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The "two-constants" theory and tensors of the microscopically-descriptive Navier-Stokes equations

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Abstract

The "two-constants" theory introduced first by Laplace in 1805 is the currently accepted theory describing isotropic, linear elasticity. The original, microscopically-descriptive Navier-Stokes [MDNS] equations were derived in the course of the development of the two-constants theory. From the viewpoint of these equations, we trace their evolution and the notion of tensor following in historical order the various contributions of Navier, Cauchy, Poisson, Saint-Venant and Stokes [1], and note the concordance between each.

Key words: the microscopically-descriptive equation, the Navier-Stokes equations, mathematical history.

1 Preliminary Remarks

In this report, we use the following definition of the stress tensor, due to I. Imai[7, p.178]: we call a 3×3 array such as P a stress tensor that returns a new vector $\mathbf{P}_n$ when multiplied from the right by the column vector of directional cosines:

$$\begin{bmatrix} P_{nx} \\ P_{ny} \\ P_{nz} \end{bmatrix} = \begin{bmatrix} p_{xx} & p_{yx} & p_{zx} \\ p_{xy} & p_{yy} & p_{zy} \\ p_{xz} & p_{yz} & p_{zz} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}_n = \mathrm{P}\cdot\mathbf{n}$$

Moreover, if $p_{ij}=p_{ji}$ for all $i,j=x,y,z$ then this tensor is said to be symmetric. If we suppose for example t_{ij} is the (i,j) element of a tensor, and $t_{ij}=-t_{ji}$ then the tensor is said to be anti-symmetric or skew-symmetric. Throughout the paper, we display for brevity a tensor by specifying its components, such as δ_{ij} of the well-known Krönecker-delta. Furthermore, we write $v_{k,k}=\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\cdots$ where we have employed the Einstein summation convention [2]. Simplifications occur as, for example, in Navier's elasticity of (1-1) in Table 4 where the tensor can be expressed as follows:

$$-\varepsilon\begin{bmatrix} \left(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ \left(\frac{dw}{dx}+\frac{du}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \left(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\right) \end{bmatrix} = -\varepsilon\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix}, \quad \text{where} \quad \epsilon=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}, \tag{1}$$

Expressions in Poisson's elasticity (3-1) in Table 4 are also of a similar form.

Moreover, we can easily express Navier's stress tensor t_{ij} of elasticity in the form: $t_{ij}=-\varepsilon(\delta_{ij}u_{k,k}+u_{i,j}+u_{j,i})$. Stokes' fluid theory (20) or (5) in Table 4 affords a second illustration: $t_{ij}=(-p-\frac{2}{3}\mu v_{k,k})\delta_{ij}+\mu(v_{i,j}+v_{j,i})$, or the equivalent expression $\sigma_{ij}=-p\delta_{ij}+\mu(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})-\frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}$. [3]

In what follows, " tensor " means the stress tensor as defined by I. Imai. [4] When referring to a " fluid ", an "elastic fluid" is implied.

2 Introduction

We have studied the original MDNS equations as formulated by their authors [5], Navier, Cauchy, Poisson, Saint-Venant and Stokes, and endeavor to ascertain their aims and conceptual thoughts in formulating these new equations. The "two-constants theory" [6] was first introduced in 1805 by Laplace [7] in regard to capillary action with constants denoted by H and K (cf. Table 2, 3).Thereafter, various pairs of constants have been proposed by their originators in formulating MSNS equations or equations describing equilibrium or capillary situations. It is commonly accepted that this theory describes isotropic, linear elasticity. [8] We argue that Poisson had

[1] Navier(1785-1836), Cauchy(1789-1857), Poisson(1781-1840), Saint-Venant(1797-1886), Stokes(1819-1903).

[2] Remark: in general, $v_{k,k}\neq v_{i,j}$ because the summation convention is in force when there is a repetition of indices.

[3] c.f. Schlichting [20], in our footnote(22).

[4] Numbers on the left-hand-side of equations refer to those given by the author in the original paper while numbers on the right-hand-side correspond to our indexing. The subscript to the original indexing, for example N^e/N^f, refer to author and type of theory, such as " elastic/fluid by Navier ". For equations indexed by section, the citation is then in the format "section no.-no. by author".

[5] The order followed is by date of proposal or publication.

[6] So-called by the author because of the prominence of two constants in the theory.

[7] Of capillary action, Laplace[8, V.4, Supplement p.2] acknowledges Clailaut[3, p.22], and Clailaut cites Maupertuis[10].

[8] Darrigol [4, p.121].

already pointed out the special aspect deduced by Laplace when, in 1831, he states, “elles renferment les deux constantes spéciales donc j'ai parlé tout à l'heure” [18, p.4]. Poisson was, we think, one of few who were aware of this issue.

3 A universal method for the two-constants theory

In this section, we would like to propose a universal method to describe the kinetic equations that arise in isotropic, linear elasticity. This is outlined as follows:

- The partial differential equations describing waves in elastic solids or flows in elastic fluids are expressed by using one constant or a pair of constants C_1 and C_2 such that:
  for elastic solids: $\frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f}$,
  for elastic fluids: $\frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f}$,
  where $T_1,\ T_2, \cdots$ are the first kind of tensors or terms constituting our equations. For example, the MDNS equations corresponding to incompressible fluids is composed of the kinetic equation along with the continuity equation and are conventionally written, in modern vector notation, as follows:

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{2}$$

- C_1 and C_2 are the two coefficients of the two-constants theory, for example, ε and E introduced by Navier, or R and G by Cauchy, k and K by Poisson, ε and $\frac{\varepsilon}{3}$ by Saint-Venant, or μ and $\frac{\mu}{3}$ by Stokes. Moreover, C_1 and C_2 can be expressed in the following form:

$$\begin{cases} C_1 \equiv \mathcal{L} r_1 g_1 S_1, \\ C_2 \equiv \mathcal{L} r_2 g_2 S_2, \end{cases} \quad \begin{cases} S_1 = \iint g_3 \to C_3, \\ S_2 = \iint g_4 \to C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3 \mathcal{L} r_1 g_1 = \frac{2\pi}{15} \mathcal{L} r_1 g_1, \\ C_2 = C_4 \mathcal{L} r_2 g_2 = \frac{2\pi}{3} \mathcal{L} r_2 g_2. \end{cases}$$

- The two coefficients are expressible in terms of either the operator $\mathcal{L}$ ($\sum_0^\infty$ or $\int_0^\infty$) depending on one's personal preference, [9] where r_1 and r_2 are radial functions related to the radius of the active sphere of the molecules, raised to some power of n for Poisson's and Navier's cases, the relationship between these functions can be expressing by a logarithm with base r such that: $\log_r \frac{r_1}{r_2} = 2$.
- g_1 and g_2 are certain functions which depend on r and are described with attraction &/or repulsion.
- S_1 and S_2 are two expressions which describe the surface of the active unit-sphere centered on a molecule through application of the double integral (or single sum in the case of Poisson's fluid).
- g_3 and g_4 are certain compound spherical harmonic functions to calculate the moments over the unit sphere.
- C_3 and C_4 are indirectly determined as the common coefficients derived from the invariant tensor. With the exception of Poisson's fluid case, C_3 of C_1 is $\frac{2\pi}{3}$, and C_4 of C_2 is $\frac{2\pi}{15}$, which on computing only the molecules, and which are independent of personal preferences. In Poisson's case, we get the same as above after multiplying by $\frac{1}{4\pi}$. integrals are calculated from the total moment of the active sphere of the
- The ratio of the two coefficients, including Poisson's case, is an invariant: $\frac{C_3}{C_4} = \frac{1}{5}$.

4 Genealogy and convergence of the stress tensor

We show in Figure 1 a genealogy tracing in particular the form of the tensor t_{ij} appearing in the Navier-Stokes equations. In Table 4, we differentiate between the tensors associated with elastic solids or elastic fluids. From this genealogy, it could be asserted that Cauchy[1, 2] was the inventor or the first user of tensors, a view supported by the admission of Poisson[17] that he received the idea of symmetric tensor from Cauchy. Moreover, the tensor idea of Saint-Venant reappears in the work of Stokes. Here, we denote the two routes as NCP and PSS, both of which are portrayed in our figure, and by which we can explain the genealogy of tensor as it applies to the MDNS equations. cf. Table 4.

(fig.1) A genealogy of stress tensors in the prototypical Navier-Stokes equations

[9] At the time, there were heated arguments over Navier ' s integration and Poisson ' s summation.

Table 1: C_1, C_2, C_3, C_4: definitions of constants and computation of total moment of molecular actions by Navier, Cauchy, Poisson, Saint-Venant & Stokes

no	name/problem	elastic solid	elastic fluid	remark
1	Navier elastic:[12] fluid:[13]	(Navier[12] only:) $C_1 = \varepsilon \equiv \frac{2\pi}{15} \int_0^\infty d\rho \cdot \rho^4 f\rho$ $C_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} \cos\varphi d\varphi\ g_3 \Rightarrow \{\frac{16}{15}, \frac{4}{15}, \frac{2}{5}\}$ $\Rightarrow \frac{1}{2}\frac{\pi}{4}\frac{16}{15} = \frac{2\pi}{15}$	(Navier[13] only:) $C_1 = \varepsilon \equiv \frac{2\pi}{15} \int_0^\infty d\rho \cdot \rho^4 f(\rho)$ $C_2 = E \equiv \frac{2\pi}{3} \int_0^\infty d\rho \cdot \rho^2 F(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi\ g_3$ $\Rightarrow \{\frac{\pi}{10}, \frac{\pi}{30}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi\ g_4 \Rightarrow \frac{2\pi}{3}$	$\alpha = \rho\cos\psi\cos\varphi$, $\beta = \rho\cos\psi\sin\varphi$, $\gamma = \rho\sin\psi$
2	Cauchy elastic and fluid[2]	(Cauchy[2]) $C_1 = R = \frac{2\pi\Delta}{15} \int_0^\infty r^3 f(r) dr$ $= \pm\frac{2\pi\Delta}{15} \int_0^\infty [r^4 \mathrm{f}'(r) - r^3 \mathrm{f}(r)] dr$ $C_2 = G = \pm\frac{2\pi\Delta}{3} \int_0^\infty r^3 \mathrm{f}(r) dr$ $C_3 = \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2\alpha\cos^2\beta dp$, $= \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2 p \sin^2 p \sin p dp = \frac{2\pi}{15}$, $C_4 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi \cos^2\alpha \sin p dq dp$ $= \pi\int_0^\pi \cos^2 p \sin\ p dp = \frac{2\pi}{3}$,	(Cauchy[2]) same as elastic solid	$\cos\alpha = \cos p$, $\cos\beta = \sin p \cos q$, $\cos\gamma = \sin p \sin q$ $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$: mass of molecules per volume.
3	Poisson elastic:[15, 17] fluid:[17]	(Poisson[15] only:) $C_1 = k \equiv \frac{2\pi}{15} \sum \frac{r^5}{\alpha^5} \frac{d.\frac{1}{r}fr}{dr}$ $C_2 = K \equiv \frac{2\pi}{3} \sum \frac{r^3}{\alpha^5} fr$ $C_3 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta\sin\beta d\beta\ g_3 \Rightarrow \{\frac{2\pi}{5}, \frac{2\pi}{15}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta\sin\beta d\beta\ g_4 \Rightarrow \frac{2\pi}{3}$ Remark: C_3 is choiced as the common factor of $\{\cdot,\cdot\}$	(Poisson[17] both elastic and fluid) $C_1 = -k \equiv -\frac{1}{30\varepsilon^3} \sum r^3 \frac{d.\frac{1}{r}fr}{dr}$ $= -\frac{2\pi}{15} \sum \frac{r^3}{4\pi\varepsilon^3} \frac{d.\frac{1}{r}fr}{dr}$ $C_2 = -K \equiv -\frac{1}{6\varepsilon^3} \sum rfr$ $= -\frac{2\pi}{3} \sum \frac{r}{4\pi\varepsilon^3} fr$ $C_3 : \begin{cases} G = \frac{1}{10} \sum r^3 \frac{d.\frac{1}{r}fr}{dr}, \\ E = F = \frac{1}{30} \sum r^3 \frac{d.\frac{1}{r}fr}{dr} \end{cases}$ $\Rightarrow \{\frac{1}{10}, \frac{1}{30}\} \Rightarrow \frac{1}{30}$ $C_4 : (3\text{-}2)_{Pf} \quad N = \frac{1}{6\varepsilon^3} \sum rfr \Rightarrow \frac{1}{6}$	In Poisson[17], he treats both elastic and fluid the same. $x_1 = r\cos\beta\cos\gamma$, $y_1 = r\sin\beta\sin\gamma$, $\zeta = -r\cos\beta$
4	Saint-Venant[19]		$C_1 = \varepsilon, \quad C_2 = \frac{\varepsilon}{3}$	
5	Stokes[21]	$C_1 = A, \quad C_2 = B$	$C_1 = \mu, \quad C_2 = \frac{\mu}{3}$	

Table 2: The two constants in the kinetic equations

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Navier [12]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
2	Navier [13]	fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
3	Cauchy [2]	system of particles in elastic and fluid	R		$\frac{2\pi}{15}\Delta$		$\int_0^\infty dr$	r^3		$f(r)$		$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}\Delta$	$\int_0^\infty dr$		r^3		$\pm\mathrm{f}(r)$	$f(r) \neq \mathrm{f}(r)$, $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$: mass of molecules per volume.
4	Poisson [15]	elastic solid	k		$\frac{2\pi}{15}$		$\sum \frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum \frac{1}{\alpha^5}$		r^3		fr	
5	Poisson [17]	elastic solid and fluid	k		$\frac{1}{30}$		$\sum \frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$		$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		fr	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
6	Saint-Venant [19]	fluid	ε	$\frac{\varepsilon}{3}$								
7	Stokes [21]	fluid	μ	$\frac{\mu}{3}$								
8	Stokes [21]	elastic solid	A	B								$A = 5B$

Table 3: The two constants in the equilibrium equation

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Laplace [8, V.4, Supplement p.9] [9, V.4, p.700]	capillary action	H		2π		$\int_0^\infty dz$	z		$\Psi(z)$		z: distance
				K		2π	$\int_0^\infty dz$				$\Psi(z)$	cf.Gauss[5]
2	Poisson [18]	capillary action	H		$\frac{\pi}{4}\rho^2$		$\int_0^\infty dr$	r^4		φr		[18, p.14]
				K		$\frac{2\pi}{3}\rho^2$	$\int_0^\infty dr$		r^3		φr	[18, p.12]
3	Navier fluid [13]	equiliblium of fluid	p		$\frac{4\pi}{3}$		$\int_0^\infty d\rho$	ρ^3		$f(\rho)$		ρ : radius
4	Poisson [17, §5, ¶46, p.104]	equiliblium of fluid	p		$\frac{1}{6}$		$\sum \frac{1}{\varepsilon^3}$		r		R	$C_3 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
				q		$\frac{1}{4}$		$\sum \frac{1}{\varepsilon^3}$	$\frac{1}{r}$		$r_i^2 z' R$	$C_4 = \frac{1}{4\pi}\pi = \frac{1}{4}$

Navier[12, 13]⊗

(Euler) $\Longrightarrow$ $\cdots$ $\Vert$ $\Updownarrow$ $\Vert$ Poisson[15, 17]• $\Longrightarrow$ Saint-Venant[19]† $\Longrightarrow$ Stokes[21]‡

$\Downarrow$ $\Uparrow\odot$

Cauchy[1, 2]§

$\longleftarrow$ molecular deduction$\longrightarrow$ $\Vert$ $\longleftarrow$ non-molecular deduction$\longrightarrow$

$\longleftarrow$ Navier-Cauchy-Poisson pattern$\longrightarrow$ $\Vert$ $\longleftarrow$Poisson-Saint-Venant-Stokes pattern

⊗ Navier: $t^e_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$, $t^f_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$

• Poisson: $t^e_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$, $t^f_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$

§ Cauchy: $t^{e,f}_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$

† Saint-Venant: $t^f_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$, $\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) = -p$

‡ Stokes: $t^f_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$,

$\odot$ Poisson states his reduction of the independent tensor elements from 9 to 6 is due to Cauchy. (cf.§5.2).

5 Derivations of the two constants and tensor

Recently Darrigol [4, p.121] has concluded: "it is called that the two-constants theory is the one now accepted for isotropic, linear elasticity," but Poisson [18, p.4] has stated already in 1831: "elles renferment les deux constantes spéciales donc j'ai parlé tout à l'heure. " Moreover, we believe that the first proposer of a " two-constants " theory was Laplace [9] in Table 3.

5.1 Navier's two constants and tensor

In his theory of elasticity, Navier deduced the single constant ε in (1). The corresponding Navier-Stokes equations derived by Navier himself for incompressible fluids (2) are as follows:

$$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\Big(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\Big) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\Big(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\Big) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\Big) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\ ; \end{cases} \tag{3}$$

along with the equation of continuity: $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$. Navier supposes two constants as follows:

$$(3\text{-}10)_{Nf} \quad \varepsilon \equiv \frac{8\pi}{30}\int_0^\infty d\rho\rho^4 f(\rho) = \frac{4\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho), \quad E \equiv \frac{4\pi}{6}\int_0^\infty d\rho\rho^2 F(\rho) = \frac{2\pi}{3}\int_0^\infty d\rho\rho^2 F(\rho). \tag{4}$$

In the case of fluid, Navier was well aware of the necessity for the equation of continuity, because from (3) he obtained $\varepsilon\Delta$ by differentiating the equation of continuity with $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$. For example, the ε-terms in (3), as well as (5), are reduced to $\varepsilon\Delta\mathbf{u}$ as for example in (6). This is solely due to the mass conservative law, according to the explanation given by Navier.

As an aside, Navier always used his well-used method involving a four-step procedure to solve three equations, such as the equilibrium equation for the fluid [13], the kinetic equation for the elastic [12], and the kinetic equation for the fluid [13] with general methods as follows: • initially, to deduce one or two constants including incomputable function such as $f\rho$, $f(\rho)$ or $F(\rho)$ in Table 2, • then to construct the indeterminate equation, • then to make a Taylor series expansion and partial integration, exchanging d and δ, and pairing with the same integral operator, • and finally, to solve the indeterminate equation from the two points of view, the interior and the boundary. We present more details of this procedure by outlining Navier's analysis of fluid flow [13].

5.1.1 Indeterminate equation

The indeterminate equation, so-called then by Navier, is as follows:

$$
\begin{aligned}
(3\text{-}24)_{Nf} \quad 0 \quad = & \iiint dxdydz \begin{cases} [P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)]\delta u \\ [Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)]\delta v \\ [R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)]\delta w \end{cases} \\
- & \varepsilon \iiint dxdydz \begin{cases} \left(3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz}\right) + \left(\frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy}\right) + \left(\frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz}\right) \\ \left(\frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx}\right) + \left(\frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz}\right) + \left(\frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy}\right) \\ \left(\frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx}\right) + \left(\frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy}\right) + \left(\frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz}\right) \end{cases} \\
+ & \mathbf{S}ds^2 E(u\delta u + v\delta v + w\delta w). \qquad (5)
\end{aligned}
$$

5.1.2 Determinate equation from a Taylor series expansion and partial integration

Putting $\mathbf{S}ds^2E(u\delta u+v\delta v+w\delta w) = 0$ in the indeterminate equation (5) and performing a Taylor series expansion to first .order and neglecting higher-order terms, we get as follows:

$$
(3\text{-}29)_{Nf} \quad 0 = \iiint dxdydz \begin{cases} [P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) + \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)]\delta u \\ [Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) + \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)]\delta v \\ [R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right)]\delta w \end{cases} \qquad (6)
$$

From (6) we obtain (3), i.e. the kinetic equation, which is equivalent to the first equation of (2).

5.1.3 Determinate equation deduced from boundary condition

As a boundary condition, Navier used two constants in one equation. In this aspect, his method is unique among the original formulators. Navier explains as follows: regarding the conditions which apply at the points on the surface of the fluid element, if we substitute

• $dydz \quad \rightarrow \quad ds^2 \cos l$, where l is the angle by which the tangent plane makes with the yz-plane of the surface frame,
• $dxdz \quad \rightarrow \quad ds^2 \cos m$, where similarly m is s the angle with the xz-plane,
• $dxdy \quad \rightarrow \quad ds^2 \cos n$, where similarly n is the angle with the xyplane,
• $\iint dydz, \iint dxdz, \iint dxdy \quad \rightarrow \quad \mathbf{S}ds^2$, where $\mathbf{S}$ is the unit normal to the surface at the point,

then, because the factors multiply $\delta u, \delta v$ and δw respectively reduce to zero, the following determinate equations should hold for any point on the surface of the fluid element:

$$
(3\text{-}32)_{Nf} \quad \begin{cases} Eu + \varepsilon[\cos l 2\frac{du}{dx} + \cos m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos n\left(\frac{du}{dz} + \frac{dw}{dx}\right)] = 0, \\ Ev + \varepsilon[\cos l\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos m 2\frac{dv}{dy} + \cos n\left(\frac{dv}{dz} + \frac{dw}{dy}\right)] = 0, \\ Ew + \varepsilon[\cos l\left(\frac{dw}{dx} + \frac{du}{dz}\right) + \cos m\left(\frac{dw}{dy} + \frac{dv}{dz}\right) + \cos n 2\frac{dw}{dz}] = 0. \end{cases} \qquad (7)
$$

Here the value of the constant E must vary in accordance with the nature of solid with which the fluid is in contact. The equations of (7) are an expression of conditions prevailing on the boundary of the surface and constitute the so-called boundary conditions. The first terms of the left-hand-side of (7) are defined in (4) for the expression that we seek for the sum of the momentum of all interactions arising between molecules on the boundary and the fluid, while the second terms are the normal derivatives. Here, derivative terms on the left-hand-side of (7) are expressible as $v_{i,j}+v_{j,i}$. If we introduce the basis of the tensor as $\left[\begin{array}{ccc} \cos l & \cos m & \cos n \end{array}\right]^T$, then the tensor part of (7) is expressible as: $t_{ij} = \varepsilon[\{2v_{i,j}-(v_{i,j}+v_{j,i})\}\delta_{ij}+(v_{i,j}+v_{j,i})] = \varepsilon\{0\delta_{ij}+(v_{i,j}+v_{j,i})\} = \varepsilon(v_{i,j} + v_{j,i})$.

5.2 Cauchy's two constants and tensor

In this section we assume the following definitions:
• $a,\ b,\ c$: the coordinate values of a molecule $\mathbf{m}$ in the rectangular axes of $x,\ y,\ z$; • $a+\Delta a,\ b+\Delta b,\ c+\Delta c$: the coordinates of an arbitrary molecule m ; • $\xi,\ \eta,\ \zeta$: three functions of $a,\ b,\ c$ representing infinitesimal displacements parallel to the axes of molecule $\mathbf{m}$; • $(x,\ y,\ z)$, $(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$: the coordinates of molecules $\mathbf{m}$ and m in the new state of the system ; • $r(1+\varepsilon)$: the distance between the molecule $\mathbf{m}$ and m ; • ε: the dilatation of the length r in the path from the first state to the second, and then we have $x=a+\xi,\ y=b+\eta,\ z=c+\zeta$; • $X,\ Y,\ Z$: the quantities of the algebraic projections.

Cauchy deduces the three following elements of material points of elasticity after calculating the interactions of molecules, the details of which are omitted for sake of brevity. Moreover, we start with the following equation of elasticity

$$(40)_C \quad \begin{cases} X=(L+G)\frac{\partial^2\xi}{\partial a^2}+(R+H)\frac{\partial^2\xi}{\partial b^2}+(Q+I)\frac{\partial^2\xi}{\partial c^2}+2R\frac{\partial^2\eta}{\partial a\partial b}+2Q\frac{\partial^2\zeta}{\partial c\partial a}, \\ Y=(R+G)\frac{\partial^2\eta}{\partial a^2}+(M+H)\frac{\partial^2\eta}{\partial b^2}+(P+I)\frac{\partial^2\eta}{\partial c^2}+2P\frac{\partial^2\zeta}{\partial b\partial c}+2R\frac{\partial^2\xi}{\partial a\partial b}, \\ Z=(Q+G)\frac{\partial^2\zeta}{\partial a^2}+(P+H)\frac{\partial^2\zeta}{\partial b^2}+(N+I)\frac{\partial^2\zeta}{\partial c^2}+2Q\frac{\partial^2\xi}{\partial c\partial a}+2P\frac{\partial^2\eta}{\partial b\partial c} \end{cases}$$

which displays all 9 components of a tensor. (The invariants of the tensor are represented by the two constants G and R.)
Cauchy says of the elements of the tensor, i.e. the invariable values: G,H,I,L,M,N,P,Q,R:
If we suppose that the molecules $m,m',m'',\cdots$ are originally allocated by the same way in relation to the three planes made by the molecule m in parallel with the plane coordinates, then the values of these quantities come to remain invariable, even though a series of changes are made among the three angles: α,β,γ.
Cauchy considers symmetric tensors such that:

$$(41)_C \quad G=H=I,\quad L=M=N,\quad P=Q=R,\quad (45)_C \quad L=3R.$$

$$(46)_C \begin{cases} X=(R+G)\left(\frac{\partial^2\xi}{\partial a^2}+\frac{\partial^2\xi}{\partial b^2}+\frac{\partial^2\xi}{\partial c^2}\right)+2R\frac{\partial\nu}{\partial a}, \\ Y=(R+G)\left(\frac{\partial^2\eta}{\partial a^2}+\frac{\partial^2\eta}{\partial b^2}+\frac{\partial^2\eta}{\partial c^2}\right)+2R\frac{\partial\nu}{\partial b}, \\ Z=(R+G)\left(\frac{\partial^2\zeta}{\partial a^2}+\frac{\partial^2\zeta}{\partial b^2}+\frac{\partial^2\zeta}{\partial c^2}\right)+2R\frac{\partial\nu}{\partial c}, \end{cases} \quad (47)_C \quad \nu=\frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}$$

Cauchy may be the inventor of the term [10] "tensor", and Poisson supports Cauchy's symmetry properties when he reduces the number of independent components from 9 to 6 elements, in the following quote:

> D'un autre côté, il faut, pour l'equilibre d'un parallélépipède rectangle d'une étendue insensible, que les neuf composantes des pressions appliquées à ses trois faces non-parallélles, se réduisent à six forces qui peuvent être inégales. Cette proposition est due à M.Cauchy, et se déduit de la considération des momens. [17, §38, p.83]

Continuing, we define the density of molecules as: $(48)_C \quad \Delta=\frac{\mathcal{M}}{\mathcal{V}}$, where $\mathcal{M}$ is the sum of the mass of molecules contained in the sphere and $\mathcal{V}$ is the volume of the sphere. We then find expressions for the two constants, G and R:

$$(50)_C \quad \begin{cases} G=\pm\frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3\mathrm{f}(r)\cos^2\alpha\sin p\,dr\,dq\,dp=\pm\frac{2\pi\Delta}{3}\int_0^\infty r^3\mathrm{f}(r)dr, \\ R=\frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3 f(r)\cos^2\alpha\cos^2\beta\sin p\,dr\,dq\,dp=\frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr=\pm\frac{2\pi\Delta}{15}\int_0^\infty\left[r^4\mathrm{f}'(r)-r^3\mathrm{f}(r)\right]dr \end{cases} \tag{8}$$

where we have used: $(51)_C \quad \cos\alpha=\cos p,\quad \cos\beta=\sin p\cos q,\quad \cos\gamma=\sin p\sin q.$ [11] When we calculate these values in the general case then (8) yields the following expressions:

$$(56)_C \begin{cases} A\equiv\left[(L+G)\frac{\partial\xi}{\partial a}+(R-G)\frac{\partial\eta}{\partial b}+(Q-G)\frac{\partial\zeta}{\partial c}\right]\Delta, \\ B\equiv\left[(R-H)\frac{\partial\xi}{\partial a}+(M+H)\frac{\partial\eta}{\partial b}+(P-H)\frac{\partial\zeta}{\partial c}\right]\Delta, \\ C\equiv\left[(Q-I)\frac{\partial\xi}{\partial a}+(P-I)\frac{\partial\eta}{\partial b}+(N+I)\frac{\partial\zeta}{\partial c}\right]\Delta, \end{cases} \quad (57)_C \begin{cases} D\equiv\left[(P+I)\frac{\partial\eta}{\partial c}+(P+H)\frac{\partial\zeta}{\partial b}\right]\Delta, \\ E\equiv\left[(Q+G)\frac{\partial\zeta}{\partial a}+(Q+I)\frac{\partial\xi}{\partial c}\right]\Delta, \\ F\equiv\left[(R+H)\frac{\partial\xi}{\partial b}+(R+G)\frac{\partial\eta}{\partial a}\right]\Delta, \end{cases}$$

[10]The editors of Hamilton's papers [6, p.237, footnote] say, " The writer believes that what originally led him to use the terms 'modulus' and 'amplitude,' was a recollection of M. Cauchy's nomenclature respecting the usual imaginaries of algebra."
[11]We obtain the following results:

$$\begin{cases} \int_0^{2\pi}\int_0^\pi\cos^2\alpha\sin p\,dq\,dp=2\pi\int_0^\pi\cos^2 p\sin\ p\,dp=2\pi\left[-\frac{\cos^3 p}{3}\right]_0^\pi=\frac{4\pi}{3}, \\ \int_0^{2\pi}\int_0^\pi\cos^2\alpha\cos^2\beta\sin\ p\,dp=\int_0^{2\pi}\cos^2 q\,dq\int_0^\pi\cos^2 p(1-\cos^2 p)\sin p\,dp=\left[\frac{q}{2}+\frac{1}{4}\sin 2q\right]_0^{2\pi}\left[-\frac{\cos^5 p}{5}\right]_0^\pi=(\frac{2\pi}{2}-0)(\frac{2}{3}-\frac{2}{5})=\frac{4\pi}{15} \\ C_3=\frac{1}{2}\frac{4\pi}{15}=\frac{2\pi}{15},\quad C_4=\frac{1}{2}\frac{4\pi}{3}=\frac{2\pi}{3}, \end{cases}$$

By $(41)_C$ and $(45)_C$, we obtain the following reduced form:
$\frac{A}{\Delta} = 2(R+G)\frac{\partial\xi}{\partial a} + (R-G)v, \quad \frac{B}{\Delta} = 2(R+G)\frac{\partial\eta}{\partial b} + (R-G)v, \quad \frac{C}{\Delta} = 2(R+G)\frac{\partial\zeta}{\partial c} + (R-G)v,$
$\frac{D}{\Delta} = (R+G)\left(\frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c}\right), \quad \frac{E}{\Delta} = (R+G)\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right), \quad \frac{F}{\Delta} = (R+G)\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right).$
For the sake of convenience, in the particular case when booth $(41)_C$ and $(45)_C$ hold, it is sufficient to have : $(59)_C \quad (R+G)\Delta \equiv \frac{1}{2}k, \quad (R-G)\Delta \equiv K, \quad \Rightarrow \quad R = \frac{k+2K}{4\Delta} \quad G = \frac{k-2K}{4\Delta}$. Equations $(56)_C$ and $(57)_C$ can be displayed in a more convenient manner

$$(60)_C \quad \Rightarrow \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} = \begin{bmatrix} k\frac{\partial\xi}{\partial a} + Kv & \frac{1}{2}k\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right) & \frac{1}{2}k\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right) \\ \frac{1}{2}k\left(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b} + Kv & \frac{1}{2}k\left(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\right) \\ \frac{1}{2}k\left(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\right) & \frac{1}{2}k\left(\frac{\partial\eta}{\partial c} + \frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c} + Kv \end{bmatrix}. \tag{9}$$

Here, we must remark that the layout of the symmetric tensor of $(58)_C$ or $(60)_C$ is Cauchy's invention. If, moreover, the condition $(54)_C$: $R = -G$ holds, then $k = 0$ holds, thus yielding the following identities: $(61)_C \quad A = B = C = Kv, \quad D = E = F = 0.$

5.2.1 Equilibrium and kinetic equation of fluid by Cauchy

In what follows, equations referring to Cauchy's work on fluids will be designated in the form $(\cdot)_{C^*}$ instead by $(\cdot)_C$ to distinguish these from equations appearing in his work on elasticity above.
(Verification of equations in fluid.)
By replacing $(a,\ b,\ c)$ of $(56)_C$ and $(57)_C$ with $(x,\ y,\ z)$, we derive an equivalent set of equations for fluids as for elasticity. We omit for the sake of brevity the precise process in leading to the two constants or equations and present the final form

$$(76)_{C^*} \quad \begin{cases} \frac{\partial A}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial E}{\partial z} + X\Delta = 0, \\ \frac{\partial F}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial D}{\partial z} + Y\Delta = 0, \\ \frac{\partial E}{\partial x} + \frac{\partial D}{\partial y} + \frac{\partial C}{\partial z} + Z\Delta = 0, \end{cases} \quad \Rightarrow \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} + \Delta \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{0}$$

We follow the layout of Cauchy's symmetric tensor as presented originally in $(76)_{C^*}$. By replacing $R+G$ and $2R$ with Cauchy's usage $C_1 \equiv R+G = \frac{k}{2\Delta}, \quad C_2 \equiv 2R = \frac{k+2K}{2\Delta}$, we can reduce these equations of fluids in motion and in equilibrium to the same form $(46)_C$ found for elasticity. However, here, we would like to adopt not Cauchy ' s C_1 and C_2 but $C_1 = R$ and $C_2 = G$, because it is more rational to do so, as can be seen by checking the reciprocal coincidence in Table 2. [12]
(Comparison with and comments on Navier's equation in elasticity.)

Cauchy states: for the reduction of equations $(79)_{C^*}$ and $(80)_{C^*}$ to Navier's equations([12]) to determine the law of equilibrium and elasticity, it is necessary to assume such as the condition which we have mentioned above: $k = 2K$. If $G = 0$ then we get as the equations of equilibrium and the kinetic equations in equal elasticity, then the tensor is equivalent with the tensor not only of the elastic but also of ε in Navier's fluid equation (3) (c.f. Table 4).

5.3 Poisson's two constants and tensor

5.3.1 Principles and equations in elastic solids

Below, we deduce K and k according to Poisson[15, pp.368-405, §1-§16]. For brevity, we introduce the following definitions:

$$\begin{cases} ax_1 + by_1 + c(z_1 - \zeta_1) \equiv \phi, \\ a'x_1 + b'y_1 + c'(z_1 - \zeta_1) \equiv \psi, \\ a''x_1 + b''y_1 + c''(z_1 - \zeta_1) \equiv \theta, \end{cases} \qquad \begin{cases} \phi\frac{du}{dx} + \psi\frac{du}{dy} + \theta\frac{du}{dz} \equiv \phi', \\ \phi\frac{dv}{dx} + \psi\frac{dv}{dy} + \theta\frac{dv}{dz} \equiv \psi', \\ \phi\frac{dw}{dx} + \psi\frac{dw}{dy} + \theta\frac{dw}{dz} \equiv \theta' \end{cases} \tag{10}$$

We assume that α is the average molecular distance, ω represents a finite surface area, and $\frac{\omega}{\alpha^2}$ is the average number of molecules in ω. We then get the pressure terms.

$$P = \sum \frac{(\phi + \phi')\zeta}{\alpha^3 r'} fr', \quad Q = \sum \frac{(\psi + \psi')\zeta}{\alpha^3 r'} fr' \quad R = \sum \frac{(\theta + \theta')\zeta}{\alpha^3 r'} fr'. \tag{11}$$

[12] Here, C_1 and C_2 are not the two-constants defined here by us but introduced temporarily by Cauchy himself.

By using his so-called *effective transformation*, [13], we get from (11) the following:

$$\begin{cases} P = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[(g+g')\sum \frac{r^3}{\alpha^5} fr + (gg'+hh'+ll')g\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ Q = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[(h+h')\sum \frac{r^3}{\alpha^5} fr + (gg'+hh'+ll')h\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ R = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[(l+l')\sum \frac{r^3}{\alpha^5} fr + (gg'+hh'+ll')l\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \end{cases} \qquad \Delta := \cos\beta \cdot \sin\beta \; d\beta \; d\gamma, \tag{12}$$

Later, Poisson recalculates this problem in another book [17] [14], in which he deduces the general principles behind elasticity and fluids, and hence derives the representative two-constants theory with K and k for both elasticity and fluids as follows:

$$\begin{cases} P = \left[K(1+\frac{du}{dx}) + k\left(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right]c + \left[K\frac{du}{dy} + k\left(\frac{du}{dy}+\frac{dv}{dx}\right)\right]c' + \left[K\frac{du}{dz} + k\left(\frac{du}{dz}+\frac{dw}{dx}\right)\right]c'', \\ Q = \left[K(1+\frac{dv}{dy}) + k\left(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\right)\right]c' + \left[K\frac{dv}{dx} + k\left(\frac{dv}{dx}+\frac{du}{dy}\right)\right]c + \left[K\frac{dv}{dz} + k\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\right]c'', \\ R = \left[K(1+\frac{dw}{dz}) + k\left(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\right)\right]c'' + \left[K\frac{dw}{dy} + k\left(\frac{dw}{dy}+\frac{dv}{dz}\right)\right]c' + \left[K\frac{dw}{dx} + k\left(\frac{dw}{dx}+\frac{du}{dz}\right)\right]c, \end{cases} \tag{13}$$

where, for abbreviation, he uses similarly K and k. Moreover, instead of α in (11), he introduces ε as the average distance between molecules, and from the following considerations:

- on voit que la pression N restera la même en tous sens autour de ce point: elle sera normale à ce plan et dirigée de dehors en dedans de A, ou de dedans en dehors, selon que sa value sera positive ou negative, [(transl): we see that the pressure N orients omni-directionally around an arbitrary point: A, and from outward to inward or from inward to outward, according to that the value will be positive or negative, (then we ought to consider as $\frac{1}{2}$) ;]
- from the assumption of isotropy and homogeneity of space, $r^2 = x^2 + y^2 + z^2$, $\Rightarrow$ $\Sigma \frac{z^2}{r} fr = \Sigma \frac{1}{3} r fr$, (cf. Poisson [17], pp. 32-34):

$$(3\text{-}8)_{P^e} \qquad K \equiv \frac{1}{6\varepsilon^3}\sum rfr = \frac{2\pi}{3}\sum \frac{rfr}{4\pi\varepsilon^3}, \quad k \equiv \frac{1}{30\varepsilon^3}\sum r^3 \frac{d.\frac{1}{r}fr}{dr} = \frac{2\pi}{15}\sum \frac{1}{4\pi\varepsilon^3} r^3 \frac{d.\frac{1}{r}fr}{dr}, \tag{14}$$

et étendant les sommes Σ à tous les points matériels du corps qui sont compris dans la sphère d'activité de M. [$\Rightarrow$ and extending the summation Σ to all the material points contained in the active sphere by M.] (cf. Poisson [17], p. 46):

5.3.2 Fluid pressure in motion

[15] Poisson's tensor of the pressures in a fluid, which he assumes compressible, reads as follows:

$$(7\text{-}7)_{Pf} \quad \begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) \\ p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{bmatrix},$$

$$(k+K)\alpha = \beta, \quad (k-K)\alpha = \beta', \quad p = \psi t = K, \quad \Rightarrow \quad \beta + \beta' = 2k\alpha,$$

where χt is the density of the fluid around the point M, and ψt is the pressure. Here K and k are the same constants as in $(3\text{-}8)_{P^e}(=(14))$ for an elastic body. Velocity and pressure are defined as follows:

$$\mathbf{u} = (u,v,w), \quad \frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w, \quad \varpi \equiv p - \alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \quad (\varpi \equiv p, \quad \text{if incompressible.})$$

which substituted into (??$\cdots$) yields

$$\begin{cases} \frac{d^2x}{dt^2} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ \frac{d^2y}{dt^2} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ \frac{d^2z}{dt^2} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}. \end{cases} \quad \Rightarrow \quad (7\text{-}9)_{Pf} \quad \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}). \end{cases} \tag{15}$$

[13] $\frac{1}{r'}fr' = \frac{1}{r}fr + (\phi\phi' + \psi\psi' + \theta\theta')\frac{d.\frac{1}{r}fr}{rdr}$ ([17, p.42]).

[14] In Poisson [17], the title of the chaper 3 reads "*Calcul des Pressions dans les Corps élastiques ; équations defferentielles de l'équilibre et du mouvement de ces Corps.*"

[15] In Poisson [17], the title of the chaper 7 reads "*Calcul des Pressions dans les Fluides en mouvement ; équations defferentielles de ce mouvement.*"

5.4 Saint-Venant's tensor

Saint-Venant [16] explains that the object of his paper [19] is to simplify the description and calculation of the molecular interactions without specifying the molecular function. We show Saint-Venant's tensor, which from the extract seems to hint Stokes [19]. For this section we introduce the following parameters: ξ, η, ζ are the Velocity components at the arbitrary point m of a fluid in motion in the coordinate directions x, y, z respectively, P_{xx}, P_{yy}, P_{zz} are the normal pressures and P_{yz}, P_{zx}, P_{xy} are the tangential pressures with sub-index pair indicating the perpendicular plane and direction of decomposition.

$$(1)_{SV} \quad \frac{P_{xx} - P_{yy}}{2\left(\frac{d\xi}{dx} - \frac{d\eta}{dy}\right)} = \frac{P_{zz} - P_{xx}}{2\left(\frac{d\zeta}{dx} - \frac{d\eta}{dz}\right)} = \frac{P_{yy} - P_{zz}}{2\left(\frac{d\eta}{dy} - \frac{d\zeta}{dz}\right)} = \frac{P_{yz}}{\frac{d\eta}{dz} + \frac{d\zeta}{dy}} = \frac{P_{zx}}{\frac{d\zeta}{dx} + \frac{d\xi}{dz}} = \frac{P_{xy}}{\frac{d\xi}{dy} + \frac{d\eta}{dx}} = \varepsilon,$$

where $\frac{1}{3}\left(P_{xx} + P_{yy} + P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = \pi$. From this last equation, we solve for normal pressure as follows: $(2)_{SV} \quad P_{xx} = \pi + 2\varepsilon\frac{d\xi}{dx}, \quad P_{yy} = \pi + 2\varepsilon\frac{d\eta}{dy}, \quad P_{zz} = \pi + 2\varepsilon\frac{d\zeta}{dz}$. From $(1)_{SV}$, we then obtain the tangential pressures: P_{yz}, P_{zx}, P_{xy}, which then reduces the tensor to symmetric form

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} \pi + 2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) & \pi + 2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) & \pi + 2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}. \tag{16}$$

Saint-Venant says that by using his theory, we can obtain concordance with Navier, Cauchy and Poisson:

> Si l'on remplace π par $\varpi - \varepsilon(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz})$, et si l'on substitue les équations $(2)_{SV}$ et $(3)_{SV}$ dans les relations connues entre les pressions et les forces accélératrices, on obtient, en supposant ε le même en tous les points du fluide, les équations différentielles données le 18 mars 1822 par M.Navier (*Mémoires de l'Institut*, t.VI), en 1828 par M.Cauchy (*Exercices de Mathématiques*, p.187) [17], et le 12 octobre 1829 par M.Poisson (même *Mémoire*, p.152) [18]. La quantité variable ϖ ou π n'est autre chose, dans les liquides, que la *pression normale moyenne* en chaque point. [19, p.1243]

Saint-Venant's paper[19] seems to provide Stokes a clue to the notion of tensor (20) and his principle, because we can see the close correspondence by comparing [19] Saint-Venant's t_{ij}:

$$\begin{aligned} t_{ij} &= (\pi + 2\varepsilon v_{i,j} - \gamma)\delta_{ij} + \gamma, \quad (\text{where,} \quad \gamma \equiv \varepsilon(v_{i,j} + v_{j,i})), \\ &= \left(\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) + 2\varepsilon v_{i,j} - \gamma\right)\delta_{ij} + \gamma \\ &= \left(\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k}\right)\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}) \quad \Leftarrow \quad 2\varepsilon v_{i,j}\delta_{ij} = \varepsilon(v_{i,j} + v_{j,i})\delta_{ij} = \gamma\delta_{ij} \end{aligned} \tag{17}$$

with Stokes' " t_{ij} " (21). Here, using (17), if we put [20] $P_{xx} = P_{yy} = P_{zz} = -p$ by assuming isotropy and homogeneity, which Stokes similarly takes as his principle as follows:

> If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle :
>
> - That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and
> - that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned.
>
> [21, p.80].

Then (17) is equivalent to Stokes' t_{ij} as follows. For example, if we put $\varepsilon \equiv \mu$, and choose the t_{xx} component of Saint-Venant's tensor from (16):

$$\begin{aligned} \pi + 2\varepsilon\frac{d\xi}{dx} &= -p + \left(2 - \frac{2}{3}\varepsilon\frac{d\xi}{dx}\right) - \frac{2\varepsilon}{3}\left(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = -p + 2\varepsilon\left\{\frac{2}{3}\frac{d\xi}{dx} - \frac{1}{3}\left(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)\right\} \\ &= -p + 2\varepsilon\left\{\frac{d\xi}{dx} - \frac{1}{3}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)\right\} = -p + 2\varepsilon\left(\frac{d\xi}{dx} - \delta\right) \quad \Rightarrow \quad P_1 \text{ of Stokes' (20).} \end{aligned}$$

[16] Adhémar Jean-Claude Barré de Saint-Venant (1797-1886).

[17] Cauchy [1, p.226]

[18] Poisson [17, p.152] $(7\text{-}9)_{Pf}$.

[19] Here, we cite the source of the tensorial description of t_{ij} of Poisson and Cauchy from C.Truesdell[23], of Navier from G.Darrigol [4], and otherwise by ourself or Schlichting[20].

[20] cf.I.Imai [7, p.185].

The other tensor components are likewise demonstrated but we omit the proof here for brevity. Moreover, Saint-Venant proposes that putting $\pi = \varpi - \varepsilon\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = \varpi - \varepsilon v_{k,k}$ then $t_{ij} = \left(\varpi - \varepsilon v_{k,k} + 2\varepsilon v_{i,j} - \gamma\right)\delta_{ij} + \gamma = (\varpi - \varepsilon v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$. This form of his tensor plays a key role in common with Navier's, Cauchy's and Poisson's constants.

5.5 Stokes' equations and tensor

In expressing the fluid equations in the following form

$$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases} \tag{18}$$

Stokes points out the coincidence with Poisson with the correspondence: $\varpi = p + \frac{\alpha}{3}(K+k)\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) \Rightarrow \nabla\varpi = \nabla p + \frac{\beta}{3}\nabla\cdot(\nabla\cdot\mathbf{u})$.

Stokes also makes the comment:

> The same equations have also been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI. p.389) [21], but his principles differ from mine still more than do Poisson's. [21, p.77, footnote]

He further states: observing that $\alpha(K+k) \equiv \beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$ (=(15) in our renumbering) to the equation $(12)_S$ of this paper. Stokes proposes his approximate equations in [21, p.93] :

$$(13)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) = 0, \end{cases} \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \tag{19}$$

which are identical to $(7\text{-}9)_{Pf}$(=(15), adding that: " these equations are applicable to the determination of the motion of water in pipes and canala, to the calculation of the effect of friction on the motions of tides and waves, and such questions. " ([21, p.93]). Here we shall trace his deduction with the Stokes tensor in the form:

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} p - 2\mu\left(\frac{du}{dx} - \delta\right) & -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - 2\mu\left(\frac{dv}{dy} - \delta\right) & -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - 2\mu\left(\frac{dw}{dz} - \delta\right) \end{bmatrix}, \quad \text{where} \quad 3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \tag{20}$$

He remarks: "it may also be very easily provided directly that the value of 3δ, the rate of cubical dilatation". We find that Stokes' tensor can be described compactly in component form as follows:

$$\begin{aligned} -t_{ij} &= \{p - 2\mu(v_{i,j} - \delta) + \gamma\}\delta_{ij} - \gamma, & &\Leftarrow \text{where,} \quad \gamma = \mu(v_{i,j} + v_{j,i}), \\ &= \{p - 2\mu v_{i,j}\}\delta_{ij} + \gamma(-\delta_{ij} + \delta_{ij} - 1) & &\Leftarrow \text{where,} \quad 2\mu v_{i,j}\delta_{ij} = \mu(v_{i,j} + v_{j,i})\delta_{ij} = \gamma\delta_{ij}, \\ &= (p + 2\mu\gamma)\delta_{ij} - \gamma = (p + \frac{2}{3}\mu v_{k,k})\delta_{ij} - \mu(v_{i,j} + v_{j,i}) \end{aligned} \tag{21}$$

Therefore, the sign of $-t_{ij}$ depends on the location of the tensor in the equation. [22]

Now, in considering the coincidence of (16) with (18), we see that Stokes' tensor may have originated with Saint-Venant's. The article by J.J.O'Connor and E.F.Robertson points out this resemblance. [23] Moreover, Stokes reports on the then academic activities within hydromechanics [22], in which he cites Saint-Venant[19]. Therefore, Stokes says: "I shall therefore suppose that for water, and by analogy for other incompressible fluids." ([21, p.93]). At any rate, we obtain $(13)_S$ (=(19)) with (20) and the following (22):

$$\begin{cases} \rho\left(\frac{Du}{Dt} - X\right) + \frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} = \rho\left(\frac{Du}{Dt} - X\right) + P = 0, \\ \rho\left(\frac{Dv}{Dt} - Y\right) + \frac{dT_3}{dx} + \frac{dP_2}{dy} + \frac{dT_1}{dz} = \rho\left(\frac{Dv}{Dt} - Y\right) + Q = 0, \\ \rho\left(\frac{Dw}{Dt} - Z\right) + \frac{dT_2}{dx} + \frac{dT_1}{dy} + \frac{dP_3}{dz} = \rho\left(\frac{Dw}{Dt} - Z\right) + R = 0, \end{cases} \quad \text{where} \quad \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix} \tag{22}$$

[21] Navier[13].

[22] Schlichting reverses the sign of the Stokes' tensor as follows: $\sigma_{ij} = -p\delta_{ij} + \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}$ [20, p.58, in footnote].

[23] cf. J.J.O'Connor, E.F.Robertson,→ http://www-groups.dcs.st-and.ac.uk/ history/Printonly/Saint-Venant.html.[14]

6 Conclusions

The "two-constants theory" is the currently-accepted theory for isotropic, homogeneous, linear elasticity. (Darrigol[4, p.121]). We have shown in our report: i) the original mathematical evidence in the genealogy of tensor; of which ii) the tensors and the corresponding equations as developed historically by Navier (1822), Cauchy (1828), Poisson (1829), Saint-Venant (1843) and Stokes (1845) (sic. in order) ; and iii) the appearance of the notion of tensors especially in the work of Saint-Venant. It is our contention that his was an epoch-making contribution, by simplifying and identifying the concordance between these pioneers of the MDNS equations, for using only tensor without the microscopically descriptions, and providing context for the contribution of Stokes.

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References

[1] A.L.Cauchy, *Sur les équations qui expriment les conditions de l'équilibre ou les lois du mouvment intérieur d'un corps solide, élastique ou non élastique*, Exercises de Mathématique, **3**(1828); Œuvres complètes D'Augustin Cauchy, (Ser. 2) **8**(1890), 195-226.

[2] A.L.Cauchy, *Sur l'équilibre et le mouvement d'un système de points matériels sollicités par des forces d'attraction ou de répulsion mutuelle*, Exercises de Mathématique, **3**(1828); Œuvres complètes D'Augustin Cauchy (Ser. 2) **8**(1890), 227-252.

[3] A.C.Clairaut, *Théorie de la figure de la terre, tirée des principes de l'hydrostatique*, Paris, 1743, second ed. 1808.

[4] O.Darrigol, *Worlds of flow: a history of hydrodynamics from the Bernoullis to Prandtl*, Oxford Univ. Press, 2005.

[5] C.F.Gauss, *Principia generalia theoriae figurae fluidrum in statu aequilibrii*, Gottingae, 1830, *Carl Friedrich Gauss Werke V*, Göttingen, 1867. (Similarly: "*Carl Friedrich Gauss Werke V*", Georg Olms Verlag, Hildesheim, New York, 1973, 29-77. Also, *Anzeigen eigner Abhandlungen, Götingische gelehrt Anzeigen*, 1829, as above in "*Werke V*", 287-293.)

[6] W.R.Hamilton, *The mathematical papers of Sir William Rowan Hamilton*, edited for the Royal Irish Academy by H.Halberstam and R.E.Ingram, Vol. III, Algebra, Cambridge, 1967.

[7] I.Imai, *Fluid dynamics*, Physical text series, Iwanami Pub. (1970), (New edition, 1997). (Japanese)

[8] P.S.Laplace, *Traité de méchanique céleste*, Ruprat, Paris, 1798-1805. (We can cite in the original by Culture et Civilisation, 1967.)

[9] P.S.Laplace, *On capillary attraction, Supplement to the tenth book of the Méchanique céleste*, translated by N. Bowditch, same as above Vol. IV 685-1018, 1806,1807.

[10] T,G.Maupertuis, *Oeuvres: avec l'Examen philosophique de la preuve de l'existence de Dieu employee dans l'essai de cosmologie*, Vol 1-4. G. Olms. (Reprint) 1965-74.

[11] C.L.M.H.Navier, *Sur les lois du mouvement des fluides, en ayant égard à l'adhésion des molécules*, Ann. chimie phys., **19**(1822), 244-260.

[12] C.L.M.H.Navier, *Mémoire sur les lois de l'équilibre et du mouvement des corps solides élastiques*, Mémoires de l'Academie des Sience de l'Institute de France, **7**(1827), 375-393. (Lu: 14/mai/1821.) → http://gallica.bnf.fr/ark:/12148/bpt6k32227, 375-393.

[13] C.L.M.H.Navier, *Mémoire sur les lois du mouvement des fluides*, Mémoires de l'Academie des Sience de l'Institute de France, **6**(1827), 389-440. (Lu: 18/mar/1822.) → http://gallica.bnf.fr/ark:/12148/bpt6k3221x, 389-440.

[14] J.J.O'Connor, E.F.Robertson,→ http://www-groups.dcs.st-and.ac.uk/ history/Printonly/Saint-Venant.html.

[15] S.D.Poisson, *Mémoire sur l'Équilibre et le Mouvement des Corps élastiques*, Mémoires de l'Academie royale des Siences, **8**(1829), 357-570, 623-27. (Lu: 14/apr/1828.) → http://gallica.bnf.fr/ark:/12148/bpt6k3223j

[16] S.D.Poisson, *Mémoire sur l'Équilibre des fluides*, Mémoires de l'Academie royale des Siences, **9**(1830), 1-88. (Lu: 24/nov/1828.) → http://gallica.bnf.fr/ark:/12148/bpt6k3224v, 1-88.

[17] S.D.Poisson, *Mémoire sur les équations générales de l'équiblibre et du mouvement des corps solides élastiques et des fluides*, (1829), J. École Polytech., **13**(1831), 1-174. (Lu: 12/oct/1829.)

[18] S.D.Poisson, *Nouvelle théorie de l'action capillaire*, Paris, 1831.

[19] A.J.C.B.de Saint-Venant, *Note à joindre au Mémoire sur la dynamique des fluides. (Extrait.)*, Académie des Sciences, *Comptes-rendus hebdomadaires des séances*, **17**(1843), 1240-1243. (Lu: 14/apr/1834.)

[20] H.Schlichting, *Boundary-Layer Theory*, 6th ed., McGraw-Hill, 1968. (English version translated by J.Kestin) (The original title: *Grenzschicht-Theorie*, 1951 in German).

[21] G.G.Stokes, *On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids, 1849*, (read 1845), (From the *Transactions of the Cambridge Philosophical Society* Vol. VIII. p.287), Johnson Reprint Corporation, New York and London, 1966, *Mathematical and physical papers* **1**, 1966, 75-129, Cambridge.

[22] G.G.Stokes, *Report on recent researches on hydrodynamics*, *Mathematical and physical papers* **1**, 1966, 157-187, Cambridge.

[23] C.Truesdell, *Notes on the History of the general equations of hydrodynamics*, Amer. Math. Monthly **60**(1953), 445-458.

Remark: we use *Lu* (: in French) in the bibliography meaning "read" date by the referees of the journals, for example MAS. In citing the original paragraphs in our paper, the underscoring are by ours.

Table 4: Concurrences and variations of tensors

1	name	tensor (3×3)	coefficient matrix (3 × 5) in equations
1-1	Navier elasticity	$t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(5\text{-}4)_{N^e}$ $-\varepsilon\begin{bmatrix} \left(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ \left(\frac{dw}{dx}+\frac{du}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \left(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\right) \end{bmatrix}$ $= -\varepsilon\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix},$ where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	We define the coefficient matrix in elasticity : C^e_T as follows: C^e_T : the coefficient of $\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 v}{\partial x\partial y} & \frac{\partial^2 w}{\partial z\partial x} \\ \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial y^2} & \frac{\partial^2 v}{\partial z^2} & \frac{\partial^2 w}{\partial y\partial z} & \frac{\partial^2 u}{\partial x\partial y} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial z^2} & \frac{\partial^2 u}{\partial z\partial x} & \frac{\partial^2 v}{\partial y\partial z} \end{bmatrix},$ then $(6\text{-}1)_{N^e} \quad\Rightarrow\quad C^e_T = -\varepsilon\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix} \quad\Rightarrow\quad (3)$
1-2	Navier fluid	$t_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$ (3) $\begin{bmatrix} \epsilon' - 2\varepsilon\frac{du}{dx} & -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) \\ -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & \epsilon' - 2\varepsilon\frac{dv}{dy} & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) \\ -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) & \epsilon' - 2\varepsilon\frac{dw}{dz} \end{bmatrix},$ where $\epsilon' = p - \varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$	Similarly, we define the coefficient matrix in fluid : C^f_T , which contains p in (1,1)-, (2,2)- and (3,3)-element. $(3) \quad\Rightarrow\quad C^f_T = \begin{bmatrix} p-3\varepsilon & -\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & p-3\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & -\varepsilon & p-3\varepsilon & -2\varepsilon & -2\varepsilon \end{bmatrix}$
2	Cauchy system (contains both elasticity and fluid)	$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(60)_C$ $\begin{bmatrix} k\frac{\partial\xi}{\partial a}+K\nu & \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) \\ \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b}+K\nu & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) \\ \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c}+K\nu \end{bmatrix},$ where $\nu = \frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}$	$(46)_C \Rightarrow C^e_T = \begin{bmatrix} L & R & Q & 2R & 2Q \\ R & M & P & 2P & 2R \\ Q & P & N & 2Q & 2P \end{bmatrix}$ $\Rightarrow R\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix},$ where $P = Q = R,\quad L = M = N,\quad L = 3R.$
3-1	Poisson elasticity	$t_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(6)_{P^e}$ $-\frac{a^2}{3}\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix},$ where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	$(6)_{P^e}$ $\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) \\ = 0, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) \\ = 0, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) \\ = 0, \end{cases}$ $\Rightarrow C^e_T = -\frac{a^2}{3}\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix}$
3-2	Poisson fluid	$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(7\text{-}7)_{Pf}$ $\begin{bmatrix} \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & \pi+2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \pi+2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) \\ \pi+2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{bmatrix},$ where $\pi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt}$	$(7\text{-}9)_{Pf} \quad\Rightarrow\quad C^f_T = \begin{bmatrix} \varpi+\beta & \beta & \beta & 0 & 0 \\ \beta & \varpi+\beta & \beta & 0 & 0 \\ \beta & \beta & \varpi+\beta & 0 & 0 \end{bmatrix}.$ According to Stokes: if we put $\varpi = p + \frac{\alpha}{3}(K+k)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$ $\Rightarrow\quad C^f_T = \begin{bmatrix} p+\frac{4\beta}{3} & \beta & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & p+\frac{4\beta}{3} & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & \beta & p+\frac{4\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \end{bmatrix}$ $\Rightarrow (12)_S$ (= (18)). Remark: $\alpha(K+k) = \beta.$
4	Saint-Venant fluid	$t_{ij} = (\frac{1}{3}(P_{xx}+P_{yy}+P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $= (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i})$ $\begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix},$ where $\pi = \frac{1}{3}\left(P_{xx}+P_{yy}+P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$ $\equiv -p - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$ (16)	no description in [19].
5	Stokes fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i}),$ tensor = $-1\,\times$ $\begin{bmatrix} p-2\mu\left(\frac{du}{dx}-\delta\right) & -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-2\mu\left(\frac{dv}{dy}-\delta\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-2\mu\left(\frac{dw}{dz}-\delta\right) \end{bmatrix}$ where $3\delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$ (20)	$(12)_S \quad\Rightarrow\quad C^f_T = \begin{bmatrix} -p+\frac{4\mu}{3} & \mu & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & -p+\frac{4\mu}{3} & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & \mu & -p+\frac{4\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} \end{bmatrix}$ $\Rightarrow$ (20). Remark: $\frac{4}{3}\mu = 2\mu(1-\frac{1}{3})$