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ENDOMORPHISMS OF PROJECTIVE VARIETIES AND THEIR INVARIANT HYPERSURFACES

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ABSTRACT. We consider surjective endomorphisms f of degree > 1 on projective manifolds X of Picard number one and their f^{-1}-stable hypersurfaces V. When $X = \mathbb{P}^n$ with $n = 3$, we show that V is a hyperplane (i.e., $\deg(V) = 1$) but with four possible exceptions; it is conjectured that $\deg(V) = 1$ for any $n \geq 2$; cf. [8], [3]. For general X, we show that V is rationally chain connected. Also given is an optimal upper bound for the number of f^{-1}-stable prime divisors on (not necessarily smooth) projective varieties.

1. ENDOMORPHISMS OF $\mathbb{P}^3$

We work over the field $\mathbb{C}$ of complex numbers. We start with the consideration of endomorphisms of the projective three space. The main result of this section is Theorem 1.1 below.

Theorem 1.1. *Let $f : \mathbb{P}^3 \to \mathbb{P}^3$ be an endomorphism of degree > 1 and V an irreducible hypersurface such that $f^{-1}(V) = V$. Then either* $\deg(V) = 1$, *i.e., V is a hyperplane, or $V = V_i := \{S_i = 0\}$ is a cubic hypersurface given by one of the following four defining equations S_i in suitable projective coordinates:*

(1) $S_1 = X_3^3 + X_0X_1X_2$;
(2) $S_2 = X_0^2X_3 + X_0X_1^2 + X_2^3$;
(3) $S_3 = X_0^2X_2 + X_1^2X_3$;
(4) $S_4 = X_0X_1X_2 + X_0^2X_3 + X_1^3$.

We are unable to rule out the four cases in Theorem 1.1 but see Examples 2.8 (for V_1).

Remark 1.2. (1) The non-normal locus of V_i ($i = 3, 4$) is a single line C and stabilized by f^{-1}. Let $\sigma : V_i' \to V_i$ ($i = 3, 4$) be the normalization. Then V_i' is the (smooth) Hirzebruch surface $\mathbb{F}_1$ (i.e., the one-point blowup of $\mathbb{P}^2$); see [1, Theorem 1.5], [17].

(2) V_1 (resp. V_2) is unique as a normal cubic (or degree three del Pezzo) surface of Picard number one and with the singular locus $\operatorname{Sing} V_1 = 3A_2$ (resp. $\operatorname{Sing} V_2 =$

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E_6); see [20, Theorem 1.2] and [10, Theorem 4.4] (for the anti-canonical embedding of V_i in $\mathbb{P}^3$). V_1 contains exactly three lines (of triangle-shaped) whose three vertices form the singular locus of V_1. And V_2 contains a single line on which lies its unique singular point. f^{-3} (replaced by its cube) fixes the singular point(s) of V_i $(i = 1, 2)$.

(3) f^{-1} (or its power) does not stabilize the only line L on V_2 by using [15, Theorem 4.3.1] since the pair (V_2, L) is not log canonical at the singular point of V_2. For V_1, we do not know whether f^{-1} (or its power) stabilizes its three lines.

We now sketch the proof of Theorem 1.1.

By [16, Theorem 1.5], we may assume that $V \subset \mathbb{P}^3$ is an irreducible rational *singular* cubic hypersurface.

We first consider the case where V is non-normal. Such V is classified in [6, Theorem 9.2.1] to the effect that either $V = V_i$ $(i = 3, 4)$ or V is a cone over a nodal or cuspidal rational planar cubic curve B. The description in Remark 1.2 on V_3, V_4 and their normalizations, is given in [17, Theorem 1.1], [1, Theorem 1.5, Case (C), (E1)].

We can rule out the case where V is a cone over B.

Next we consider the case where $V \subset \mathbb{P}^3$ is a normal rational *singular* cubic hypersurface. By the adjunction formula, $-K_V = -(K_{\mathbb{P}^3} + V)|V \sim H|V$ which is ample, where $H \subset \mathbb{P}^3$ is a hyperplane. Since K_V is a Cartier divisor, V has only Du Val (or rational double, or ADE) singularities. Let $\sigma : V' \to V$ be the minimal resolution. Then $K_{V'} = \sigma^* K_V \sim \sigma^*(-H|V)$. For $f : \mathbb{P}^3 \to \mathbb{P}^3$, we can apply $f|V$ to the result below.

Lemma 1.3. *Let $V \subset \mathbb{P}^3$ be a normal cubic surface, and $f_V : V \to V$ an endomorphism such that $f_V^*(H|V) \sim qH|V$ for some $q > 1$ and the hyperplane $H \subset \mathbb{P}^3$. Let $S(V) = \{(\text{irreducible})\, G \subset V \mid G^2 < 0\}$ be the set of negative curves on V, and set $E_V := \sum_{E \in S(V)} E$. Replacing f_V by its power, we have:*

1. *If $f_V^* G \equiv aG$ for some Weil divisor $G \not\equiv 0$, then $a = q$. $f_V^*(L|V) \sim q(L|V)$ for every divisor L on $\mathbb{P}^3$. Especially, $\deg(f_V) = q^2$; $K_V \sim -H|V$ satisfies $f_V^* K_V \sim qK_V$.*
2. *$S(V)$ is a finite set. $f_V^* E = qE$ for every $E \in S(V)$. So $f_V^* E_V = qE_V$.*
3. *A curve $E \subset V$ is a line in $\mathbb{P}^3$ if and only if E is equal to $\sigma(E')$ for some (-1)-curve $E' \subset V'$.*
4. *Every curve $E \in S(V)$ is a line in $\mathbb{P}^3$.*
5. *We have $K_V + E_V = f_V^*(K_V + E_V) + \Delta$ for some effective divisor Δ containing no line in $S(V)$, so that the ramification divisor $R_{f_V} = (q-1)E_V + \Delta$. In particular, the cardinality $\#S(V) \le 3$, and the equality holds exactly when $K_V + E_V \sim_{\mathbb{Q}} 0$; in this case, f_V is étale outside the three lines of $S(V)$ and $f_V^{-1}(\operatorname{Sing} V)$.*

Remark 1.4. In the proof of Theorem 1.1, we can actually show: if $f_V : V \to V$ is an endomorphism (not necessarily the restriction of some $f : \mathbb{P}^3 \to \mathbb{P}^3$) of $\deg(f_V) > 1$ of a Gorenstein normal del Pezzo surface with $K_V^2 = 3$ (i.e., a normal cubic surface), then V is equal to V_1 or V_2 in Theorem 1.1 in suitable projective coordinates.

2. Summary of main results

Below is the summary of our recent paper [23]. Theorem 2.1 $\sim$ Theorem 2.4 are our main results.

Theorem 2.1. *Let X be a locally factorial normal projective variety of dimension $n \ge 2$ and Picard number one, and with only log canonical singularities, and let $f : X \to X$ be a surjective endomorphism with* $\deg(f) = q^n > 1$. *Then we have:*

(1) *There are at most $n+1$ prime divisors $V_i \subset X$ with $f^{-1}(V_i) = V_i$.*
(2) *There are $n+1$ of such V_i if and only if: $X = \mathbb{P}^n$, $V_i = \{X_i = 0\}$ $(1 \le i \le n+1)$ (in suitable projective coordinates), and f is given by*

$$f : [X_0, \ldots, X_n] \longrightarrow [X_0^q, \ldots, X_n^q].$$

We refer to S. -W. Zhang [21, Conjecture 1.3.1] for the Dynamic Manin-Mumford conjecture etc. solved for the (X, f) in Theorem 2.1 (2).

A projective variety X is *rationally chain connected* if every two points $x_i \in X$ are contained in a connected chain of rational curves on X. When X is smooth, X is rationally *chain* connected if and only if X is *rationally connected*, in the sense of Campana, and Kollár-Miyaoka-Mori.

Theorem 2.2. *Let X be a projective manifold of dimenion $n \ge 2$ and Picard number one, $f : X \to X$ an endomorphism of degree > 1, and $V \subset X$ a prime divisor with $f^{-1}(V) = V$. Then X, V and the normalization V' of V are all rationally chain connected.*

In Theorem 2.2, the smoothness and Picard number one assumption on X are necessary (cf. Remark 2.6 and Example 2.9). Theorem 2.2 is known for $X = \mathbb{P}^n$ with $n \le 3$ (cf. [8], [16]). In Theorem 2.2, X is indeed a Fano manifold. See Remark 2.6 for the case when X is singular.

Corollary 2.3. *With the notation and assumptions in Theorem 2.2, both X and V are simply connected, while V' has a finite (topological) fundamental group.*

A morphism $f : X \to X$ is *polarized* (by H) if $f^*H \sim qH$ for some ample line bundle H and some $q > 0$; then $\deg(f) = q^{\dim X}$. For instance, every non-constant endomorphism of a projective variety X of Picard number one, is polarized; an f-stable subvariety

$X \subset \mathbb{P}^n$ for a non-constant endomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$, has the restriction $f|X : X \to X$ polarized by the hyperplane; the multiplication map $m_A : A \to A$, $x \mapsto mx$ (with $m \neq 0$) of an abelian variety A is polarized by any $H = L + (-1)^*L$ with L an ample divisor, so that $m_A^*H \sim m^2H$.

In Theorems 2.1 and 2.4, we give upper bounds for the number of f^{-1}-stable prime divisors on a (not necessarily smooth) projective variety; the bounds are optimal, and the second possibility in Theorem 2.4(2) does occur (cf. Examples 2.8 and 2.9). One may remove the condition (*) in Theorem 2.4, when $\rho(X) = 1$, or X is a weak $\mathbb{Q}$-Fano variety, or the closed cone $\overline{\mathrm{NE}}(X)$ of effective curves has only finitely many extremal rays (cf. Remark 2.6); here $N^1(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is the *Néron-Severi group* (over $\mathbb{R}$) and $\rho(X) := \mathrm{rank}_{\mathbb{R}} N^1(X)$ is the *Picard number* of X. We refer to [11, Definition 2.34] for the definitions of Kawamata log terminal (klt) and log canonical singularities.

Theorem 2.4. *Let X be a projective variety of dimension n with only $\mathbb{Q}$-factorial Kawamata log terminal singularities, and $f : X \to X$ a polarized endomorphism with* $\deg(f) = q^n > 1$. *Suppose* $(*)$: *either* $f^* \mid N^1(X) = q\,\mathrm{id}$, *or* $n \leq 3$. *Then we have (with* $\rho := \rho(X)$*):*

(1) *Let $V_i \subset X$ $(1 \leq i \leq c)$ be prime divisors with $f^{-1}(V_i) = V_i$. Then $c \leq n + \rho$. Further, if $c \geq 1$, then the pair $(X, \sum V_i)$ is log canonical and X is uniruled.*

(2) *Suppose that $c \geq n + \rho - 2$. Then either X is rationally connected, or there is a fibration $X \to E$ onto an elliptic curve E so that every fibre is normal rationally connected and some positive power f^k descends to an $f_E : E \to E$ of degree q.*

(3) *Suppose that $c \geq n + \rho - 1$. Then X is rationally connected.*

(4) *Suppose that $c \geq n + \rho$. Then $c = n + \rho$, (for some $t > 0$)*

$$K_X + \sum_{i=1}^{n+\rho} V_i \sim_{\mathbb{Q}} 0, \qquad (f^t)^* \mid \mathrm{Pic}(X) = q^t\,\mathrm{id}\,,$$

f is étale outside $(\cup V_i) \cup f^{-1}(\mathrm{Sing}\,X)$ (and X is a toric surface with $\sum V_i$ its boundary divisor, when $\dim X = 2$).

Theorems 2.4 and 2.1 motivate the question below (without assuming the condition (*) in Theorem 2.4), where the last part is also Shokurov's conjecture (cf. [18, Theorem 6.4]).

Question 2.5. Suppose that a projective n-fold $(n \geq 3)$ X has only $\mathbb{Q}$-factorial Kawamata log terminal singularities, $f : X \to X$ a polarized endomorphism of degree > 1, and $V_i \subset X$ $(1 \leq i \leq s)$ prime divisors with $f^{-1}(V_i) = V_i$. Then, is it true that $s \leq n + \rho(X)$, and equality holds only when X is a toric variety with $\sum V_i$ its boundary divisor?

Remark 2.6. (1) In Theorem 2.2, it is necessary to assume that $\rho(X) = 1$ (cf. Example 2.9), and X is smooth or at least Kawamata log terminal (klt). Indeed, for every projective cone Y over an elliptic curve and every section $V \subset Y$ (away from the vertex), there is an endomorphism $f : Y \to Y$ of $\deg(f) > 1$ and with $f^{-1}(V) = V$ (cf. [15, Theorem 7.1.1, or Proposition 5.2.2]). The cone Y has Picard number one and a log canonical singularity at its vertex. Of course, V is an elliptic curve, and is not rationally chain connected. By the way, Y is rationally *chain* connected, but is not rationally connected.

(2) Let X be a projective variety with only klt singularities. If the closed cone $\overline{\mathrm{NE}}(X)$ of effective curves has only finitely many extremal rays, then every polarized endomorphism $f : X \to X$ satisfies $f^* \mid N^1(X) = q \,\mathrm{id}$ with $\deg(f) = q^{\dim X}$, after replacing f by its power, so that we can apply Theorem 2.4 (cf. [16, Lemma 2.1]). For instance, if X or (X, Δ) is $\mathbb{Q}$-Fano, i.e., X (resp. (X, Δ)) has only klt singularities and $-K_X$ (resp. $-(K_X + \Delta)$) is nef and big, then $\overline{\mathrm{NE}}(X)$ has only finitely many extremal rays.

(3) By Example 2.8, it is necessary to assume the local factoriality of X or the Cartierness of V_i in Theorem 2.1 (2) even when X has only klt singularities. We remark that a $\mathbb{Q}$-factorial Gorenstein terminal threefold is locally factorial. For Theorem 2.1(2), one may also use Fujita's theory to prove $X \simeq \mathbb{P}^n$, but our method is useful even when V_i's are only $\mathbb{Q}$-Cartier (cf. Theorem 2.4).

2.7. **A motivating conjecture.** Here are some motivations for our paper. It is conjectured that *every hypersurface $V \subset \mathbb{P}^n$ stabilized by the inverse f^{-1} of an endomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$ of* $\deg(f) > 1$, *is linear.* This conjecture is still open when $n \geq 3$ and V is singular, since the proof of [3] is incomplete as we were informed by an author. The smooth hypersurface case was settled in the affirmative (in any dimension) by Cerveau - Lins Neto [4] and independently by Beauville [2]. See also [16, Theorem 1.5 in its arXiv version: arXiv:0908.1688v1].

From the dynamics point of view, as seen in Dinh-Sibony [5, Theorem 1.3, Corollary 1.4], $f : \mathbb{P}^n \to \mathbb{P}^n$ behaves nicely *exactly* outside those f^{-1}-stabilized subvarieties. We refer to Fornaess-Sibony [8], and [5] for further references.

A smooth hypersurface X in $\mathbb{P}^{n+1}$ with $\deg(X) \geq 3$ and $n \geq 2$, has no endomorphism $f_X : X \to X$ of degree > 1 (cf. [2, Theorem]). However, singular X may have plenty of endomorphisms f_X of arbitrary degrees as shown in Example 2.8 below. Conjecture 2.7 asserts that such f_X can not be extended to an endomorphism of $\mathbb{P}^{n+1}$.

Example 2.8. We now construct many polarized endomorphisms for some degree $n+1$ hypersurface $X \subset \mathbb{P}^{n+1}$, with X isomorphic to the V_1 in Theorem 1.1 when $n = 2$. Let $f = (F_0, \dots, F_n) : \mathbb{P}^n \to \mathbb{P}^n$ $(n \geq 2)$, with $F_i = F_i(X_0, \dots, X_n)$ homogeneous, be any endomorphism of degree $q^n > 1$, such that $f^{-1}(S) = S$ for a reduced degree

$n+1$ hypersurface $S = \{S(X_0, \ldots, X_n) = 0\}$. So S must be normal crossing and linear: $S = \sum_{i=0}^{n} S_i$ (cf. [16, Thm 1.5 in arXiv version]). Thus we may assume that $f = (X_0^q, \ldots, X_n^q)$ and $S_i = \{X_i = 0\}$. The relation $S \sim (n+1)H$ with $H \subset \mathbb{P}^n$ a hyperplane, defines

$$\pi : X = Spec \oplus_{i=0}^{n} \mathcal{O}(-iH) \to \mathbb{P}^n$$

which is a Galois $\mathbb{Z}/(n+1)$-cover branched over S so that $\pi^* S_i = (n+1)T_i$ with the restriction $\pi|T_i : T_i \to S_i$ an isomorphism.

This X is identifiable with the degree $n+1$ hypersurface $\{Z^{n+1} = S(X_0, \ldots, X_n)\} \subset \mathbb{P}^{n+1}$ and has singularity of type $z^{n+1} = xy$ over the intersection points of S locally defined as $xy = 0$. Thus, when $n = 2$, we have $\operatorname{Sing} X = 3A_2$ and X is isomorphic to the V_1 in Theorem 1.1 (cf. Remark 1.2). We may assume that $f^* S(X_0, \ldots, X_n) = S(X_0, \ldots, X_n)^q$ after replacing $S(X_0, \ldots, X_n)$ by a scalar multiple, so f lifts to an endomorphism $g = (Z^q, F_0, \ldots, F_n)$ of $\mathbb{P}^{n+1}$ (with homogeneous coordinates $[Z, X_0, \ldots, X_n]$), stabilizing X, so that $g_X := g|X : X \to X$ is a polarized endomorphism of $\deg(g_X) = q^n$ (cf. [16, Lemma 2.1]). Note that $g^{-1}(X)$ is the union of q distinct hypersurfaces $\{Z = \zeta^i S(X_0, \ldots, X_n)\} \subset \mathbb{P}^{n+1}$ (all isomorphic to X), where $\zeta := \exp(2\pi\sqrt{-1}/q)$.

This X has only Kawamata log terminal singularities and $\operatorname{Pic} X = (\operatorname{Pic} \mathbb{P}^{n+1})|X$ ($n \geq 2$) is of rank one (using Lefschetz type theorem [12, Example 3.1.25] when $n \geq 3$). We have $f^{-1}(S_i) = S_i$ and $g_X^{-1}(T_i) = T_i$, where $0 \leq i \leq n$. Note that $(n+1)T_i = \pi^* S_i$ is Cartier, but T_i is not Cartier (cf. Theorems 2.1).

When $n = 2$, the relation $(n+1)(T_1 - T_0) \sim 0$ gives rise to an étale-in-codimenion-one $\mathbb{Z}/(n+1)$-cover $\tau : \mathbb{P}^n \simeq \widetilde{X} \to X$ so that $\sum_{i=0}^{n} \tau^* T_i$ is a union of $n+1$ normal crossing hyperplanes; indeed, τ restricted over $X \setminus \cup T_i$, is its universal cover (cf. [13, Lemma 6]), so that g_X lifts up to $\widetilde{X}$. A similar result *seems* to be true for $n \geq 3$, by considering the 'composite' of the $\mathbb{Z}/(n+1)$-covers given by $(n+1)(T_i - T_0) \sim 0$ ($1 \leq i < n$); see Question 2.5.

The simple Example 2.9 below shows that the conditions in Theorem 2.4 (2) (3), or the condition $\rho(X) = 1$ in Theorem 2.2, is necessary.

Example 2.9. Let $m_A : A \to A$ ($x \mapsto mx$) with $m \geq 2$, be the multiplication map of an abelian variety A of dimension $u \geq 1$ and Picard number one, and let $g : \mathbb{P}^v \to \mathbb{P}^v$ ($[X_0, \ldots, X_v] \mapsto [X_0^q, \ldots, X_v^q]$) with $v \geq 1$ and $q := m^2$. Then $f = (m_A \times g) : X = A \times \mathbb{P}^v \to X$ is a polarized endomorphism with $f^* \,|\, N^1(X) = \operatorname{diag}[q, q]$, and f^{-1} stabilizes $v+1$ prime divisors $V_i = A \times \{X_i = 0\} \subset X$ and no others; indeed, f is étale outside $\cup V_i$. Note that X and $V_i \simeq A \times \mathbb{P}^{v-1}$ are not rationally chain connected, and $v + 1 = \dim X + \rho(X) - (1 + \dim A)$.

2.10. The results of Favre [7], Nakayama [15] and Wahl [19] are very inspiring about the restriction of the singularity type of a normal surface imposed by the existence of an endomorphism of degree > 1 on the surface. For the proof of our results, the basic ingredients are: a log canonical singularity criterion, a rational connectedness criterion of Qi Zhang [24] and its generalization in Hacon-McKernan [9], the equivariant MMP in our early paper [22], and the characterization in Mori [14] on hypersurfaces in weighted projective spaces.

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