(h_1)) \subset \mathrm{int}(K(h_1))$, $h_1^{-1}(A) \cup h_2^{-1}(A) \subset A$, and $h_1^{-1}(A) \cap h_2^{-1}(A) = \emptyset$. Therefore $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$ and $\emptyset \neq K(h_1) \subset \hat{K}(G)$. Moreover, using the argument in the proof of Proposition 4.1, we obtain that G is hyperbolic. By Lemma 3.73, $J_{\ker}(G) = \emptyset$. By Theorem 3.22 and Lemma 3.75, we obtain that $T_{\infty,\tau}$ is continuous on $\hat{\mathbb{C}}$ and the set of varying points of $T_{\infty,\tau}$ is equal to $J(G)$. Moreover, by Theorem 3.82, $\dim_H(J(G)) < 2$ and for each non-empty open subset U of $J(G)$ there exists an uncountable dense subset A_U of U such that for each $z \in A_U$, $T_{\infty,\tau}$ is not differentiable at z. See Figures 2, 3, and 4. $T_{\infty,\tau}$ is called a devil's coliseum. It is a complex analogue of the devil's staircase.

We now present a way to construct examples of $(h_1, h_2) \in \mathcal{P}^2$ such that $G = \langle h_1, h_2 \rangle$ is hyperbolic, $\bigcap_{j=1}^2 h_j^{-1}(J(G)) = \emptyset$, and $\hat{K}(G) \neq \emptyset$.

Figure 2: The Julia set of $G = \langle h_1, h_2 \rangle$, where $g_1(z) := z^2 - 1, g_2(z) := z^2/4, h_1 := g_1^2, h_2 := g_2^2$. We have $J_{\ker}(G) = \emptyset$ and $\dim_H(J(G)) < 2$.

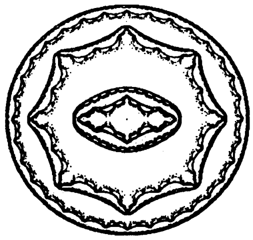


Figure 3: The graph of $T_{\infty,\tau}$, where $\tau = \sum_{i=1}^{2} \frac{1}{2}\delta_{h_i}$ with the same h_i as in Figure 2. $T_{\infty,\tau}$ is continuous on $\hat{\mathbb{C}}$. The set of varying points of $T_{\infty,\tau}$ is equal to $J(G)$ in Figure 2. A "devil's coliseum" (A complex analogue of the devil's staircase).

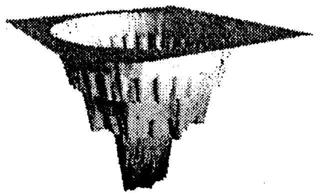
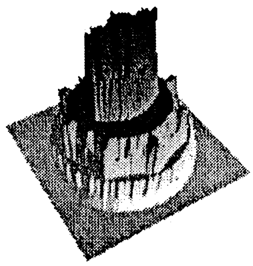


Figure 4: Figure 3 upside down. A "fractal wedding cake".



Proposition 4.3. *Let $g_1, g_2 \in \mathcal{P}$ be hyperbolic. Suppose that $(J(g_1) \cup J(g_2)) \cap (P(g_1) \cup P(g_2)) = \emptyset$, $K(g_1) \subset \mathrm{int}(K(g_2))$, and the union A of attracting cycles of g_2 in $\mathbb{C}$ is included in $\mathrm{int}(K(g_1))$. Then, there exists an $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq m$, setting $h_{i,n} = g_i^n$ and $G_n = \langle h_{1,n}, h_{2,n} \rangle$, we have that G_n is hyperbolic, $h_{1,n}^{-1}(J(G_n)) \cap h_{2,n}^{-1}(J(G_n)) = \emptyset$, and $\emptyset \neq K(g_1) \subset \hat{K}(G_n)$.*

Proposition 4.4. *Let $m \in \mathbb{N}$ and let $g = (g_1, \ldots, g_m) \in \mathcal{P}^m$. Let $G = \langle g_1, \ldots, g_m \rangle$. Suppose that $g_i^{-1}(J(G)) \cap g_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$, that G is hyperbolic, and that $\hat{K}(G) \neq \emptyset$. Then, there exists a neighborhood U of g in $\mathcal{P}^m$ such that for each $h = (h_1, \ldots, h_m) \in U$, setting $H = \langle h_1, \ldots, h_m \rangle$, we have that $h_i^{-1}(J(H)) \cap h_j^{-1}(J(H)) = \emptyset$ for each (i,j) with $i \neq j$, that H is hyperbolic, and that $\hat{K}(H) \neq \emptyset$.*

Proposition 4.5. *Let $(g_1, g_2) \in \mathcal{P}^2$ and let $(p_1, p_2) \in \mathcal{W}_2$. For each $n \in \mathbb{N}$, we set $h_{1,n} := g_1^n, h_{2,n} := g_2^n$, $G_n := \langle h_{1,n}, h_{2,n} \rangle$, and $\tau_n := \sum_{j=1}^2 p_j \delta_{h_{i,n}}$. Suppose that $\bigcap_{j=1}^2 g_j^{-1}(J(G_1)) = \emptyset$, G_1 is hyperbolic and $\hat{K}(G_1) \neq \emptyset$. Then, there exists an $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq m$, (1) G_n is hyperbolic, (2) $\bigcap_{j=1}^2 h_{j,n}^{-1}(J(G_n)) = \emptyset$, (3) $\hat{K}(G_n) \neq \emptyset$, (4) $(\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc} \neq \emptyset$, (5) for each $j = 1, 2,$, $1 < p_j \min\{\|h_{j,n}'(z)\|_s \mid z \in h_{j,n}^{-1}(J(G_n))\}$, and (6) for each $z_0 \in J(G_n)$ and for each $\varphi \in (\mathrm{LS}(\mathcal{U}_{f,\tau}(\hat{\mathbb{C}})))_{nc}$, $\limsup_{z \to z_0} \frac{|\varphi(z) - \varphi(z_0)|}{d(z,z_0)} = \infty$ and φ is not differentiable at z_0.*

Remark 4.6. Combining Proposition 4.1, Proposition 4.3, Proposition 4.4, Proposition 3.63, and Remark 3.42, we obtain many examples to which we can apply Theorem 3.15, Lemma 3.75, Theorem 3.82, Theorem 3.84, Corollary 3.87, and Theorem 3.88.

We now give an example of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ such that $J_{\ker}(G_\tau) = \emptyset$ and such that there exists a minimal set $L \in \mathrm{Min}(G_\tau, \hat{\mathbb{C}})$ with $L \cap J(G_\tau) \neq \emptyset$.

Example 4.7. Let $f_1 \in \mathcal{P}$ and suppose that f_1 has a parabolic cycle α. Let b be a point of the immediate basin of α. Let $d \in \mathbb{N}$ with $d \geq 2$ such that $(\deg(f_1), d) \neq (2,2)$. Then by Proposition 4.1, there exists a $c > 0$ such that for each $a \in \mathbb{C}$ with $0 < |a| < c$, setting $f_2 := a(z-b)^d + b$ and $G = \langle f_1, f_2 \rangle$, we have $f_1^{-1}(J(G)) \cap f_2^{-1}(J(G)) = \emptyset$ and $G(K(f_1)) \subset K(f_1) \subset \mathrm{int}(K(f_2))$. Let $p = (p_1, p_2) \in \mathcal{W}_2$ and let $\tau = \sum_{i=1}^2 p_i \delta_{f_i}$. Then by Lemma 3.73, $J_{\ker}(G_\tau) = J_{\ker}(G) = \emptyset$. Since $G(K(f_1)) \subset K(f_1) \subset \mathrm{int}(K(f_2))$, there exists a minimal set L for $(G_\tau, \hat{\mathbb{C}})$ such that $L \subset K(f_1)$. Since b belongs to the immediate basin of α for f_1, it follows that $\alpha \subset L$. In particular, $L \cap J(G_\tau) \neq \emptyset$.

We now give an example of small perturbation of a single map.

Example 4.8. Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. Let $R : \hat{\mathbb{C}} \times \mathbb{D} \to \hat{\mathbb{C}}$ be a holomorphic map such that for each $z \in \hat{\mathbb{C}}$, $c \mapsto R(z,c)$ is non-constant on $\mathbb{D}$. We set $R_c(z) := R(z,c)$ for each $(z,c) \in \hat{\mathbb{C}} \times \mathbb{D}$. Let $m \in \mathbb{N}$ and suppose that R_0 has exactly m attracting cycles $\alpha_1, \ldots, \alpha_m$. For each j, let A_j be the immediate basin of α_j for R_0. Then by [9, Theorem 0.1] and Theorem 3.15, there exists a $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$, denoting by τ_δ the normalized 2-dimensional Lebesgue measure on $\overline{D(0,\delta)}$, we have (1) τ_δ is mean stable, (2) $J_{\ker}(G_{\tau_\delta}) = \emptyset$, (3) $\sharp\mathrm{Min}(G_{\tau_\delta}, \hat{\mathbb{C}}) = m$, (4) for each $L \in \mathrm{Min}(G_{\tau_\delta}, \hat{\mathbb{C}})$, there exists a j such that $L \subset A_j$, and (5) for each $L \in \mathrm{Min}(G_{\tau_\delta}, \hat{\mathbb{C}})$, $r_L := \dim_{\mathbb{C}}(\mathrm{LS}(\mathcal{U}_{f,\tau}(L)))$ is equal to the period of α_j for R_0.

We now give an example of higher dimensional random complex dynamics to which we can apply Theorem 3.14.

Example 4.9. Let $h \in \mathrm{NHM}(\mathbb{CP}^n)$. Suppose that $\mathrm{int}(J(h)) = \emptyset$ and there exist finitely many attracting periodic cycles $\alpha_1, \dots, \alpha_m$ such that for every $z \in F(h)$, $d(h^n(z), \bigcup_{j=1}^m \alpha_j) \to 0$ as $n \to \infty$. Then, there exists a compact neighborhood Γ of h in $\mathrm{NHM}(\mathbb{CP}^n)$ such that Γ is mean stable, such that $J_{\ker}(\langle \Gamma \rangle) = \emptyset$, and such that for any $\tau \in \mathfrak{M}_1(\mathrm{NHM}(\mathbb{CP}^n))$ with $\Gamma_\tau = \Gamma$, $\mathrm{Leb}_{2n}(J_\gamma) = 0$ for $\tilde{\tau}$-a.e. $\gamma \in (\mathrm{NHM}(\mathbb{CP}^n))^{\mathbb{N}}$. For, if Γ is small enough, then there exists a neighborhood U of $\bigcup_{j=1}^m \alpha_j$ such that $\overline{\langle \Gamma \rangle (U)} \subset U \subset \overline{U} \subset F(\langle \Gamma \rangle)$. Moreover, for each $z \in \mathbb{CP}^n$, there exists a $g \in \Gamma$ such that $g(z) \in F(h)$. Thus Γ is mean stable and $J_{\ker}(\langle \Gamma \rangle) = \emptyset$. By Theorem 3.14, it follows that for each $\tau \in \mathfrak{M}_1(\mathrm{NHM}(\mathbb{CP}^n))$ with $\Gamma_\tau = \Gamma$, $\mathrm{Leb}_{2n}(J_\gamma) = 0$ for $\tilde{\tau}$-a.e. $\gamma \in (\mathrm{NHM}(\mathbb{CP}^n))^{\mathbb{N}}$.

We now give an example of τ with $J_{\ker}(G_\tau) \neq \emptyset$ to which we can apply Theorem 3.71.

Example 4.10. Let $0 < a < 1$ and let $g_1(z) = z^2$. Let $g_2 \in \mathcal{P}$ be such that $J(g_2) = \{z \in \mathbb{C} \mid |z + a| = |1 + a|\}$, $g_2(1) = 1$ and $g_2([1, \infty)) \subset [1, \infty)$. Let $l \in \mathbb{N}$ with $l \geq 2$ and let $\alpha \subset J(g_2)$ be a repelling cycle of g_2 of period l. Then there exists an $m \in \mathbb{N}$ such that $P(\langle g_1^m, g_2^m \rangle) \subset F(\langle g_1^m, g_2^m \rangle)$ and $g_1^m(\alpha) \subset F_\infty(\langle g_1, g_2 \rangle) \subset F_\infty(\langle g_1^m, g_2^m \rangle)$. Let $h_1 := g_1^m$ and $h_2 := g_2^m$. Let $(p_1, p_2) \in \mathcal{W}_2$ and let $\tau := \sum_{i=1}^2 p_i \delta_{h_i}$. Then we have $1 \in J_{\ker}(G_\tau) \cap \partial F_\infty(G_\tau)$, G_τ is hyperbolic, and $\alpha \subset F_{pt}^0(\tau)$. Thus $T_{\infty,\tau}$ is discontinuous at 1, $1 \in J_{pt}^0(\tau)$, and $T_{\infty,\tau}$ is continuous at each point of α. Moreover, by Theorem 3.71, we have $\dim_H(J_{pt}^0(\tau)) \leq \mathrm{MHD}(\tau) < 2$, $J_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$, and $J_{pt}(\tau) = J(G_\tau)$.

We now give an example of τ with $J_{\ker}(G_\tau) \neq \emptyset$ to which we can apply Theorem 3.48.

Example 4.11. Let $g_1(z) = z^2 - 1$. Let $a = \frac{1+\sqrt{5}}{2}$. Then $g_1(a) = a \in J(g_1)$. Moreover, -1 is a superattracting fixed point of g_1^2. Let $b := \frac{a+(-1)}{2}$. Then it is easy to see that b belongs to the immediate basin A_1 of 0 for the dynamics of g_1^2. Let $g_2 \in \mathcal{P}$ be such that $J(g_2) = \{z \in \mathbb{C} \mid |z - b| = a - b\}$, $g_2(a) = a$ and $g_2(-1) = -1$. Let $\epsilon > 0$ be a small number so that $b - \epsilon$ belongs to A_1. Let $c = b - \epsilon$. Let $g_3 \in \mathcal{P}$ be such that $J(g_3) = \{z \in \mathbb{C} \mid |z - c| = a - c\}$ and $g_3(a) = a$. Then b is an attracting fixed point of g_2, c is an attracting fixed point of g_3, $\{b, c\}$ is included in A_1, $\{0, c\}$ is included in the immediate basin A_2 of b for g_2, and $\{0, b, -1\}$ is included in the immediate basin A_3 of c for g_3.

Let $m \in \mathbb{N}$ be sufficiently large and let $h_1 = g_1^{2m}$, $h_2 = g_2^m$, and $h_3 = g_3^m$. Let $G = \langle h_1, h_2, h_3 \rangle$. Then $UH(G) \cap J(G) = P(G) \cap J(G) = \{-1\}$, $-1 \notin J_{\ker}(G)$ and $a \in J_{\ker}(G)$. Let $(p_1, p_2, p_3) \in \mathcal{W}_3$ and let $\tau = \sum_{i=1}^3 p_i \delta_{h_i}$. By Theorem 3.48, we obtain that (1) for $\tilde{\tau}$-a.e. $\gamma \in \mathcal{P}^{\mathbb{N}}$, $\mathrm{Leb}_2(J_\gamma) = \mathrm{Leb}_2(\hat{J}_{\gamma,\Gamma_\tau}) = 0$, (2) $\mathrm{Leb}_2(J_{pt}^0(\tau)) = 0$, and (3) for Leb_2-a.e. $y \in \hat{\mathbb{C}}$, $T_{\infty,\tau}$ is continuous at y. Moreover, since -1 is a superattracting fixed point of h_1 and $-1 \in J(h_2)$, setting $\rho = (h_1, h_1, h_1, \dots) \in X_\tau$, we have $-1 \in \mathrm{int}(\hat{J}_{\rho,\Gamma_\tau})$ (see [33, Theorem 1.6(2)]). Therefore for each $\beta \in \bigcup_{n \in \mathbb{N}} \sigma^{-n}(\rho)$, $\mathrm{int}(\hat{J}_{\beta,\Gamma_\tau}) \neq \emptyset$. Note that $\bigcup_{n \in \mathbb{N}} \sigma^{-n}(\rho)$ is dense in X_τ. Thus, (I) for $\tilde{\tau}$-a.e. $\gamma \in X_\tau$, $\mathrm{Leb}_2(\hat{J}_{\gamma,\Gamma_\tau}) = 0$, and (II) there exists a dense subset B of X_τ such that for each $\beta \in B$, $\mathrm{int}(\hat{J}_{\beta,\Gamma_\tau}) \neq \emptyset$.

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