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On the fluid balancer

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Abstract

The paper is concerned with the dynamics of the so-called fluid balancer; a hula hoop ring-like structure containing a small amount of liquid which, during rotation, is spun out to form a thin liquid layer on the inner surface of the ring. The liquid is able to counteract unbalanced mass in an elastically mounted rotor. The paper gives the equations of motion for the coupled fluid-structure system and derives the fluid forces to a first approximation.

1 Introduction

A fluid balancer is used on rotating machinery to eliminate the undesirable effects of unbalanced mass. It has become a standard feature on most household washing machines, but is also used on industrial rotating machinery. Taking the washing machine fluid balancer as example, it consists of a hollow ring, like a hula hoop ring but typically with rectangular cross sections, which contains a small amount of liquid. When the ring is rotating at a high angular velocity Ω the liquid will form a thin liquid layer on the inner surface of the outermost wall, as sketched in Fig. 1. Consider the situation where an unbalanced mass m is present; for example the clothes in a washing machine. The rotor has a critical angular velocity ω_{cr} where the centripetal forces are in balance with the forces due to the restoring springs. Below this velocity ($\Omega < \omega_{cr}$) the mass center of the fluid will be located 'on the same side' as the unbalanced mass, as shown in the left part of Fig. 1. At a certain supercritical angular velocity ($\Omega > \omega_{cr}$) the mass center of the liquid will however move to the 'opposite side' of the unbalanced mass, as shown in the right part of Fig. 1, resulting in 'mass balance' and thus in a reduced oscillation amplitude of the rotor.

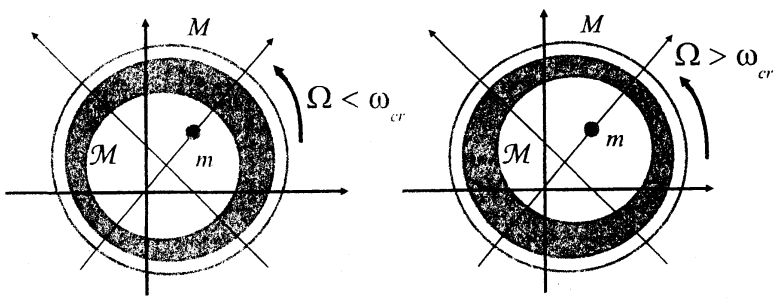

Figure 1: Sketch of the working principle of the fluid balancer.

This is the working principle of the fluid balancer. It has been verified experimentally; but no analytical explanation has been given yet. It is the aim of the present project to do this. The present paper gives the mathematical formulation of the problem; but no definite conclusions regarding the 'fundamental mechanism' of the fluid balancer can be given as yet.

The fluid balancer is, basically, an application of 'the rotor partially filled with liquid', a problem which has been intensively studied. The motivations for these studies have come from a variety of practical problems, such as the design of centrifuges, turbines, rockets, and spacecraft.

The first theoretical study appears to be due to Schmidt [20]. He discovered a very interesting phenomenon, namely that the critical speed of a partially filled rotor (with fluid just sufficient to cover the wall) is identical with that of a completely filled rotor.

Philips [15] and Miles & Troesch [13] investigated the dynamics of a rotating liquid layer from a purely fluid dynamics point-of-view, ignoring the dynamics and stability of the rotor.

Experiments and comprehensive analytical studies of the fluid-structure interaction problem were carried out by Kollmann [12] and Wolf [21]. Both assumed the fluid to be inviscid in their models. Ehrich [4] considered the case where the rotor is only partly covered with liquid. Hendricks & Morton [5] included fluid viscosity via a boundary layer model.

Saito and co-workers [16, 17, 18, 19] uncovered many interesting phenomena via a comprehensive series of experiments, and contributed to the mathematical modeling of the problem as well.

Kaneko & Hayama [8, 9] considered the flow structure in the longitudinal direction of the fluid vessel, also both experimentally and theoretically.

Brommundt & Ostermeyer [2] studied the three-dimensional dynamics of an anisotropically mounted rotor with a fluid container attached in an overhung configuration.

Holm-Christensen & Träger [6] performed a stability analysis via a linearization of the full Navier-Stokes equations; this was the first study dealing with a viscous liquid that did not rely on a boundary layer approximation. This study considered also some interesting effects related to structural damping and fluid viscosity: (1) Attaching a dashpot damper to the (flexible) rotor implies that the equilibrium position of the rotor will be unstable at any angular velocity if the liquid is inviscid. (2) Inclusion of viscosity of the liquid re-stabilizes the equilibrium position.

These phenomena were analyzed in detail by Zhang et al. [26]. The main result of their analysis is that 'external damping' (a dashpot) always is a destabilizing factor, while 'internal damping' (viscous dissipation in the fluid) can be a stabilizing factor, but not always.

Yasuo et al. [22, 23] investigated the effects of flow-limiting boards, both experimentally and theoretically, considering a variety of configurations.

Derendyaev et al. [3] presented an interesting 'analogous model' consisting of a ring (as a simplified model of the liquid) sliding over a disk (as a model of the liquid vessel).

Finite amplitude wave phenomena, such as hydraulic jumps, undular bores, solitary waves (solitons), and sloshing, have been considered by Berman et al. [1], Jinnouchi et al. [7], Kasahara et al. [10, 11], and Yoshizumi [24, 25]. These studies typically rely on approximations from the theory of shallow water waves, perturbation methods, and numerical computations via, e.g., finite difference approximations.

2 Equations of motion

2.1 Rotor equations

Consider a rotating vessel of mass M equipped with a small unbalance mass m located a distance s from the geometric center, and containing a small amount of liquid, as sketched in Fig. 2. The inner radius of the vessel is a; the radius vector to the fluid surface is $R(t, \theta)$, while the radius of the undisturbed (mean) fluid surface is b. The rotor is supported by springs, with spring constant K, in the X and Y directions. The structural damping forces in these directions are proportional to the parameter C. Let the coordinate system $(\hat{x}, \hat{y})$ rotate with the constant

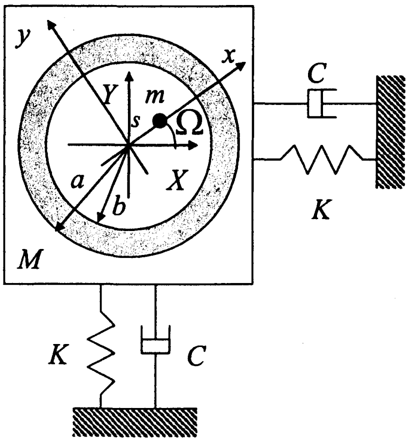

Figure 2: Definition of the coordinates and symbols used.

angular velocity Ω about the fixed system (X, Y). In terms of the rotating coordinate system the equation of motion of the rotor is given by

$$\begin{bmatrix} M+m & 0 \\ 0 & M+m \end{bmatrix} \frac{d^2}{dt^2} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{bmatrix} C & -2(M+m)\Omega \\ 2(M+m)\Omega & C \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{1}$$
$$+ \begin{bmatrix} K-(M+m)\Omega^2 & -C\Omega \\ C\Omega & K-(M+m)\Omega^2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} ms\Omega^2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}.$$

Here x and y denote the deflections of the rotor in the $(\hat{x}, \hat{y})$ system. F_x and F_y are the fluid force components acting on the rotor.

2.2 Fluid equations

The fluid motion in the rotating vessel is described by a shallow water approximation. In terms of cylindrical coordinates (r, θ), rotating with angular velocity Ω as the $(\hat{x}, \hat{y})$ system, the equations of motion are

$$\frac{\partial u}{\partial t} - a\Omega^2 - 2v\Omega = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\partial^2 x}{\partial t^2}\cos(\Omega t + \theta) - \frac{\partial^2 y}{\partial t^2}\sin(\Omega t + \theta), \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} + 2u\Omega = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 x}{\partial t^2}\sin(\Omega t + \theta) - \frac{\partial^2 y}{\partial t^2}\cos(\Omega t + \theta). \tag{3}$$

Here u and v are the fluid velocity components in the radial (r) and circumferential (θ) directions, p is the fluid pressure, ρ is the fluid density, and ν the kinematic viscosity of the fluid.

The continuity equation is

$$\frac{\partial}{\partial r}(ur) + \frac{\partial v}{\partial \theta} = 0. \tag{4}$$

The boundary conditions are

$$\begin{aligned} u(a) &= 0, \quad v(a) = 0, \\ u(R) &= u_R = \left(\frac{\partial R}{\partial t} + \frac{v}{r}\frac{\partial R}{\partial \theta}\right)_{r=R}, \quad p(R) = 0, \end{aligned} \tag{5}$$

where, again, $R(t, \theta)$ specifies the free surface of the fluid layer.

In the shallow water approximation it is assumed that

$$u(r) = \frac{r-a}{R-a}u_R, \quad \text{with} \quad u_R \approx \frac{\partial R}{\partial t}. \tag{6}$$

It will be furthermore assumed that

$$\frac{\partial u}{\partial t} \approx \frac{r-a}{R-a}\frac{\partial^2 R}{\partial t^2} \tag{7}$$

Applying an 'averaging operator' on the form $(a-R)\int_R^a \cdots dr$ to (3), and defining the mean circumferential flow velocity as

$$\bar{v} = \frac{1}{a-R}\int_R^a v(r)dr, \tag{8}$$

we get

$$\begin{aligned}&\frac{\partial \bar{v}}{\partial t} + \frac{a\ln(a/b)}{(a-b)^2}\bar{v}\frac{\partial R}{\partial t} + \bar{v}\frac{\partial \bar{v}}{a\partial\theta} - \frac{a-b}{3}\frac{\partial^3 R}{a\partial\theta\partial t^2} - \Omega^2\frac{\partial R}{\partial\theta} + 2\Omega\frac{\partial R}{\partial t} + \frac{\eta}{4}\frac{1}{a-b}\bar{v}^2 = \\ &\frac{\partial^2 x}{\partial t^2}\left\{\sin(\Omega t+\theta) - \frac{\partial R}{a\partial\theta}\cos(\Omega t+\theta)\right\} - \frac{\partial^2 y}{\partial t^2}\left\{\cos(\Omega t+\theta) + \frac{\partial R}{a\partial\theta}\sin(\Omega t+\theta)\right\}.\end{aligned} \tag{9}$$

Here the boundary term $-\nu\partial v/\partial r|_{r=a}$ has been approximated by $\frac{1}{4}\eta\bar{v}^2$, where η is a friction factor. [This is similar to the fluid friction model used in pipe flow.]

Finally, integration of (2) gives

$$p(a) = \rho(a-b)\left[-\frac{1}{2}\frac{\partial^2 R}{\partial t^2} + a\Omega^2 + 2\Omega\bar{v} - \frac{a-R}{a-b}\left\{\frac{\partial^2 x}{\partial t^2}\cos(\Omega t+\theta) + \frac{\partial^2 y}{\partial t^2}\sin(\Omega t+\theta)\right\}\right]. \tag{10}$$

2.3 Fluid forces

The fluid force components on the right hand side of (1) can be split up into pressure- and friction-related parts, indicated by subscripts p and f respectively, as follows:

$$F_{xp} = aL\int_0^{2\pi} p(a)\cos\theta d\theta, \quad F_{yp} = aL\int_0^{2\pi} p(a)\sin\theta d\theta, \tag{11}$$

$$F_{xf} = aL\int_0^{2\pi} \rho\frac{\eta}{4}\bar{v}^2\sin\theta d\theta, \quad F_{yf} = -aL\int_0^{2\pi} \rho\frac{\eta}{4}\bar{v}^2\cos\theta d\theta. \tag{12}$$

3 Non-dimensional equations

Introducing the non-dimensional parameters

$$\begin{aligned}\mu &= \frac{m}{M}, \quad x_* = \frac{x}{a}, \quad y_* = \frac{y}{a}, \quad \sigma = \frac{s}{a}, \quad \zeta = \frac{C}{M\omega_*}, \\ \omega_* &= \sqrt{\frac{K}{M}}, \quad \tau = \omega_* t, \quad v_* = \frac{\bar{v}}{a\omega_*}, \quad R_* = \frac{R}{a}, \\ \Omega_* &= \frac{\Omega}{\omega_*}, \quad \Lambda = \frac{a^2\ln(a/b)}{(a-b)^2}, \quad \eta_* = \frac{\eta a^2}{4(a-b)^2},\end{aligned} \tag{13}$$

we obtain the non-dimensional rotor equation

$$\begin{aligned}&\begin{bmatrix} 1+\mu & 0 \\ 0 & 1+\mu \end{bmatrix}\frac{\partial^2}{\partial\tau^2}\begin{Bmatrix} x_* \\ y_* \end{Bmatrix} + \begin{bmatrix} \zeta & -2(1+\mu)\Omega_* \\ 2(M+m) & \zeta \end{bmatrix}\frac{\partial}{\partial\tau}\begin{Bmatrix} x_* \\ y_* \end{Bmatrix} \\ &+ \begin{bmatrix} 1-(1+\mu)\Omega_*^2 & -\zeta\Omega_* \\ \zeta\Omega_* & 1-(1+\mu)\Omega_*^2 \end{bmatrix}\begin{Bmatrix} x_* \\ y_* \end{Bmatrix} = \begin{Bmatrix} \mu\sigma\Omega_*^2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} F_{*x} \\ F_{*y} \end{Bmatrix},\end{aligned} \tag{14}$$

and the non-dimensional fluid equation

$$\begin{aligned}&\frac{\partial v_*}{\partial\tau} + \Lambda v_*\frac{\partial R_*}{\partial\tau} + v_*\frac{\partial v_*}{\partial\theta} - \epsilon\frac{1}{3}\frac{\partial^3 R_*}{\partial\theta\partial\tau^2} - \Omega_*^2\frac{\partial R_*}{\partial\theta} + 2\Omega_*\frac{\partial R_*}{\partial\tau} + \eta_* v_*^2 = \\ &\frac{\partial^2 x_*}{\partial\tau^2}\left\{\sin(\Omega_*\tau+\theta) - \frac{\partial R}{\partial\theta}\cos(\Omega\tau+\theta)\right\} - \frac{\partial^2 y_*}{\partial\tau^2}\left\{\cos(\Omega_*\tau+\theta) + \frac{\partial R_*}{\partial\theta}\sin(\Omega_*\tau+\theta)\right\}.\end{aligned} \tag{15}$$

Here

$$\epsilon = \frac{a-b}{a} \tag{16}$$

will be used in the perturbation analysis to follow as a small parameter.

The non-dimensional pressure equation takes the form

$$\begin{aligned} p_*(1) &= (1-R_*)\left\{\Omega_*^2 - \frac{\partial^2 x_*}{\partial\tau^2}\cos(\Omega_*\tau+\theta) - \frac{\partial^2 y_*}{\partial\tau^2}\sin(\Omega_*\tau+\theta)\right\} \\ &+ \epsilon\left\{2\Omega_* v_* - \frac{1}{2}\frac{\partial^2 R_*}{\partial\tau^2}\right\}. \end{aligned} \tag{17}$$

In the following the subscript asterisks will be dropped. Differentiating (15) with respect to time τ, and using the relation

$$\frac{\partial R}{\partial\tau} = \beta\frac{\partial v}{\partial\theta}, \tag{18}$$

where $\beta = (a-b)/b$, we get

$$\begin{aligned} &\frac{\partial^2 v}{\partial\tau^2} + 2\Omega\beta\frac{\partial^2 v}{\partial\tau\partial\theta} - \Omega^2\beta\frac{\partial^2 v}{\partial\theta^2} - \epsilon\beta\frac{1}{3}\frac{\partial^4 v}{\partial\tau^2\partial\theta^2} + \Gamma\left(\frac{\partial v}{\partial\tau}\frac{\partial v}{\partial\theta} + v\frac{\partial^2 v}{\partial\tau\partial\theta}\right) + 2\eta v\frac{\partial v}{\partial\tau} = \\ &\frac{1}{2}e^{i(\omega\tau+\theta)}\left[-\frac{\partial R}{\partial\theta}\left(\frac{d^3x}{d\tau^3} + \Omega\frac{d^2y}{d\tau^2}\right) - \frac{\partial^2 R}{\partial\tau\partial\theta}\frac{d^2x}{d\tau^2} - \frac{d^3y}{d\tau^3} + \Omega\frac{d^2x}{d\tau^2}\right. \\ &\left.+i\left\{\frac{\partial R}{\partial\theta}\left(\frac{d^3y}{d\tau^3} - \Omega\frac{d^2x}{d\tau^2}\right) + \frac{\partial^2 R}{\partial\tau\partial\theta}\frac{d^2y}{d\tau^2} - \frac{d^3x}{d\tau^3} - \Omega\frac{d^2y}{d\tau^2}\right\}\right] + c.c. \end{aligned} \tag{19}$$

where *c.c.* indicates the complex conjugate of the preceding terms on the right hand side.

4 Perturbation analysis

Following the method of multiple scales [14], we consider expansion of the variables as

$$\begin{aligned} v &= 0 + \epsilon v_1(t_0, t_1, \cdots) + \epsilon^2 v_2(t_0, t_1, \cdots) + \cdots, \\ R &= \mathfrak{b} + \epsilon R_1(t_0, t_1, \cdots) + \epsilon^2 R_2(t_0, t_1, \cdots) + \cdots, \\ p &= p_0 + \epsilon p_1(t_0, t_1, \cdots) + \epsilon^2 p_2(t_0, t_1, \cdots) + \cdots, \end{aligned} \tag{20}$$

where $\mathfrak{b} = b/a$. The time parameters are defined as

$$t_0 = \tau, \qquad t_1 = \epsilon\tau, \qquad t_2 = \epsilon^2\tau, \qquad \cdots \tag{21}$$

The time derivatives then take the forms

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \cdots, \qquad \frac{\partial^2}{\partial\tau^2} = \frac{\partial^2}{\partial t_0^2} + 2\epsilon\frac{\partial^2}{\partial t_0\partial t_1} + \cdots \tag{22}$$

The rotor displacement vector is assumed to have the form $\{x(\tau), y(\tau)\} = \{x(t_0), y(t_0)\} = \{x_0, y_0\}e^{\lambda t_0}$, with $\lambda = \alpha + i\omega$, and are assumed to act at the ϵ-order. The problem of order ϵ is thus

$$\begin{aligned} &\frac{\partial^2 v_1}{\partial t_0^2} + 2\Omega\beta\frac{\partial^2 v_1}{\partial t_0\partial\theta} - \Gamma^2\beta\frac{\partial^2 v_1}{\partial\theta^2} = \\ &\frac{1}{2}e^{i(\omega\tau+\theta)}\left[\Omega\frac{d^2x}{dt_0^2} - \frac{d^3y}{dt_0^3} - i\left\{\Omega\frac{d^2y}{dt_0^2} + \frac{d^3x}{dt_0^3}\right\}\right] + c.c. \end{aligned} \tag{23}$$

The problem of order ϵ^2 is

$$
\begin{aligned}
&\frac{\partial^2 v_2}{\partial t_0^2} + 2\Omega\beta\frac{\partial^2 v_2}{\partial t_0 \partial\theta} - \Omega^2\beta\frac{\partial^2 v_2}{\partial\theta^2} = \\
&-2\frac{\partial^2 v_1}{\partial t_0 \partial t_1} - 2\Omega\beta\frac{\partial^2 v_1}{\partial t_1 \partial\theta} + \beta\frac{1}{3}\frac{\partial^4 v_1}{\partial t_0^2 \partial\theta^2} - \Gamma\left(\frac{\partial v_1}{\partial t_0}\frac{\partial v_1}{\partial\theta} + v_1\frac{\partial^2 v_1}{\partial t_0 \partial\theta}\right) - 2\eta v_1\frac{\partial v_1}{\partial t_0} \\
&+\frac{1}{2}\mathrm{e}^{\mathrm{i}(\omega\tau+\theta)}\left[-\frac{\partial R_1}{\partial\theta}\left(\frac{d^3x}{dt_0^3} + \Omega\frac{d^2y}{dt_0^2}\right) - \frac{\partial^2 R_1}{\partial t_0 \partial\theta}\frac{d^2x}{dt_0^2}\right. \\
&\left.+\mathrm{i}\left\{\frac{\partial R_1}{\partial\theta}\left(\frac{d^3y}{dt_0^3} - \Omega\frac{d^2x}{dt_0^2}\right) + \frac{\partial^2 R_1}{\partial t_0 \partial\theta}\frac{d^2y}{dt_0^2}\right\}\right] + c.c.
\end{aligned}
\tag{24}
$$

The ϵ^0 order part of the pressure is just $p_0(1) = \Omega^2(1-\mathfrak{b})$; the ϵ order part is given by

$$
p_1(1) = -\Omega^2 R_1(t_0, t_1, \theta) - (1-\mathfrak{b})\left\{\frac{\partial^2 x}{\partial t_0^2}\cos(\Omega t_0 + \theta) + \frac{\partial^2 y}{\partial t_0^2}\sin(\Omega t_0 + \theta)\right\} \tag{25}
$$

The homogeneous form of (23) has the complete solution

$$
v_1(t_0, t_1, \theta) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\theta}\left\{A_n(t_1)\mathrm{e}^{\mathrm{i}n\Omega\{-1+\sqrt{1+1/\beta}\}t_0} + B_n(t_1)\mathrm{e}^{\mathrm{i}n\Omega\{-1-\sqrt{1+1/\beta}\}t_0}\right\}. \tag{26}
$$

Considering next (24), it will be seen that the three first terms on the right hand side will generate secular terms in the solution. The secular terms are eliminated by satisfying the solvability conditions

$$
\frac{dA_n}{dt_1} + \mathrm{i}a_n A_n = 0, \quad a_n = \frac{\beta^2}{6}n^3\Omega\upsilon_-, \quad \upsilon_- = \frac{(-1+\sqrt{1+1/\beta})^2}{\sqrt{1+1/\beta}}, \tag{27}
$$

$$
\frac{dB_n}{dt_1} - \mathrm{i}b_n B_n = 0, \quad b_n = \frac{\beta^2}{6}n^3\Omega\upsilon_+, \quad \upsilon_+ = \frac{(1+\sqrt{1+1/\beta})^2}{\sqrt{1+1/\beta}}. \tag{28}
$$

These equations have the solutions $A_n = \mathcal{A}_n e^{-\mathrm{i}a_n t_1}$ and $B_n = \mathcal{B}_n e^{\mathrm{i}b_n t_1}$, where $\mathcal{A}_n$ and $\mathcal{B}_n$ are constants. The complete solution to (23) can thus be written as

$$
\begin{aligned}
v_1(t_0, t_1, \theta) &= \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\theta}\left\{\mathcal{A}_n\mathrm{e}^{\mathrm{i}n\Omega\beta(r_+ t_0 - n^2\frac{\beta}{6}\upsilon_- t_1)} + \mathcal{B}_n\mathrm{e}^{\mathrm{i}n\Omega\beta(r_- t_0 + n^2\frac{\beta}{6}\upsilon_+ t_1)}\right\} \\
&+ \frac{1}{2}\mathrm{e}^{\sigma t_0 + \mathrm{i}\theta}\lambda^2\frac{\gamma y - \mathrm{i}\sigma x}{\sigma^2 + \mathrm{i}2\beta\Omega\sigma + \beta\Omega^2} + c.c.,
\end{aligned}
\tag{29}
$$

where

$$
\sigma = \lambda + \mathrm{i}\Omega, \qquad \gamma = \sigma^* = \lambda - \mathrm{i}\Omega. \tag{30}
$$

The constants $\mathcal{A}_n$ and $\mathcal{B}_n$ can be determined by specification of initial conditions for v_1 (or for R, via (18)).

The pressure-related fluid forces can now be evaluated as

$$
\begin{aligned}
F_{xp} &= -\epsilon\chi\left[\pi\Omega\left\{\frac{\mathcal{A}_1}{r_\beta - 1}\mathrm{e}^{\mathrm{i}\Omega\beta(r_\beta - 1)w_- \tau} - \frac{\mathcal{B}_1}{r_\beta + 1}\mathrm{e}^{-\mathrm{i}\Omega\beta(r_\beta + 1)w_+ \tau}\right\}\right. \\
&+ \left.\frac{\pi}{2}\lambda^2\mathrm{e}^{\sigma\tau}\left\{\frac{\Omega^2}{\sigma}\frac{\sigma x + \mathrm{i}\gamma y}{\sigma^2 + \mathrm{i}2\beta\Omega\sigma + \beta\Omega^2} + (1-\mathfrak{b})(x - \mathrm{i}y)\right\} + c.c.\right], \\
F_{yp} &= \mathrm{i}F_{xp},
\end{aligned}
\tag{31}
$$

where

$$\begin{aligned}\chi &= \frac{\rho a^2 L}{M}, \qquad r_\beta = \sqrt{1+1/\beta}, \\ w_+ &= 1-\epsilon\frac{\beta}{6}\frac{r_\beta+1}{r_\beta}, \quad w_- = 1-\epsilon\frac{\beta}{6}\frac{r_\beta-1}{r_\beta}.\end{aligned} \tag{32}$$

In terms of the original time parameter τ the pressure-related part of the fluid force vector in (14) can be expressed as

$$\begin{aligned}\left\{\begin{array}{c} F_{xp} \\ F_{yp}\end{array}\right\} &= -\epsilon\chi\pi\Omega\left\{\frac{\mathcal{A}_1}{r_\beta-1}\mathrm{e}^{\mathrm{i}\Omega\beta(r_\beta-1)w_-\tau} - \frac{\mathcal{B}_1}{r_\beta+1}\mathrm{e}^{-\mathrm{i}\Omega\beta(r_\beta+1)w_+\tau}\right\}\left\{\begin{array}{c} 1 \\ \mathrm{i}\end{array}\right\} \\ &+ \frac{\pi}{2}\lambda^2\mathrm{e}^{\sigma\tau}\Omega^2\frac{1}{\sigma^2+\mathrm{i}2\beta\Omega\sigma+\beta\Omega^2}\left[\begin{array}{cc} 1 & \mathrm{i}\gamma/\sigma \\ \mathrm{i} & -\gamma/\sigma\end{array}\right]\left\{\begin{array}{c} x \\ y\end{array}\right\} \\ &+ \frac{\pi}{2}\lambda^2\mathrm{e}^{\sigma\tau}(1-\mathfrak{b})\left[\begin{array}{cc} 1 & -\mathrm{i} \\ \mathrm{i} & 1\end{array}\right]\left\{\begin{array}{c} x \\ y\end{array}\right\} + c.c.\end{aligned} \tag{33}$$

The friction-related fluid forces enter only in the next approximation, that is, in the solution of (24).

5 Concluding remarks

The force vector $\{F_{xp}\ F_{yp}\}^T$ should equal $\{-\mu\sigma\Omega^2\ 0\}^T$ for complete cancelation of the forced excitation by the unbalance mass; this appears not to be possible.

The present paper has given a formulation of the problem in a form that appears to produce manageable equations; but still, much work remains to be done.

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