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ON FATOU–JULIA DECOMPOSITIONS OF PSEUDOSEMIGROUPS

TARO ASUKE

ABSTRACT. According to Sullivan's dictionary, the Julia sets for iterations of rational mappings and the limit sets of Kleinian groups are in a close relationship. In this article, we will give a rough idea which relates these two notions more concretely. This is an announcement of [2], and based on a talk given at '2009 Complex Dynamics conference – Integrated Research on Complex Dynamics and its Related Fields –' held at Kyoto University.

Introduction

According to Sullivan's dictionary, the Julia sets for iterations of rational mappings and the limit sets of Kleinian groups are in a close relationship [13]. Recently, the notion of Julia sets is also introduced for complex codimension-one transversely holomorphic foliations of closed manifolds [4], [1], and it is shown that they have some common properties to the Julia sets and the limit sets as above. It is quite natural to expect there is a concept which unifies these notions. In order to find such a concept, we will need to deal with semigroups, groups and pseudogroups. Hence one way will be to consider pseudosemigroups and define their Julia sets (another approach can be found in [3]). We propose in [2] a definition of such Julia sets. In this article, we will sketch a rough idea of the definition in the case where the actions have a certain compactness called 'compact generation'.

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1. Fatou-Julia decompositions

Let f be a rational mappings on $\mathbb{C}P^1$ and $F(f)$ the Fatou set of f. We denote by $\langle f\rangle$ the semigroup generated by f, namely, we set $\langle f\rangle = \{f^n\}_{n=0}^{\infty}$, where f^n denotes the n-th iteration of f and $f^0 = \mathrm{id}$. Then,

$$(1.1)\qquad F(f) = \left\{ x \in \mathbb{C}P^1 \;\middle|\; \begin{array}{l} \text{there exists an open neighborhood } U \text{ of } x \\ \text{such that } \{g|_U\}_{g\in\langle f\rangle} \text{ is a normal family} \end{array} \right\}.$$

Hence we can regard $F(f)$ as the Fatou set of $\langle f\rangle$. Indeed, if Γ is a semigroup generated by a finite number of rational maps, then we can define the Fatou set $F(\Gamma)$ of Γ in the same way (cf. [7], [14]).

On the other hand, if G is a Kleinian group, namely, a finitely generated discrete subgroup of $\mathrm{PSL}(2;\mathbb{C})$ and if we denote by $\Omega(G)$ the domain of discontinuity of G, then it can be shown that

$$(1.2)\qquad \Omega(G) = \left\{ x \in \mathbb{C}P^1 \;\middle|\; \begin{array}{l} \text{there exists an open neighborhood } U \text{ of } x \\ \text{such that } \{g|_U\}_{g\in G} \text{ is a normal family} \end{array} \right\}.$$

Recently, Ghys, Gomez-Mont and Saludes [4] and the author [1] introduced Fatou–Julia decompositions of complex codimension-one transversely holomorphic foliations of closed manifolds. Those foliations can be viewed as one-dimensional complex dynamical systems as follows. Let $\mathcal{F}$ be a complex codimension-one transversely holomorphic foliation of a closed manifold M. Then, we can find a relatively compact, embedded real 2-dimensional manifold, say T, such that T is transversal to $\mathcal{F}$, T meets every leaf of $\mathcal{F}$, and that the holonomy along the leaves induce biholomorphic diffeomorphisms from the domains to the ranges (such mappings are called local biholomorphic diffeomorphisms), where the complex structure of T is induced from the transversal complex structure of $\mathcal{F}$. Thus obtained pseudogroup is called the *holonomy pseudogroup of* $\mathcal{F}$ *with respect to* T, and T is called a *complete transversal.* We may assume that T is the disjoint union of a finite number of open discs in $\mathbb{C}$. Such a pseudogroup inherits a certain compactness called 'compact generation' from the ambient manifold.

In short, a pseudogroup is a group but each element is equipped with its domain and range. If Γ is a pseudogroup and if $\gamma_1, \gamma_2 \in \Gamma$, then the product (composite) $\gamma_2\gamma_1$ is defined only if the range of γ_1 is contained in the domain of γ_2. As pointed out by Haefliger [6], the Fatou–Julia decomposition in the sense of [4] can be defined for compactly generated holomorphic pseudogroups on one-dimensional complex manifolds. On the other hand, a Fatou set of such a pseudogroup is defined in [1] almost in the same form as (1.1) and (1.2). A difficulty is that we cannot consider the family $\{\gamma|_U\}_{\gamma\in\Gamma}$ because the domain of $\gamma \in \Gamma$ can arbitrarily small. This leads to the following definition

Definition 1.3. An connected open subset U of T' is an F-open set ('Fatou'-open set) if the following conditions are satisfied:

1) If γ_x is the germ of an element of Γ' at x, γ is defined on U as an element of Γ, where (Γ', T') is a reduction of (Γ, T) which is explained below.
2) Let Γ^U be the subset of Γ which consists of elements of Γ obtained as in 1). Then Γ^U is a normal family.

Pseudogroups (Γ, T) and (Δ, S) are said to be equivalent if they correspond to the same dynamical systems. For example, if (Γ, T) and (Δ, S) are the holonomy pseudogroups of a foliation $\mathcal{F}$ associated with different complete transversals, then they are not the same but equivalent. See [5] for a precise definition of equivalence.

If (Γ, T) is a compactly generated pseudogroup, then by definition there is a relatively compact subset T' of T such that if we set

$$\Gamma' = \{\gamma \in \Gamma \mid \operatorname{dom}\gamma \subset T',\ \operatorname{range}\gamma \subset T'\},$$

then (Γ', T') is equivalent to (Γ, T), where $\operatorname{dom}\gamma$ and $\operatorname{range}\gamma$ denote the domain and the range of γ, respectively. Such a (Γ', T') is called a *reduction* of (Γ, T). Note that (Γ', T') is also a pseudogroup.

Definition 1.4. Let (Γ, T) be a compactly generated pseudogroup and (Γ', T') a reduction.

1) The Fatou set of (Γ', T') is the union of F-open sets, and denoted by $F(\Gamma')$.
2) The Fatou set of (Γ, T) is the image of $F(\Gamma')$ under the equivalence from (Γ', T'), and denoted by $F(\Gamma)$.

It can be shown that the decomposition is independent of the choice of reductions so that it is well-defined. It can be also shown that the decomposition is invariant under equivalences. It follows that Fatou–Julia decompositions of complex codimension-one transversely holomorphic foliations of closed manifolds can be defined via holonomy pseudogroups.

It is shown in [4] and [1] that the Fatou–Julia decomposition of compactly generated pseudogroups and that of transversely holomorphic foliations have common properties to those of the (classical) Julia sets and the limit sets.

2. Pseudosemigroups

In order to unify the (classical) Julia sets and the limit sets, we will need semigroups and their Julia sets. If we would like to add the Julia sets of compactly generated pseudogroups, we will need pseudosemigroups and their Julia sets. The notion of pseudosemigroups has already appeared (cf. [9], [15] and [8]). We will make use of a similar but different one.

Definition 2.1. Let T be a topological space and Γ be a family of continuous mappings from open subsets of T into T. Then, Γ is a *pseudosemigroup* (psg for short) if the following conditions are satisfied.

1) $\mathrm{id}_T \in \Gamma$, where id_T denotes the identity map of T.
2) If $\gamma \in \Gamma$, then $\gamma|_U \in \Gamma$ for any open subset U of $\operatorname{dom}\gamma$.
3) If $\gamma_1, \gamma_2 \in \Gamma$ and $\operatorname{range}\gamma_1 \subset \operatorname{dom}\gamma_2$, then $\gamma_2 \circ \gamma_1 \in \Gamma$.
4) Let U be an open subset of T and γ continuous mapping defined on U. If for each $x \in U$, there is an open neighborhood, say U_x, of x such that $\gamma|_{U_x}$ belongs to Γ, then $\gamma \in \Gamma$.

Example 2.2. Let f be a rational mapping on $\mathbb{C}P^1$. Let Γ be the set of mappings from an open subset of $\mathbb{C}P^1$ into $\mathbb{C}P^1$ such that $\gamma \in \Gamma$ if and only if for each $x \in \operatorname{dom}\gamma$ there is an open neighborhood $U \subset \operatorname{dom}\gamma$ of x and an $n \in \mathbb{N}$ such that $\gamma|_U = f^n|_U$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Then Γ is a pseudosemigroup which acts on $\mathbb{C}P^1$. Indeed, Γ is the pseudosemigroup generated by f.

Example 2.3. Let G be a finitely generated Kleinian group on $\mathbb{C}P^1$. Let Γ be the set of mappings from an open subset of $\mathbb{C}P^1$ into $\mathbb{C}P^1$ such that $\gamma \in \Gamma$

if and only if for each $x \in \operatorname{dom}\gamma$ there is an open neighborhood $U \subset \operatorname{dom}\gamma$ of x and a $g \in G$ such that $\gamma|_U = g|_U$. Then Γ is a pseudosemigroup which acts on $\mathbb{C}P^1$. Indeed, Γ is the pseudosemigroup generated by G. If we work on the category of homeomorphisms and require γ to be a homeomorphism, then we obtain a *pseudogroup* generated by G.

Example 2.4. Let Γ be the holonomy pseudogroup of a complex codimension-one transversely holomorphic foliation of a closed manifold with respect to a complete transversal T. Let Γ_{psg} be the set of mappings from an open subset of T into T such that $\gamma \in \Gamma_{\mathrm{psg}}$ if and only if for each $x \in \operatorname{dom}\gamma$ there is an open neighborhood $U \subset \operatorname{dom}\gamma$ of x and a $\gamma' \in \Gamma$ such that $\gamma|_U = \gamma'|_U$. Then Γ is a pseudosemigroup which acts on T. Indeed, Γ_{psg} is the pseudosemigroup generated by Γ. If we work on the category of homeomorphisms and require γ to be a homeomorphism, then we obtain the same pseudogroup as Γ instead of Γ_{psg}.

Remark 2.5. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and define $\gamma\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ by $\gamma(z) = e^{2\pi\sqrt{-1}\theta}z$, where we regard $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. Let Γ be the pseudogroup generated by γ, which is defined in a similar way as above but Γ consists of homeomorphisms. If we set $U = \{z \in \mathbb{C} \mid |z-1| < \epsilon\}$, where ϵ is a small positive number, then $\gamma|_U \in \Gamma$. We set $V = \{z \in \mathbb{C} \mid \left|z - \sqrt{-1}\right| < \epsilon\}$. We may assume that $U \cap V = \varnothing$, however, for a suitable choice of n, we have $\gamma^n(V) \cap U \neq \varnothing$. Let γ' be the mapping from $U \amalg V$ to $\mathbb{C}P^1$ by $\gamma'|_U = \gamma$ and $\gamma'|_V = \gamma^{n+1}$, then $\gamma' \notin \Gamma$ because γ' is not a homeomorphism but $\gamma' \in \Gamma_{\mathrm{psg}}$.

The semigroups which appeared in this section are *compactly generated.* Roughly speaking, a semigroup is compactly generated if it is derived from a dynamical system on a closed manifold. We refer to [2] for a precise definition.

3. A Fatou–Julia decomposition of pseudosemigroups

If (Γ, T) is a compactly generated pseudosemigroup, we can introduce a Fatou set of (Γ, T) in the same way as in Definitions 1.3 and 1.4. Even if (Γ, T) is not compactly generated, we can introduce a Fatou set, however, the construction is much more involved (see [2]). In the both cases, we can introduce the notion of *equivalence* also for pseudosemigroups, and show that Fatou–Julia decompositions are invariant under equivalences.

The Julia sets and the limit sets are unified as follows.

Theorem 3.1. *If Γ is a compactly generated pseudosemigroup, we denote by $J_{\mathrm{psg}}(\Gamma)$ its Julia set. Then we have the following.*

1) *If f is a rational mapping on $\mathbb{C}P^1$, then $J(f) = J_{\mathrm{psg}}(\langle f \rangle)$, where $\langle f \rangle$ denotes the pseudosemigroup generated by f. More generally, if $f_1, \ldots, f_r$ are rational mappings on $\mathbb{C}P^1$ and if G is the semigroup generated by $f_1, \ldots, f_r$, then $J(G) = J_{\mathrm{psg}}(\langle f_1, \ldots, f_r \rangle)$, where $\langle f_1, \ldots, f_r \rangle$ denotes the pseudosemigroup generated by $f_1, \ldots, f_r$ (or by G).*
2) *If G is a finitely generated Kleinian group, then $\Lambda(G) = J_{\mathrm{psg}}(\Gamma)$, where Γ is the pseudosemigroup generated by G.*
3) *If Γ is the holonomy pseudogroup of a complex codimension-one foliation of a closed manifold with respect to a complete transversal. If we denote by Γ_{psg} the pseudosemigroup generated by Γ, then $J(\Gamma) = J_{\mathrm{psg}}(\Gamma_{\mathrm{psg}})$.*

Some of common properties of the Julia sets and the limit sets can be regarded as properties of Julia sets of compactly generated pseudosemigroups.

Lemma 3.2. *Let Γ be a compactly generated pseudosemigroup. If we denote by $F(\Gamma)$ and $J(\Gamma)$ Fatou and Julia sets of Γ, then we have the following.*

1) *$F(\Gamma)$ is forward Γ-invariant, i.e., $\Gamma(F(\Gamma)) = \Gamma$, where $\Gamma(F(\Gamma)) = \{x \in T \mid \exists \gamma \in \Gamma,\ \exists y \in F(\Gamma) \text{ s.t. } x = \gamma(y)\}$.*
2) *$J(\Gamma)$ is backward Γ-invariant, i.e., $\Gamma^{-1}(J(\Gamma)) = J(\Gamma) = \{x \in T \mid \exists \gamma \in \Gamma, \text{ s.t. } \gamma(x) \in J(\Gamma)\}$.*

Remark 3.3. 1) We can construct a metric on $F(\Gamma)$ which is adapted to the Γ-action. This suggests that the Γ-action on $F(\Gamma)$ is tame.

2) A Fatou–Julia decomposition of singular holomorphic foliations of complex codimension one can be introduced by using Fatou–Julia decompositions of non-compactly generated pseudogroups, In [2], some properties of those decompositions will be studied.

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Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1, Komaba, Meguro-ku, Tokyo 153-8914, Japan
E-mail address: `asuke@ms.u-tokyo.ac.jp`