数理解析研究所講究録
第 1700 巻 2010 年 118-131

Darboux transformations for twisted derivations

C.X. Li[1*] and J.J.C. Nimmo[2†]
[1]School of Mathematical Sciences,
Capital Normal University, Beijing 100048, CHINA
[2]Department of Mathematics,
University of Glasgow, Glasgow G12 8QW, UK

Abstract

This paper is concerned with a generalized type of Darboux transformations defined in terms of a twisted derivation D satisfying $D(AB)=D(A)+\sigma(A)B$ where σ is a homomorphism. Such twisted derivations include regular derivations, difference and q-difference operators and superderivatives as special cases. Remarkably, the formulae for the iteration of Darboux transformations are identical with those in the standard case of a regular derivation and are expressed in terms of quasideterminants. As an example, we revisit the Darboux transformations for the Manin-Radul super KdV equation.

1 Introduction

Recently noncommutative versions of integrable systems have received much attention [1–14]. It has been shown that such systems often have solutions expressed in terms of quasideterminants [15]. The prototypical example of this is the class of solutions of the noncommutative KP equation found using Darboux transformations [16]. In [12] also, a second type of quasideterminant solutions for this equation were found using binary Darboux transformations.

Supersymmetric integrable systems are a particular noncommutative extension of integrable systems. Among these, the Manin-Radul super KdV equation [17] is perhaps the best known example. Motivated in part by the properties of superderivatives, we consider a generalized derivation which has regular derivations, difference operators, q-difference operators and superderivatives as some of its special cases. We call this a twisted derivation, following the terminology used in [18,19]. We show that one can formulate Darboux transformations for such twisted derivations and the iteration formulae are expressed in terms of quasideterminants in which one simply replaces the derivative with the twisted derivation.

In [20, 21] solutions for the Manin-Radul super KdV equation were constructed by means of Darboux transformations and binary Darboux transformations. In this paper, we use an alternative approach to such Darboux transformations using quasideterminants. This is successful in obtaining simple unified formulae for the solutions. From these quasideterminant solutions, we recover the superdeterminant solutions given in [20, 21] and also get a superdeterminant representation in the cases not considered in the earlier work.

The paper is organized as follows. In Section 2, we give a brief review of relevant properties of quasideterminants. In Section 3, in order to introduce the basic ideas, we discuss Darboux transformations for the noncommutative KP equation. Then, in Section 4, the main results are

*trisha_li2001@yahoo.com
†j.nimmo@maths.gla.ac.uk

described. Twisted derivation and related Darboux transformation are defined and a quasideterminant iteration formula for twisted Darboux transformation is obtained. Section 5 contains some basic facts about supersymmetric objects. In Section 6 the Darboux and binary Darboux transformations to the Manin-Radul super KdV system are discussed. Finally, in Section 7, the solutions obtained using iterated Darboux and binary Darboux transformations are reexpressed in terms of superdeterminants. Proofs of the results stated in this paper are given in [22].

2 Properties of quasideterminants

In this section, we record some basic facts about quasideterminants [15, 16, 23]. The reader is referred to the above mentioned literature for more details.

An $n \times n$ matrix $M = (m_{i,j})$ over a ring $\mathcal{R}$ (noncommutative, in general) has n^2 *quasideterminants* written as $|M|_{i,j}$ for $i, j = 1, \ldots, n$, which are also elements of $\mathcal{R}$. They are defined recursively by

$$|M|_{i,j} = m_{i,j} - r_i^j (M^{i,j})^{-1} c_j^i, \quad M^{-1} = (|M|_{j,i}^{-1})_{i,j=1,\ldots,n}. \tag{1}$$

In the above r_i^j represents the ith row of M with the jth element removed, c_j^i the jth column with the ith element removed and $M^{i,j}$ the submatrix obtained by removing the ith row and the jth column from M. Quasideterminants can be also denoted as shown below by boxing the entry about which the expansion is made

$$|M|_{i,j} = \begin{vmatrix} M^{i,j} & c_j^i \\ r_i^j & \boxed{m_{i,j}} \end{vmatrix}.$$

Note that if the entries in M commute then

$$|M|_{i,j} = (-1)^{i+j} \frac{\det(M)}{\det(M^{i,j})}. \tag{2}$$

Noncommutative Jacobi Identity There is a quasideterminant version of the Jacobi identity for determinants [15]. The simplest version of this identity is given by

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}, \tag{3}$$

where $f, g, h, i \in \mathcal{R}$, A is an $n \times n$ matrix and B, C (resp. D, E) are column (resp. row) n-vectors over $\mathcal{R}$.

Quasi-Plücker coordinates Given an $(n+k) \times n$ matrix A, denote the ith row of A by A_i, the submatrix of A having rows with indices in a subset I of $\{1, 2, \ldots, n+k\}$ by A_I and $A_{\{1,\ldots,n+k\}\setminus\{i\}}$ by $A_{\hat{i}}$. Given $i, j \in \{1, 2, \ldots, n+k\}$ and I such that $\#I = n-1$ and $j \notin I$, one defines the *(right) quasi-Plücker coordinates*

$$r_{ij}^I = r_{ij}^I(A) := \begin{vmatrix} A_I \\ A_i \end{vmatrix}_{ns} \begin{vmatrix} A_I \\ A_j \end{vmatrix}_{ns}^{-1} = - \begin{vmatrix} A_I & 0 \\ A_i & \boxed{0} \\ A_j & 1 \end{vmatrix}, \tag{4}$$

for any column index $s \in \{1, \ldots, n\}$. The final equality in (4) comes from an identity of the form (3) and proves that the definition is independent of the choice of s.

Derivatives of quasideterminants Consider the derivative of an arbitrary quasideterminant

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B' \tag{5}$$

where A is an $n \times n$ matrix, C is a row vector and B a column vector. Let I denote the $n \times n$ identity matrix and let Z^k and Z_k denote the kth row and the kth column of a matrix Z, respectively. Then

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & I_k \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ (A^k)' & \boxed{(B^k)'} \end{vmatrix}. \tag{6}$$

3 Darboux transformations for the ncKP equation

To introduce the key aspects of Darboux transformations we consider the standard example of the noncommutative KP (ncKP) equation [1–9, 12, 16]

$$(v_t + v_{xxx} + 3v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = 0. \tag{7}$$

Its Lax pair is

$$L = \partial_x^2 + v_x - \partial_y, \tag{8}$$
$$M = 4\partial_x^3 + 6v_x\partial_x + 3v_{xx} + 3v_y + \partial_t. \tag{9}$$

Let θ be such that $L(\theta) = M(\theta) = 0$, and we call θ an eigenfunction. Define the operator

$$G_\theta = \theta\partial_x\theta^{-1} = \partial_x - \theta_x\theta^{-1}. \tag{10}$$

The Lax pair is covariant with respect to G_θ in the sense that

$$\widetilde{L} = G_\theta L G_\theta^{-1}, \quad \widetilde{M} = G_\theta M G_\theta^{-1},$$

have the same form as L and M with v changed to $\widetilde{v} = v + 2\theta_x\theta^{-1}$. This transformation is called a *Darboux transformation.* Since the form of L and M is preserved, it induces a Bäcklund transformation for the ncKP equation.

This transformation may be iterated as follows. Let $\phi_{[0]} = \phi$ be a generic eigenfunction and let $\theta_0, \ldots, \theta_{n-1}$ be invertible eigenfunctions of $(L[0], M[0]) = (L, M)$. Define $\theta[0] = \theta_0$. Then $\phi[1] := G_{\theta[0]}(\phi[0])$ and $\theta[1] = \phi[1]|_{\phi\to\theta_1}$ are eigenfunctions for

$$(L[1], M[1]) = (G_{\theta[0]}L[0]G_{\theta[0]}^{-1}, G_{\theta[0]}M[0]G_{\theta[0]}^{-1}).$$

In general, for $n \geq 0$ define the nth Darboux transform of ϕ by

$$\phi[n+1] = \phi[n]^{(1)} - \theta[n]^{(1)}\theta[n]^{-1}\phi[n],$$

in which

$$\theta[k] = \phi[k]|_{\phi\to\theta_k}.$$

After n Darboux transformations the change of the Lax pair is that

$$v[n] = v + 2\sum_{i=0}^{n-1} \theta[i]_x\theta[i]^{-1}. \tag{11}$$

Further, it may be proved by induction that

$$\sum_{i=0}^{n-1} \theta[i]_x \theta[i]^{-1} = - \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}, \tag{12}$$

where $\Theta = (\theta_0, \ldots, \theta_{n-1})$ and $\Theta^{(k)}$ is its kth derivative with respect to x.

To define a binary Darboux transformation one needs to consider the adjoint Lax pair

$$L^\dagger = \partial_x^2 + v_x^\dagger + \partial_y, \tag{13}$$
$$M^\dagger = -4\partial_x^3 - 6v_x^\dagger \partial_x - 3v_{xx}^\dagger + 3v_y^\dagger - \partial_t. \tag{14}$$

Following the standard construction of a binary Darboux transformation (see [24, 25]) one introduces a potential $\Omega(\phi, \psi)$ satisfying

$$\Omega(\phi,\psi)_x = \psi^\dagger \phi, \quad \Omega(\phi,\psi)_y = \psi^\dagger \phi_x - \psi_x^\dagger \phi, \quad \Omega(\phi,\psi)_t = -4(\psi^\dagger \phi_{xx} - \psi_x^\dagger \phi_x + \psi_{xx}^\dagger \phi) - 6\psi^\dagger v_x \phi. \tag{15}$$

The definition is consistent whenever $L(\phi) = M(\phi) = 0$ and $L^\dagger(\psi) = M^\dagger(\psi) = 0$. More generally, we can define $\Omega(\Phi, \Psi)$ for any row vectors Φ and Ψ such that $L(\Phi) = M(\Phi) = 0$ and $L^\dagger(\Psi) = M^\dagger(\Psi) = 0$. If Φ is an n-vector and Ψ is an m-vector then $\Omega(\Phi, \Psi)$ is an $m \times n$ matrix.

A binary Darboux transformation is then defined by

$$\phi_{[n+1]} = \phi_{[n]} - \theta_{[n]} \Omega(\theta_{[n]}, \rho_{[n]})^{-1} \Omega(\phi_{[n]}, \rho_{[n]})$$

and

$$\psi_{[n+1]} = \psi_{[n]} - \rho_{[n]} \Omega(\theta_{[n]}, \rho_{[n]})^{-\dagger} \Omega(\theta_{[n]}, \psi_{[n]})^\dagger,$$

where

$$\theta_{[n]} = \phi_{[n]}\big|_{\phi \to \theta_n}, \quad \rho_{[n]} = \psi_{[n]}\big|_{\psi \to \rho_n}.$$

Using the notation $\Theta = (\theta_0, \ldots, \theta_{n-1})$ (as above) and $P = (\rho_0, \ldots, \rho_{n-1})$ it is can be shown that for $n \geq 1$,

$$\phi_{[n]} = \begin{vmatrix} \Omega(\Theta, P) & \Omega(\phi, P) \\ \Theta & \boxed{\phi} \end{vmatrix}, \tag{16}$$

$$\psi_{[n]} = \begin{vmatrix} \Omega(\Theta, P)^\dagger & \Omega(\Theta, \psi)^\dagger \\ P & \boxed{\psi} \end{vmatrix}, \tag{17}$$

and

$$\Omega(\phi_{[n]}, \psi_{[n]}) = \begin{vmatrix} \Omega(\Theta, P) & \Omega(\phi, P) \\ \Omega(\Theta, \psi) & \boxed{\Omega(\phi, \psi)} \end{vmatrix}. \tag{18}$$

The effect of this transformations on the Lax pair is to give new coefficients defined in terms of

$$\widehat{v} = v + 2\theta \Omega(\theta, \rho)^{-1} \rho^\dagger.$$

Thus after n binary Darboux transformations we obtain

$$v_{[n]} = v + 2 \sum_{k=0}^{n-1} \theta_{[k]} \Omega(\theta_{[k]}, \rho_{[k]})^{-1} \rho_{[k]}^\dagger, \tag{19}$$

and this may be reexpressed in terms of a single quasideterminant as

$$v_{[n]} = v - 2\begin{vmatrix} \Omega(\Theta, P) & P^\dagger \\ \Theta & \boxed{0} \end{vmatrix}. \tag{20}$$

In this way one obtains a second expression for solutions of the ncKP equation in terms of quasideterminants.

4 Darboux transformations for twisted derivations

Suppose that $\mathcal{A}$ is an associative, unital algebra over ring K. Suppose that there is a homomorphism $\sigma\colon \mathcal{A} \to \mathcal{A}$ (i.e. for all $\alpha \in K$, $a, b \in \mathcal{A}$, $\sigma(\alpha a) = \alpha\sigma(a)$, $\sigma(a+b) = \sigma(a) + \sigma(b)$ and $\sigma(ab) = \sigma(a)\sigma(b)$) and a *twisted derivation* or σ-derivation [18, 19] $D\colon \mathcal{A} \to \mathcal{A}$ satisfying $D(K) = 0$ and $D(ab) = D(a)b + \sigma(a)D(b)$.

Important particular examples of such a set-up arise when elements $a \in \mathcal{A}$ depend on a variable x, say.

Derivative Here $D = \partial/\partial x$ satisfies $D(ab) = D(a)b + aD(b)$ and σ is the identity mapping.

Forward difference The homomorphism is the shift operator T, where $T(a(x)) = a(x+1)$ and the twisted derivation is

$$\Delta(a(x)) = \frac{a(x+h) - a(x)}{h},$$

satisfying $\Delta(ab) = D(a)b + T(a)D(b)$.

Jackson derivative The homomorphism is a q-shift operator defined by $S_q(a(x)) = a(qx)$ and the twisted derivation is

$$D_q(a(x)) = \frac{a(qx) - a(x)}{(q-1)x}.$$

satisfying $D_q(ab) = D_q(a)b + S_q(a)D_q(b)$.

Superderivative As described in Section 5, for $a, b \in \mathcal{A}$, a superalgebra, $D(ab) = D(a)b + \hat{a}D(b)$ where $\hat{}$ is the grade involution.

4.1 Darboux transformations

Here we consider a more abstract situation modelled on the Darboux transformation for the KP equation. Let $\theta_0, \theta_1, \theta_2, \dots$ be a sequence in $\mathcal{A}$. Consider the sequence $\theta[0], \theta[1], \theta[2], \dots$ in $\mathcal{A}$, generated from the first sequence by Darboux transformations of the form

$$G_\theta = \sigma(\theta)D\theta^{-1} = D - D(\theta)\theta^{-1}, \tag{21}$$

where D and σ are the twisted derivation and homomorphism defined above. To be specific, $\theta[0] = \theta_0$ and $G[0] = G_{\theta[0]}$, then let

$$\theta[1] = G[0](\theta_1) = D(\theta_1) - D(\theta_0)\theta_0^{-1}\theta_1 \tag{22}$$

and $G[1] = G_{\theta[1]}$, $\theta[2] = G[1] \circ G[0](\theta_2)$ and $G[2] = G_{\theta[2]}$ and so on. In general, for $k \in \mathbb{N}$,

$$\theta[k] = G[k-1] \circ G[k-2] \circ \cdots \circ G[0](\theta_k), \quad G[k] = \sigma(\theta[k])D\theta[k]^{-1}, \tag{23}$$

in which we require that each $\theta[k]$ is invertible.

In the standard case of a derivation, $D = \partial$ and $\sigma = \mathrm{Id}$, it is well known that the terms in the sequence of Darboux transformations have closed form expressions in terms of the original sequence. In the case that $\mathcal{A}$ is commutative, they are expressed as ratios of wronskian determinants [26],

$$\theta[n] = \frac{\begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} & \theta_n \\ \theta_0^{(1)} & \cdots & \theta_{n-1}^{(1)} & \theta_n^{(1)} \\ \vdots & & \vdots & \\ \theta_0^{(n-1)} & \cdots & \theta_{n-1}^{(n-1)} & \theta_n^{(n-1)} \\ \theta_0^{(n)} & \cdots & \theta_{n-1}^{(n)} & \theta_n^{(n)} \end{vmatrix}}{\begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} \\ \theta_0^{(1)} & \cdots & \theta_{n-1}^{(1)} \\ \vdots & & \vdots \\ \theta_0^{(n-1)} & \cdots & \theta_{n-1}^{(n-1)} \end{vmatrix}}, \quad n \in \mathbb{N}, \tag{24}$$

where $\theta_j^{(i)}$ denotes $\partial^i(\theta_j)$. In the case that $\mathcal{A}$ is not commutative, the terms in the sequence are expressed as quasideterminants [16],

$$\theta[n] = \begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} & \theta_n \\ \theta_0^{(1)} & \cdots & \theta_{n-1}^{(1)} & \theta_n^{(1)} \\ \vdots & & \vdots & \\ \theta_0^{(n-1)} & \cdots & \theta_{n-1}^{(n-1)} & \theta_n^{(n-1)} \\ \theta_0^{(n)} & \cdots & \theta_{n-1}^{(n)} & \boxed{\theta_n^{(n)}} \end{vmatrix}, \quad n \in \mathbb{N}. \tag{25}$$

The following theorem gives a generalisation of this formula to the case of general D and σ. Note in particular that the expressions do not depend on σ and are obtained simply by replacing ∂ with D. It is proved by induction.

Theorem 1. *Let $\phi[0] = \phi$ and for $n \in \mathbb{N}$ let*

$$\phi[n] = D(\phi[n-1]) - D(\theta[n-1])\theta[n-1]^{-1}\phi[n-1],$$

where $\theta[n] = \phi[n]|_{\phi \to \theta_n}$. Then, for $n \in \mathbb{N}$,

$$\phi[n] = \begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} & \phi \\ D(\theta_0) & \cdots & D(\theta_{n-1}) & D(\phi) \\ \vdots & & \vdots & \vdots \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_{n-1}) & D^{n-1}(\phi) \\ D^n(\theta_0) & \cdots & D^n(\theta_{n-1}) & \boxed{D^n(\phi)} \end{vmatrix}. \tag{26}$$

As an application of this theorem, we will apply it to the super KdV equation in which the twisted derivation is a superderivative. Before that, we will recall the definition of a superalgebra and related concepts.

5 Superalgebras and superderivatives

In this section, we collect together some basic facts about supersymmetric objects such as superderivatives, supermatrices, supertranspose and superdeterminants [27, 28] and about the relationship between superdeterminants and quasideterminants [29].

Let $\mathcal{A}$ be a supercommutative, associative, unital superalgebra over a (commutative) ring K. There is a standard $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ such that $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$. Elements of $\mathcal{A}$ that belong to either $\mathcal{A}_0$ or $\mathcal{A}_1$ are called *homogeneous*; those in $\mathcal{A}_0$ are called *even* and those in $\mathcal{A}_1$ are called *odd*. The *parity* $|a|$ of a homogeneous element a is 0 if it is even and 1 if it is odd. It follows that if a, b are homogeneous then $|ab| = |a| + |b|$. Supercommutativity means that all homogeneous elements a, b satisfy $ba = (-1)^{|a||b|}ab$, i.e. even elements commute with all elements, and odd elements anticommute. In particular, this implies that $a_1^2 = 0$, for all $a_1 \in \mathcal{A}_1$.

Grade involution and superderivative The homomorphism $\widehat{\ }\colon \mathcal{A} \to \mathcal{A}$ satisfying $\widehat{a}_i = (-1)^i a_i$ for $a_i \in \mathcal{A}_i$ is called the *grade involution.* For general $a \in \mathcal{A}$, expressed as $a = a_0 + a_1$ where $a_i \in \mathcal{A}_i$, we have $\widehat{a} = a_0 - a_1$. Also for any matrix $M = (m_{ij})$ over $\mathcal{A}$, $\widehat{M} := (\widehat{m}_{ij})$. It is easy to see that $\widehat{\widehat{a}} = a$.

A *superderivative* D is a linear mapping $D\colon \mathcal{A} \to \mathcal{A}$ such that $D(K) = 0$ and $D(\mathcal{A}_i) \subseteq \mathcal{A}_{i+1}$ and satisfying $D(ab) = D(a)b + \widehat{a}D(b)$. One way to obtain a superderivative is as $D = \partial_\theta + \theta\partial_x$ where x is an even variable and θ is an odd (Grassmann) variable. For such a superderivative $D^2 = \partial_x$.

Note that since $D(\mathcal{A}_0) \subseteq \mathcal{A}_1$ and $D(\mathcal{A}_1) \subseteq \mathcal{A}_0$, it follows that $D(\widehat{a}) = D(a_0) - D(a_1) = -\widehat{D(a)}$ and so grade involution and superderivatives anticommute.

Even and odd supermatrices A block matrix $\mathcal{M} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ over $\mathcal{A}$ where X is $r \times m$, Y is $r \times n$, Z is $s \times m$ and T is $s \times n$ for integers r, s, m and n with $r, m \geq 1$ and $s, n \geq 0$ is called an $(r|s) \times (m|n)$ *supermatrix.* It is said to be *even*, and has parity 0, if X and T (if not empty) have even entries and Y and Z (if non-empty) have odd entries. One the other hand, if X and T have odd entries and Y, Z have even entries then $\mathcal{M}$ is said to be *odd*, and has parity 1. It is said to be homogeneous if it is either even or odd.

Supertranspose The *supertranspose* of a homogeneous supermatrix $\mathcal{M}$, is defined to be

$$\mathcal{M}^{st} = \begin{pmatrix} X^t & (-1)^{|\mathcal{M}|}Z^t \\ -(-1)^{|\mathcal{M}|}Y^t & T^t \end{pmatrix}, \tag{27}$$

where ^{t} denotes the normal matrix transpose. In particular, an even $(m|n)$-row vector has the form $(a_{01}, a_{02}, \ldots, a_{0m}, a_{11}, a_{12}, \ldots, a_{1n})$, where $a_{ij} \in \mathcal{A}_i$, and its supertranspose is

$$(a_{01}, a_{02}, \ldots, a_{0m}, a_{11}, a_{12}, \ldots, a_{1n})^{st} = (a_{01}, a_{02}, \ldots, a_{0m}, -a_{11}, -a_{12}, \ldots, -a_{1n})^t. \tag{28}$$

On the other hand, an odd $(m|n)$-row vector has the form $(a_{11}, a_{12}, \ldots, a_{1m}, a_{01}, a_{02}, \ldots, a_{0n})$, and the supertranspose

$$(a_{11}, a_{12}, \ldots, a_{1m}, a_{01}, a_{02}, \ldots, a_{0n})^{st} = (a_{11}, a_{12}, \ldots, a_{1m}, a_{01}, a_{02}, \ldots, a_{0n})^t. \tag{29}$$

For homogenous supermatrices $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$, it is known that

$$(\mathcal{M}\mathcal{N})^{st} = (-1)^{|\mathcal{M}||\mathcal{N}|}\mathcal{N}^{st}\mathcal{M}^{st}, \tag{30}$$

$$(\mathcal{M}^{st})^{st} = (-1)^{|\mathcal{M}|}\widehat{\mathcal{M}}. \tag{31}$$

Supertranspose commutes with the grade involution but not with a superderivative; for a homogeneous matrix $\mathcal{M}$,

$$(\widehat{\mathcal{M}})^{st} = \widehat{(\mathcal{M})^{st}}, \quad (D(\mathcal{M}))^{st} = (-1)^{|\mathcal{M}|}D(\widehat{\mathcal{M}}^{st}). \tag{32}$$

Superdeterminants Consider an even $(m|n) \times (m|n)$ supermatrix $\mathcal{M} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ in which X and T are non-singular. The *superdeterminant*, or *Berezinian*, of $\mathcal{M}$ is defined to be

$$\mathrm{Ber}(\mathcal{M}) = \frac{\det(X - YT^{-1}Z)}{\det(T)} = \frac{\det(X)}{\det(T - ZX^{-1}Y)}.$$

It is also convenient to define

$$\mathrm{Ber}^*(\mathcal{M}) = \frac{1}{\mathrm{Ber}(\mathcal{M})}.$$

Relationship between quasideterminants and superdeterminants The basic formulae connecting quasideterminants of even supermatrices with their Berezinians are given in [29].

Theorem 2. *Let $\mathcal{M}$ be an $(m|n) \times (m|n)$-supermatrix. Then*

$$|\mathcal{M}|_{i,j} = \begin{cases} (-1)^{i+j} \dfrac{\mathrm{Ber}(\mathcal{M})}{\mathrm{Ber}(\mathcal{M}^{i,j})} & 1 \le i,j \le m, \\ (-1)^{i+j} \dfrac{\mathrm{Ber}^*(\mathcal{M})}{\mathrm{Ber}^*(\mathcal{M}^{i,j})} & m+1 \le i,j \le m+n, \end{cases} \tag{33}$$

(*cf.* (2).)

Roughly speaking, a quasideterminant with indices in one of the even blocks of $\mathcal{M}$ is given as a ratio of Berezinians. A quasideterminant with its indices in the one of the odd blocks is not well-defined.

6 The Manin-Radul super KdV equation

The Manin-Radul supersymmetric KdV (MRSKdV) system [17] is

$$\alpha_t = \tfrac{1}{4}(\alpha_{xx} + 3\alpha D(\alpha) + 6\alpha u)_x, \quad u_t = \tfrac{1}{4}(u_{xx} + 3u^2 + 3\alpha D(u))_x, \tag{34}$$

where u and α are even and odd dependent variables respectively, x, t are even independent variables and D is the superderivative defined by $D = \partial_\theta + \theta\partial_x$, where θ is a Grassmann odd variable, satisfying $D^2 = \partial_x$. This system has the Lax pair

$$L = \partial_x^2 + \alpha D + u, \tag{35}$$

$$M = \partial_x^3 + \tfrac{3}{4}\big((\alpha\partial_x + \partial_x\alpha)D + u\partial_x + \partial_x u\big), \tag{36}$$

in the sense that $L_t + [L, M] = 0$ implies (34). Eigenfunctions satisfy

$$L(\phi) = \lambda\phi, \quad \phi_t = M(\phi), \tag{37}$$

for eigenvalue λ.

6.1 Darboux transformations

A Darboux transformation for this system [21] is

$$\phi \to D(\phi) - D(\theta)\theta^{-1}\phi, \tag{38}$$

$$\alpha \to -\alpha + 2(D(\theta)\theta^{-1})_x, \tag{39}$$

$$u \to u + D(\alpha) - 2D(\theta)\theta^{-1}(\alpha - (D(\theta)\theta^{-1})_x), \tag{40}$$

where θ is an invertible, and hence necessarily even, solution of (37). Note that it is an example of the general type of Darboux transformation discussed in Section 4.1. As discussed there, this transformation may be iterated by taking solutions $\theta_0, \theta_1, \theta_2, \ldots$ of (37) to obtain

$$\phi[k+1] = D(\phi[k]) - D(\theta[k])\theta[k]^{-1}\phi[k], \tag{41}$$
$$\theta[k] = \phi[k]|_{\phi\to\theta_k}. \tag{42}$$

The requirement that each $\theta[k]$ is invertible means that it must be even and consequently that θ_i must have parity i. The corresponding solutions of MRSKdV are $\alpha[0] = \alpha$, $u[0] = u$ and

$$\alpha[k+1] = -\alpha[k] + 2(D(\theta[k])\theta[k]^{-1})_x, \tag{43}$$
$$u[k+1] = u[k] + D(\alpha[k]) - 2D(\theta[k])\theta[k]^{-1}(\alpha[k] - (D(\theta[k])\theta[k]^{-1})_x). \tag{44}$$

From Theorem 1, we have a closed-form expression (26) for $\phi[n]$ as a quasideterminant and the corresponding expressions for $\alpha[n]$ and $u[n]$ may also be found. For $i, j \geq 0$ define the quasideterminants

$$Q_n(i,j) = \begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} & 0 \\ D(\theta_0) & \cdots & D(\theta_{n-1}) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ D^{n-j-2}(\theta_0) & \cdots & D^{n-j-2}(\theta_{n-1}) & 0 \\ D^{n-j-1}(\theta_0) & \cdots & D^{n-j-1}(\theta_{n-1}) & 1 \\ D^{n-j}(\theta_0) & \cdots & D^{n-j}(\theta_{n-1}) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_{n-1}) & 0 \\ D^{n+i}(\theta_0) & \cdots & D^{n+i}(\theta_{n-1}) & \boxed{0} \end{vmatrix} \tag{45}$$

$$= -\begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} \\ \vdots & \ddots & \vdots \\ D^{n-j-2}(\theta_0) & \cdots & D^{n-j-2}(\theta_{n-1}) \\ D^{n-j}(\theta_0) & \cdots & D^{n-j}(\theta_{n-1}) \\ \vdots & \ddots & \vdots \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_{n-1}) \\ D^{n+i}(\theta_0) & \cdots & D^{n+i}(\theta_{n-1}) \end{vmatrix}_{n,s} \begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} \\ \vdots & \ddots & \vdots \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_{n-1}) \end{vmatrix}_{n-j,s}^{-1}, \tag{46}$$

for any $s = 1, \ldots, n$ (see (4)).

Theorem 3. *After n repeated Darboux transformations, the MRSKdV system has new solutions $\alpha[n]$ and $u[n]$ expressed in terms of $Q_n(0,0)$ and $Q_n(0,1)$.*

$$\alpha[n] = (-1)^n\alpha - 2Q_n(0,0)_x, \tag{47}$$
$$u[n] = u - 2Q_n(0,1)_x - 2Q_n(0,0)((-1)^n\alpha - Q_n(0,0)_x) + \frac{1-(-1)^n}{2}D(\alpha). \tag{48}$$

6.2 Binary Darboux transformations

Binary Darboux transformations for the MRSKdV system were discussed in [20,30]. In these articles, solutions expressed in terms of determinants were obtained. As discussed in connection with Darboux transformations it is to be expected that solutions for this supersymmetric system should be *superdeterminants* in general. In this section, we will construct a more general type of binary Darboux transformation which will be shown to give these superdeterminants solutions and includes the solutions found in [20, 30] as a special case.

First we recall the definition of the adjoint for supersymmetric linear operators. For a linear operator P, $|P|$ denotes its parity. For example, $|D| = 1$ and $|\partial| = 0$, where ∂ denotes any derivative with respect to an even variable, and the parity of multiplication by a homogeneous element is the parity of that element (in the usual sense). The rules defining the superadjoint are

$$D^\dagger = -D, \quad \partial^\dagger = -\partial, \quad \mathcal{M}^\dagger = \mathcal{M}^{st}, \tag{49}$$

where $\mathcal{M}$ denotes any matrix over $\mathcal{A}$, together with the product rule

$$(PQ)^\dagger = (-1)^{|P||Q|}Q^\dagger P^\dagger, \tag{50}$$

where P and Q are operators (cf. (30) for the case of matrices). In particular, this gives $(D^n)^\dagger = (-1)^{n(n+1)/2}D^n$ and, consistently, $(\partial^n)^\dagger = (-1)^n\partial^n$. For any $a \in \mathcal{A}$, $a^\dagger = a$.

The Lax pair (35), (36) has the adjoint form

$$L^\dagger = \partial_x^2 + D\alpha + u, \tag{51}$$

$$M^\dagger = -\partial_x^3 - \tfrac{3}{4}\big(D(\alpha\partial_x + \partial_x\alpha) + u\partial_x + \partial_x u\big), \tag{52}$$

and adjoint eigenfunctions satisfy

$$L^\dagger(\psi) = \xi\psi, \quad -\psi_t = M^\dagger(\psi), \tag{53}$$

for eigenvalue ξ. Given an (eigenfunction, adjoint) eigenfunction pair (θ, ρ), the binary Darboux transformation [20, 30] is given by

$$\phi \to \phi - \theta\Omega(\theta,\rho)^{-1}\Omega(\phi,\rho), \tag{54}$$

$$\psi \to \psi - \rho\Omega(\theta,\rho)^{-1}\Omega(\theta,\psi), \tag{55}$$

$$\alpha \to \alpha + 2(\theta\Omega(\theta,\rho)^{-1}\widehat{\rho})_x \tag{56}$$

$$u \to u - 2(\alpha + (\theta\Omega(\theta,\rho)^{-1}\widehat{\rho})_x)\theta\Omega(\theta,\rho)^{-1}\widehat{\rho} + 2(\theta\Omega(\theta,\rho)^{-1}D(\rho))_x, \tag{57}$$

where eigenfunction θ and adjoint eigenfunction ρ have opposite parities. Since $D\big(\Omega(\phi,\psi) = \psi\phi$, Ω is even and assumed to be invertible. When iterating this transformation, both previous papers [20, 30] on this topic considered the case that all eigenfunctions are even and all adjoint eigenfunctions are odd. We will show that this is not the most general possibility however.

Consider an even $(m|n)$-row vector eigenfunction $\mathcal{E} = (\theta_0, \ldots \theta_{m+n-1})$ and an odd $(m|n)$-row vector adjoint eigenfunction $\mathcal{O} = (\rho_0, \ldots, \rho_{m+n-1})$, where θ_i for $i = 0, \ldots, m-1$ and ρ_{m+j} for $j = 0, \ldots, n-1$ are even and ρ_i for $i = 0, \ldots, m-1$ and θ_{m+j} for $j = 0, \ldots, n-1$ are odd. These row vectors satisfy

$$L(\mathcal{E}) = \mathcal{E}\Lambda, \quad \mathcal{E}_t = M(\mathcal{E}), \tag{58}$$

$$L^\dagger(\mathcal{O}) = \mathcal{O}\Xi, \quad -\mathcal{O}_t = M^\dagger(\mathcal{O}), \tag{59}$$

where Λ and Ξ are constant $(m+n) \times (m+n)$ diagonal matrices containing the eigenvalues. Then $\Omega = \Omega(\mathcal{E}, \mathcal{O})$ is an even $(m|n) \times (m|n)$-supermatrix defined up to a constant by

$$D(\Omega) = \mathcal{O}^\dagger\mathcal{E}, \qquad \Omega\Lambda - \Xi\Omega = D(\mathcal{O}^\dagger\mathcal{E}_x - \mathcal{O}_x^\dagger\mathcal{E}) - \widehat{\mathcal{O}}^\dagger\alpha\mathcal{E}, \tag{60}$$

$$\Omega_t = D(\mathcal{O}_{xx}^\dagger\mathcal{E} - \mathcal{O}_x^\dagger\mathcal{E}_x + \mathcal{O}^\dagger\mathcal{E}_{xx}) + \tfrac{3}{2}\widehat{\mathcal{O}}_x^\dagger\alpha\mathcal{E} + \tfrac{3}{4}\widehat{\mathcal{O}}^\dagger\alpha_x\mathcal{E} + \tfrac{3}{2}D(\mathcal{O}^\dagger\alpha D(\mathcal{E})) + \tfrac{3}{2}D(\mathcal{O}^\dagger u\mathcal{E}). \tag{61}$$

The closed form expressions for the results of iterated binary Darboux transformations are stated in the following theorems.

Theorem 4. *Iterating the binary Darboux transformations* (54), (55) *for* $m+n \geq 1$, *one obtains*

$$\phi[m+n] = \begin{vmatrix} \Omega(\mathcal{E},\mathcal{O}) & \Omega(\phi,\mathcal{O}) \\ \mathcal{E} & \boxed{\phi} \end{vmatrix}, \quad \psi[m+n] = \begin{vmatrix} \Omega(\mathcal{E},\mathcal{O})^\dagger & \Omega(\mathcal{E},\psi)^\dagger \\ \mathcal{O} & \boxed{\psi} \end{vmatrix} = \begin{vmatrix} \widehat{\Omega(\mathcal{E},\mathcal{O})} & \mathcal{O}^\dagger \\ \widehat{\Omega(\mathcal{E},\psi)} & \boxed{\psi} \end{vmatrix} \tag{62}$$

with

$$\Omega(\phi[m+n],\psi[m+n]) = \begin{vmatrix} \Omega(\mathcal{E},\mathcal{O}) & \Omega(\phi,\mathcal{O}) \\ \Omega(\mathcal{E},\psi) & \boxed{\Omega(\phi,\psi)} \end{vmatrix}. \tag{63}$$

Theorem 5. *Let* (α,u) *be a solution of MRSKdV and let* $\mathcal{E}$ *and* $\mathcal{O}$ *respectively be even and odd* $(m|n)$*-row vectors satisfying* (58) *and* (59). *Then for any integers* $m+n \geq 0$

$$\alpha[m+n] = \alpha - 2A[m+n]_x, \quad u[m+n] = u + 2(\alpha - A[m+n]_x)A[m+n] - 2U[m+n]_x, \tag{64}$$

where

$$A[m+n] = \begin{vmatrix} \Omega(\mathcal{E},\mathcal{O}) & \widehat{\mathcal{O}}^\dagger \\ \mathcal{E} & \boxed{0} \end{vmatrix}, \quad U[m+n] = \begin{vmatrix} \Omega(\mathcal{E},\mathcal{O}) & D(\mathcal{O}^\dagger) \\ \mathcal{E} & \boxed{0} \end{vmatrix}, \tag{65}$$

are also solutions of MRSKdV.

6.3 From quasideterminants to superdeterminants

It is usual in a supersymmetric integrable system for the solutions to be expressable in terms of superdeterminants. Indeed, in this section we will show that this can be done here. The expressions we will obtain coincide with the superdeterminant solutions found in [21] and we also find the superdeterminant expressions in the case that they did not.

Let us therefore introduce the relabeling of the eigenfunctions used in the Darboux transformations

$$\theta_{2k} = E_k, \quad \theta_{2k+1} = O_k. \tag{66}$$

Recall that θ_i has parity i so that all E_k are even and all O_k are odd. Also, we write $D^{2j}(\theta) = \theta^{(j)}$ and $D^{2j+1}(\theta) = D(\theta^{(j)})$, where ${}^{(j)}$ denotes the jth derivative with respect to x.

Consider the matrix

$$W_n = \begin{bmatrix} \theta_0 & \cdots & \theta_{n-1} \\ \vdots & \ddots & \vdots \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_{n-1}) \end{bmatrix}, \tag{67}$$

appearing in the definition (46) of $Q_n(i,j)$. There is a natural reordering of the rows and columns

$$W_n \to \mathcal{W}_n = \begin{bmatrix} X_n & Y_n \\ Z_n & T_n \end{bmatrix}, \tag{68}$$

which gives an even matrix $\mathcal{W}_n$. This reordering does not change the value of any associated quasideterminant, as long as the expansion point in each refers to the same element. In the case that n is even,

$$X_{2k} = \begin{bmatrix} E_0 & \cdots & E_{k-1} \\ \vdots & \ddots & \vdots \\ E_0^{(k-1)} & \cdots & E_{k-1}^{(k-1)} \end{bmatrix}, \quad Y_{2k} = \begin{bmatrix} O_0 & \cdots & O_{k-1} \\ \vdots & \ddots & \vdots \\ O_0^{(k-1)} & \cdots & O_{k-1}^{(k-1)} \end{bmatrix}$$

and $Z_{2k} = D(X_{2k})$ and $T_{2k} = D(Y_{2k})$ are all $k \times k$ matrices. In the case that n is odd, X_{2k+1} is $(k+1) \times (k+1)$, Y_{2k+1} is $(k+1) \times k$, Z_{2k+1} is $k \times (k+1)$ and T_{2k+1} is a $k \times k$ matrix whose precise form can be easily deduced from the above description.

Similarly, consider the matrix

$$W'_n = \begin{bmatrix} \theta_0 & \cdots & \theta_{n-1} \\ \vdots & \ddots & \vdots \\ D^{n-3}(\theta_0) & \cdots & D^{n-3}(\theta_{n-1}) \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_{n-1}) \\ D^n(\theta_0) & \cdots & D^n(\theta_{n-1}) \end{bmatrix}, \tag{69}$$

appearing in the definition (46) of $Q_n(0,1)$. A similar reordering of this matrix

$$W'_n \to \mathcal{W}'_n = \begin{bmatrix} X'_n & Y'_n \\ Z'_n & T'_n \end{bmatrix}, \tag{70}$$

gives another even matrix $\mathcal{W}'_n$ where, for example,

$$X'_{2k} = \begin{bmatrix} E_0 & \cdots & E_{k-1} \\ \vdots & \ddots & \vdots \\ E_0^{(k-2)} & \cdots & E_{k-1}^{(k-2)} \\ E_0^{(k)} & \cdots & E_{k-1}^{(k)} \end{bmatrix}.$$

The solutions obtained by use of Darboux transformations (47)–(48) are expressed in terms of two particular quasideterminants $Q_n(0,0)$ and $Q_n(0,1)$, The following theorem given the superdeterminant expressions for these.

Theorem 6. *For $n \in \mathbb{N}$,*

$$Q_n(0,0) = D(\log(\mathrm{B}(\mathcal{W}_n))), \quad Q_n(0,1) = -\frac{\mathrm{B}(\mathcal{W}'_n)}{\mathrm{B}(\mathcal{W}_n)}, \tag{71}$$

where $\mathrm{B} = \mathrm{Ber}$ *if n is even, and* $\mathrm{B} = \mathrm{Ber}^*$ *if n is odd.*

Next we will show how the quasideterminant solutions $(A[m+n], U[m+n])$ obtained using binary Darboux transformations can also be expressed in terms of superdeterminants. To do this, it is necessary to introduce a more detailed notation for row vector eigenfunctions and adjoint eigenfunctions. Recall that for the general transformation we use $(m|n)$-row vectors $\mathcal{E}$ and $\mathcal{O}$ which are even and odd with entries θ_i and ρ_i respectively. Here we will also write $\mathcal{E}^i = (\theta_0, \ldots, \theta_{i-1})$ and $\mathcal{O}^i = (\rho_0, \ldots, \rho_{i-1})$ for the row vectors containing the first i entries of $\mathcal{E}$ and $\mathcal{O}$ respectively, and denote by subscript 0 and 1 the even and odd element parts of $\mathcal{E}$ and $\mathcal{O}$ respectively. Thus $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1)$ and $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_0)$.

Theorem 7. *The expressions $(A[m+n], U[m+n])$ can expressed as*

$$A[m+n] = D\left(\log \mathrm{Ber}(\mathcal{G}_{(m|n)})\right), \quad U[m+n] = \frac{\mathrm{Ber}(\mathcal{G}'_{(m+1|n)})}{\mathrm{Ber}(\mathcal{G}_{(m|n)})}, \tag{72}$$

where

$$\mathcal{G}_{(m|n)} = \left(\begin{array}{c:c} \Omega(\mathcal{E}_0, \mathcal{O}_1) & \Omega(\mathcal{E}_1, \mathcal{O}_1) \\ \hdashline \Omega(\mathcal{E}_0, \mathcal{O}_0) & \Omega(\mathcal{E}_1, \mathcal{O}_0) \end{array}\right),$$

in an even $(m|n) \times (m|n)$-supermatrix and

$$\mathcal{G}'_{(m+1|n)} \left(\begin{array}{cc:c} \Omega(\mathcal{E}_0, \mathcal{O}_1) & D(\mathcal{O}_1^{\dagger}) & \Omega(\mathcal{E}_1, \mathcal{O}_1) \\ \hdashline \mathcal{E}_0 & 0 & \mathcal{E}_1 \\ \hdashline \Omega(\mathcal{E}_0, \mathcal{O}_0) & D(\mathcal{O}_0^{\dagger}) & \Omega(\mathcal{E}_1, \mathcal{O}_0) \end{array}\right),$$

in an even $(m+1|n) \times (m+1|n)$-supermatrix.

Remark 1. The earlier papers on this topic [20,30] deal with the case $n = 0$ only. In this case, $\mathcal{E} = \mathcal{E}_0$ and $\mathcal{O} = \mathcal{O}_1$ and the solutions can be expressed in terms of determinants rather than the more general superdeterminants

$$A[m] = D(\log\det\Omega(\mathcal{E}_0, \mathcal{O}_1)),$$

and

$$U[m] = \frac{\det\begin{pmatrix} \Omega(\mathcal{E}_0, \mathcal{O}_1) & D(\mathcal{O}_1^{t}) \\ \mathcal{E}_0 & 0 \end{pmatrix}}{\det(\Omega(\mathcal{E}_0, \mathcal{O}_1))}.$$

7 Conclusions

In this paper, we considered a twisted derivation which includes normal derivative, forward difference operator, q-difference operator and superderivatives as special cases. Darboux transformations defined in terms of twisted derivations have an quasideterminant iteration formula very similar to the known one for the untwisted case. This result gives a framework for a unified approach to Darboux transformations for differential, superdifferential, difference and q-difference operators. As an example we showed how this is achieved for superderivatives in the Manin-Radul super KdV equation.

Acknowledgement

Much of the work presented in this paper was carried out during the authors' visit to the Newton Institute for Mathematical Sciences as part of the Programme on Discrete Integrable Systems (January-July 2009). The authors are grateful to the organizers for their invitation and to the Isaac Newton Institute for financial support and hospitality. This work was supported by the National Natural Science Foundation of China (grant No. 10601028) and the Project-sponsored by SRF for ROCS, SEM.

References

[1] Kupershmidt, B. A. KP or mKP, vol. 78 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI (2000). Noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems.

[2] Paniak, L. D. Exact Noncommutative KP and KdV Multi-solitons (2001). hep-th/0105185.

[3] Sakakibara, M. *J. Phys. A* **37** (2004) L599–L604.

[4] Wang, N. and Wadati, M. *J. Phys. Soc. Japan* **72** (2003) 1366–1373.

[5] Wang, N. and Wadati, M. *J. Phys. Soc. Japan* **72** (2003) 1881–1888.

[6] Wang, N. and Wadati, M. J. Phys. Soc. Japan **73** (2004) 1689–1698.

[7] Hamanaka, M. Noncommutative solitons and D-branes. Ph.D. thesis (2003). hep-th/0303256.

[8] Hamanaka, M. and Toda, K. *Phys. Lett. A* **316** (2003) 77–83.

[9] Dimakis, A. and Müller-Hoissen, F. *J. Phys. A* **38** (2005) 5453–5505.

[10] Nimmo, J. J. C. *J. Phys. A* **39** (2006) 5053–5065.

[11] Li, C. X., Nimmo, J. J. C. and Tamizhmani, K. M. *Proc. R. Soc. A* **465** (2009) 1441–1451.

[12] Gilson C.R. and Nimmo J.J.C. *J. Phys. A: Math. Theor.* **40**(2007) 3839–3850.

[13] Gilson, C. R., Nimmo, J. J. C. and Ohta, Y. *J. Phys. A: Math. Theor.* **40** (2007) 12607–12617.

[14] Li, C. X. and Nimmo, J. J. C., *Proc. R. Soc. A* **464**(2008) 951–966.

[15] Gelfand I.M. and Retakh V.S., Determinants of matrices over noncommutative rings. *Funct. Anal. Appl.* **25**(1991) 13–25.

[16] Etingof P., Gelfand I. and Retakh V., *Math. Res. Lett.* **4** (1997) 413–425.

[17] Manin Y.I. and Radul A.O., *Comm. Math. Phys.* **98** (1985) 65–77.

[18] Hartwig J. T., Larsson D. and Silvestrov S. D., *J. Algebra* **295** (2006) 314–361.

[19] De Concini S. and Procesi C., *Quantum Groups*, Lecture Notes in Mathematics **1565** (1993) 31–140.

[20] Liu Q.P. and Mañas M., *Phys. Lett. B* **394** (1997) 337–342.

[21] Liu Q.P. and Mañas M., *Phys. Lett. B* **396** (1997) 133–140.

[22] Li C.X. and Nimmo J.J.C., arXiv: solv-int/0911.1413.

[23] Gelfand I., Gelfand S., Retakh V. and Wilson R.L., *Adv. in Math.* **193** (2005) 56–141.

[24] Matveev, V.B. and Salle, M.A. *Darboux transformations and solitons.* Springer-Verlag, Berlin, (1991).

[25] Oevel, W. and Schief, W. "Darboux theorems and the KP hierarchy" in *Applications of analytic and geometric methods to nonlinear differential equations* (Exeter, 1992), 193–206, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 413, Kluwer Acad. Publ., Dordrecht, (1993).

[26] Crum M.M., *Q, J. Math.* **6** (1955) 121–127.

[27] Berezin F.A., Introduction to superanalysis (D. Reidel Publishing Company, Dordrecht, 1987).

[28] DeWitt B., Supermanifolds (Cambridge University Press, 1984).

[29] Bergvelt M.J. and Rabin J.M., Super curves, their Jacobians and super KP equations. arXiv: alg-geom/9601012v1.

[30] Shaw J.C. and Tu M.H., *J. Math. Phys.* **39** (1998) 4773–4784.