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音波の障害物による散乱の逆問題における探針法と囲い込み法の過去と現在

The probe and enclosure methods for inverse obstacle scattering problems. The past and present.

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Contents

1 The probe method for inverse obstacle scattering problems at a fixed wave number

In this paper we consider inverse problems for partial differential equations. We restrict ourself to the reconstruction issue of the problems and refer the reader to [29] for several aspects and uniqueness results in inverse problems for partial differential equations.

More than ten years ago Ikehata discovered two methods for the purpose of extracting information about the location and shape of unknown *discontinuity* embedded in a known background medium from observation data. The methods are called the *probe* and *enclosure* methods. This paper presents their past and recent applications to inverse obstacle scattering problems of acoustic wave.

The probe method was originally introduced in 1997 and published in [5]. Since then the method has been applied to several inverse problems for partial differential equations [6, 7, 13, 19, 20] and still now some new knowledge on the method itself added in [16, 21].

In this section we present one of typical applications of the probe method published in [7]. Therein the author considered an inverse obstacle scattering problem at *a fixed wave number.* We denote by D and B_R an unknown obstacle in $\mathbf{R}^3$ and open ball radius R, respectively. We assume that: D is an open set with smooth boundary satisfying $\overline{D} \subset B_R$ and that $B_R \setminus \overline{D}$ is connected. ∂B_R indicates the location of the emitters and the receivers.

Let $k > 0$. Given $y \in \partial B_R$ let $\Phi(x) = \Phi_D(x, y; k), x \in \mathbf{R}^3 \setminus \overline{D}$ denote the solution of the problem:

$$(\Delta + k^2)\Phi + \delta(\,\cdot\, - y) = 0 \text{ in} \, \mathbf{R}^3 \setminus \overline{D}, \ \frac{\partial \Phi}{\partial \nu} = 0 \text{ on} \, \partial D$$

and the outgoing Sommerfeld radiation condition $\lim_{r \longrightarrow \infty} r(\partial\Phi/\partial\nu - ik\Phi) = 0$, where $r = |x|$ and ν is the outward normal relative to D.

Inverse Problem 1.1. *Fix* k. Reconstruct D from the surface data $\Phi_D(x, y; k)$ given at all $x \in \partial B_R$ and $y \in \partial B_R$.

The Φ_D has the form $\Phi_D(x, y; k) = \Phi_0(x, y; k) + E_D(x, y; k)$, where $E(x) = E_D(x, y; k)$ satisfies

$$(\Delta + k^2)E = 0 \text{ in} \, \mathbf{R}^3 \setminus \overline{D}, \ \frac{\partial E}{\partial \nu} = -\frac{\partial \Phi_0}{\partial \nu} \text{ on} \, \partial D$$

and the outgoing Sommerfeld radiation condition $\lim_{r \longrightarrow \infty} r(\partial E/\partial\nu - ikE) = 0$; $\Phi_0(x, y; k) = e^{ik|x-y|}/(4\pi|x - y|)$. The $E_D(x, y; k)$ is called the *scattered* wave field generated by the *point source* $\delta(\,\cdot\, - y)$ located at y. $\Phi_D(x, y; k)$ is called the *total* wave field.

In [7] the author has established the following result.

Theorem 1.1. *Assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ on B_R nor an eigenvalue for $-\Delta$ on $B_R \setminus \overline{D}$ with homogeneous Dirichlet boundary condition on ∂B_R and Neumann boundary condition on ∂D. Then one can reconstruct D from $\Phi_D(x, y; k)$ given at all $x \in \partial B_R$ and $y \in \partial B_R$.*

A brief outline of the proof is as follows. Set $\Omega = B_R$. We starts with introducing two Dirichlet-to-Neumann maps for the Helmholtz equation in $\Omega \setminus \overline{D}$ and Ω.

Given $f \in H^{1/2}(\partial\Omega)$ let $u \in H^1(\Omega \setminus \overline{D})$ be the weak solution of the elliptic problem

$$(\Delta + k^2)u = 0 \text{ in} \, \Omega \setminus \overline{D}, \ \frac{\partial u}{\partial \nu} = 0 \text{ on} \, \partial D, \ u = f \text{ on} \, \partial\Omega. \tag{1.1}$$

The map $\Lambda_D : f \longmapsto \partial u/\partial\nu|_{\partial\Omega}$ is called the *Dirichlet-to-Neumann map* associated with the elliptic problem. Set also $\Lambda_D = \Lambda_0$ for $D = \emptyset$.

Theorem 1.1 is divided into two steps.

Step 1. *One can calculate $\Lambda_0 - \Lambda_D$ from $\Phi_D(x, y; k)$ given at all $x \in \partial\Omega$ and $y \in \partial\Omega$.*

Step 2. *One can reconstruct D itself from the integral $\int_{\partial\Omega}(\Lambda_0 - \Lambda_D)f \cdot \overline{f}dS$ for infinitely many fs independent of D.*

Note that the integral in Step 2 has the form

$$\int_{\partial\Omega}(\Lambda_0 - \Lambda_D)f \cdot \overline{f}dS = \int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\overline{u} - \frac{\partial u}{\partial\nu}\overline{v}\right)dS$$

where $v = v(x)$, $x \in \Omega$ solves $(\Delta + k^2)v = 0$ in Ω, $v = f$ on $\partial\Omega$; $u = u(x)$, $x \in \Omega \setminus \overline{D}$ solves (1.1) with $f = v|_{\partial\Omega}$. Thus *infinitely many f* means *infinitely many v*.

The step 1 consists of two parts.

(i) Given f find the solutions g and h of the integral equations

$$\int_{\partial\Omega}\Phi_0(x, y; k)g(y)dS(y) = f(x), \ \int_{\partial\Omega}\Phi_D(x, y; k)h(y)dS(y) = f(x), \ x \in \partial\Omega.$$

(ii) Compute $(\Lambda_0 - \Lambda_D)f$ by using solutions g and h in (i) by the formula $(\Lambda_0 - \Lambda_D)f = g - h$.

This type of procedure, like (i) and (ii) has been known for the stationary Schrödinger equation [35] and the proof is an adaptation of the argument. Thus the point is Step 2.

1.1 Step 2.

In this subsection we explain Step 2. Instead of the original formulation of the probe method we employ a new one developed in [16, 21].

1.1.1 Needle, Needle sequence

Definition 1.1. Given a point $x \in \Omega$ we say that a non self-intersecting piecewise linear curve σ in $\overline{\Omega}$ is a *needle with tip at x* if σ connects a point on $\partial\Omega$ with x and other points of σ are contained in Ω. We denote by N_x the set of all needles with tip at x.

Let $\boldsymbol{b}$ be a nonzero vector in $\mathbf{R}^3$. Given $x \in \mathbf{R}^3$, $\rho > 0$ and $\theta \in]0, \pi[$ set $C_x(\boldsymbol{b}, \theta/2) = \{y \in \mathbf{R}^3 \,|\, (y - x) \cdot \boldsymbol{b} > |y - x||\boldsymbol{b}|\cos(\theta/2)\}$ and $B_\rho(x) = \{y \in \mathbf{R}^3 \,|\, |y - x| < \rho\}$. A set having the form $V = B_\rho(x) \cap C_x(\boldsymbol{b}, \theta/2)$ for some ρ, $\boldsymbol{b}$, θ and x is called a *finite cone with vertex* at x.

Let $G(y)$ be a solution of the Helmholtz equation in $\mathbf{R}^3 \setminus \{0\}$ such that, for any finite cone V with vertex at 0

$$\int_V |\nabla G(y)|^2 dy = \infty.$$

Hereafter we *fix* this G.

Definition 1.2. Let $\sigma \in N_x$. We call the sequence $\{v_n\}$ of $H^1(\Omega)$ solutions of the Helmholtz equation a *needle sequence* for (x, σ) if it satisfies, for any compact set K of $\mathbf{R}^3$ with $K \subset \Omega \setminus \sigma$

$$\lim_{n\longrightarrow\infty}(\|v_n(\,\cdot\,) - G(\,\cdot\, - x)\|_{L^2(K)} + \|\nabla\{v_n(\,\cdot\,) - G(\,\cdot\, - x)\}\|_{L^2(K)}) = 0.$$

The *existence* of the needle sequence is a consequence of the *Runge approximation property* (cf.[30]) for the Helmholtz equation under the assumption on k: k^2 is not a Dirichlet eigenvalue for $-\Delta$ on Ω. See the appendix of [7] and A.1.Remark in the appendix of [16] for the proof. The *unique continuation property* of the solution of the Helmholtz equation is essential.

1.1.2 Special behaviour of the needle sequence

In the following we do not assume that k^2 is not an eigenvalue for $-\Delta$ in Ω with Dirichlet boundary condition.

Lemma 1.1. *Let $x \in \Omega$ be an arbitrary point and $\sigma \in N_x$. Let $\{v_n\}$ be an arbitrary needle sequence for (x,σ). Then, for any finite cone V with vertex at x we have $\|\nabla v_n\|_{L^2(V\cap\Omega)} \longrightarrow \infty$ as $n \longrightarrow \infty$.*

Lemma 1.2. *Let $x \in \Omega$ be an arbitrary point and $\sigma \in N_x$. Let $\{v_n\}$ be an arbitrary needle sequence for (x,σ). Then for any point $z \in \sigma$ and open ball B centered at z we have $\|\nabla v_n\|_{L^2(B\cap\Omega)} \longrightarrow \infty$ as $n \longrightarrow \infty$.*

Note that from Definition 1.2 and Lemmas 1.1 and 1.2 one can recover $\sigma \in N_x$ itself from the behaviour of any needle sequence for (x,σ).

Summing up, we see that $\{v_n\}$ has two different sides:
(A) *converges* to singular solution $G(y-x)$ with singularity at $y = x$ outside σ;
(B) *blows up* on σ.
These different sides of needle sequences yield two sides of the probe method which we call Side A and Side B.

1.1.3 Indicator function and Side A of the probe method

Let v satisfy $(\Delta + k^2)v = 0$ in Ω and u solve (1.1) with $f = v|_{\partial\Omega}$. Set $w = u - v$ in $\Omega\setminus\overline{D}$. The w satisfies

$$(\Delta + k^2)w = 0 \text{ in } \Omega\setminus\overline{D},\ w = 0 \text{ on } \partial\Omega,\ \frac{\partial w}{\partial\nu} = -\frac{\partial v}{\partial\nu} \text{ on } \partial D. \tag{1.2}$$

Integration by parts yields

$$\begin{aligned}\int_{\partial\Omega}(\Lambda_\emptyset - \Lambda_D)(v|_{\partial\Omega})\cdot\overline{v}dS &= \int_D|\nabla v|^2dy - k^2\int_D|v|^2dy \\ &+ \int_{\Omega\setminus\overline{D}}|\nabla w|^2dy - k^2\int_{\Omega\setminus\overline{D}}|w|^2dy.\end{aligned} \tag{1.3}$$

This motivates
Definition 1.3. The *indicator function* $I(x)$, $x \in \Omega\setminus\overline{D}$ is defined by the formula

$$I(x) = \int_D|\nabla G(y-x)|^2dy - k^2\int_D|G(y-x)|^2dy + \int_{\Omega\setminus\overline{D}}|\nabla w_x|^2dy - k^2\int_{\Omega\setminus\overline{D}}|w_x|^2dy,$$

where w_x is the unique weak solution of the problem:

$$(\Delta + k^2)w = 0 \text{ in } \Omega\setminus\overline{D},\ \frac{\partial w}{\partial\nu} = -\frac{\partial}{\partial\nu}(G(\cdot\ - x)) \text{ on } D,\ w = 0 \text{ on } \partial\Omega.$$

The function w_x is called the *reflected solution* by D.

The following theorem is based on the convergence property of needle sequences and says that
• one can calculate the value of the indicator function at an arbitrary point outside D from $\Lambda_0 - \Lambda_D$;
• the indicator function can not be continued across ∂D as a bounded function in the whole domain.

Thus one can reconstruct ∂D as the *singularity* of the field $I(x)$ which can be computed from the data with needles and needle sequences. That is the meaning of the following result.

Theorem A. *It holds that*
• *(A.1) given $x \in \Omega \setminus \overline{D}$ and needle σ with tip at x if $\sigma \cap \overline{D} = \emptyset$, then for any needle sequence $\{v_n\}$ for (x, σ) we have $I(x) = \lim_{n \longrightarrow \infty} \int_{\partial\Omega} (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}) \cdot \overline{v_n} dS$;*
• *(A.2) for each $\epsilon > 0$ $\sup\{I(x) \,|\, dist(x, D) > \epsilon\} < \infty$;*
• *(A.3) for any point $a \in \partial D$ $\lim_{x \longrightarrow a} I(x) = \infty$.*

The key for (A.3) is to establish $\limsup_{x \longrightarrow a} \|w_x\|_{L^2(\Omega \setminus \overline{D})} < \infty$. An outline of the proof is as follows. Using the solution of the boundary value problem: $(\Delta + k^2)p = w_x$ in $\Omega \setminus \overline{D}$, $p = 0$ on $\partial\Omega$ and $\partial p/\partial\nu = 0$ on ∂D, we have the expression

$$\int_{\Omega \setminus \overline{D}} |w_x|^2 dy = \int_{\partial D} (p(x) - p(y)) \overline{\frac{\partial \Phi_0}{\partial \nu}(y - x)} dS(y) + k^2 p(x) \int_D \overline{\Phi_0(y - x)} dy.$$

Applying a standard regularity estimate of p: $\|p\|_{H^2(\Omega \setminus \overline{D})} \le C \|w_x\|_{L^2(\Omega \setminus \overline{D})}$ and the Sobolev imbedding: $|p(x) - p(y)| \le C|x - y|^{1/2} \|p\|_{H^2(\Omega \setminus \overline{D})}$, $x, y \in \Omega \setminus \overline{D}$ and $\|p\|_{L^\infty(\Omega \setminus \overline{D})} \le C\|p\|_{H^2(\Omega \setminus \overline{D})}$ to this right-hand side, one gets an upper bound of $\|w_x\|_{L^2(\Omega \setminus \overline{D})}$ which involves integrals of weakly singular kernels over ∂D and D.

1.2 Remark I. Side B of the probe method and an open problem

Since mathematically Theorem A is enough for establishing a reconstruction formula, in the previous applications of the probe method we did not consider the following natural question.
• Let $x \in \Omega$ and $\sigma \in N_x$. Let $\xi = \{v_n\}$ be a needle sequence for (x, σ). What happens on the sequence

$$I(x, \sigma, \xi)_n \equiv \int_{\partial\Omega} (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}) \cdot \overline{v_n} dS, n = 1, 2, \cdots$$

when x is just located on the boundary of obstacles, inside or passing through the obstacles? We call sequence $\{I(x, \sigma, \xi)_n\}$ the *indicator sequence* for (x, σ) and ξ.

In practice the tip of the needle can not move forward with infinitely small step and therefore in the scanning process with needle there is a possibility of skipping the unknown boundary of obstacles, entering inside or passing through obstacles. So for the practical use of the probe method we have to clarify the behaviour of the indicator sequence in those cases. The answer to this question is

Theorem B. *Assume that k^2 is sufficiently small (not specify here). Let $x \in \Omega$ and $\sigma \in N_x$. If $x \in \Omega \setminus \overline{D}$ and $\sigma \cap D \neq \emptyset$ or $x \in \overline{D}$, then for any needle sequence $\xi = \{v_n\}$ for (x, σ) we have* $\lim_{n \longrightarrow \infty} I(x, \sigma, \xi)_n = \infty$.

In the proof the blowing up property of needle sequences is essential.

A sketch of the proof. For simplicity, we consider here only a *single* obstacle case. We make use of two well known Poincaré's inequalities:

(I) $\|w\|^2_{L^2(\Omega\setminus\overline{D})} \le C(\Omega \setminus \overline{D})\|\nabla w\|^2_{L^2(\Omega\setminus\overline{D})}$ for all $w \in H^1(\Omega \setminus \overline{D})$ with $w = 0$ on $\partial\Omega$;

(II) $\|v - v_D\|^2_{L^2(D)} \le C(D)\|\nabla v\|^2_{L^2(D)}$ for all $v \in H^1(D)$, where $v_D = \int_D v dy/|D|$.

Let A be an arbitrary Lebesgue measurable set with $A \subset D$, $|A| > 0$ and $v \in L^2(D)$. A simple argument in [42] gives $\|v - v_A\|^2_{L^2(D)} \le 2K_A\|v - v_D\|^2_{L^2(D)}$, where $v_A = \int_A v dy/|A|$ and $K_A = 1 + |D|/|A|$. A combination of this and (II) yields

$$\int_D |v|^2 dy \le 4K_A C(D) \int_D |\nabla v|^2 dy + 2|D||v_A|^2. \tag{1.4}$$

Let $u = u_n$ solve (1.1) with $f = v_n|_{\partial\Omega}$ and set $w_n = u_n - v_n$. It follows from (1.3), (I) and (1.4) that

$$I(x, \sigma, \xi)_n \ge (1 - k^2 C(\Omega \setminus \overline{D})) \int_{\Omega\setminus\overline{D}} |\nabla w_n|^2 dy$$

$$+(1 - 4k^2 K_A C(D)) \int_D |\nabla v_n|^2 dy - 2k^2|D||(v_n)_A|^2.$$

Thus if k satisfies $k^2 C(\Omega \setminus \overline{D}) \le 1$, then we have

$$I(x, \sigma, \xi)_n \ge (1 - 4k^2 K_A C(D)) \int_D |\nabla v_n|^2 dy - 2k^2|D||(v_n)_A|^2.$$

Write $1 - 4k^2 K_A C(D) = 1 - 8k^2 C(D) - 4k^2(K_A - 2)C(D)$. Here we make k smaller in such a way that $8k^2 C(D) < 1$. Using an exhaustion of $\Omega \setminus \sigma$, one can construct $A \subset D$ in such a way that $|A| \approx |D|$ and $\overline{A} \subset \Omega \setminus \sigma$. Since $K_A - 2 = |D|/|A| - 1$, one gets $1 - 4k^2 K_A C(D) > 0$. Note also that the sequence $\{(v_n)_A\}$ is always *convergent* for a fixed A. Thus the blowing up property of the indicator sequence is governed by that of the sequence $\{\|\nabla v_n\|^2_{L^2(D)}\}$.

A combination of Theorems A and B yields another characterization of the obstacle.

Corollary 1.1. *Assume the smallness of k^2 same as Theorem B. A point $x \in \Omega$ belongs to $\Omega \setminus \overline{D}$ if and only if there exists a needle σ with tip at x and needle sequence ξ for (x, σ) such that the indicator sequence is bounded from above.*

Needless to say, this automatically gives a uniqueness theorem, too.

An open problem in the foundation of the probe method is the following.

Open problem 1.1. Can one remove the smallness of k^2 in Theorem B?

Here are some closely related technical questions.

- Is it true ?: if $x \in \Omega \setminus \overline{D}$ and $\sigma \cap D \neq \emptyset$ or $x \in \overline{D}$, then

$$\lim_{n \longrightarrow \infty} \frac{\|v_n\|_{L^2(D)}}{\|\nabla v_n\|_{L^2(D)}} = 0. \tag{1.5}$$

• Let $u = u_n$ solve (1.1) with $f = v_n|_{\partial\Omega}$ and set $w_n = u_n - v_n$. We know that if $x \in \overline{D}$, then $\|\nabla w_n\|_{L^2(\Omega\setminus\overline{D})} \longrightarrow \infty$ as $n \longrightarrow \infty$ ([20]). The question is: identify the points in $\overline{\Omega} \setminus D$ that really contribute the blowing up of ∇w_n. See [16] for an example in the case when $k = 0$.
• Is it true ?: if $x \in \Omega \setminus \overline{D}$ and $\sigma \cap D \neq \emptyset$ or $x \in \overline{D}$, then

$$\lim_{n\longrightarrow\infty} \frac{\|w_n\|_{L^2(\Omega\setminus\overline{D})}}{\|\nabla v_n\|_{L^2(D)}} = 0.$$

See [16, 19, 20] for more information on these questions.

1.3 Remark II. An explicit needle sequence

From Lemmas 1.1 and 1.2 we know that given $\sigma \in N_x$ the energy of an arbitrary needle sequence $\{v_n\}$ for (x, σ) blows up on σ. However, it will be difficult to understand the behaviour of $v_n(y)$ at each $y \in \sigma$. In this subsection, we give a family of special solutions of the Helmholtz equation with two parameters that yields an explicit needle sequence for a straight needle. We call such a family a generator of needle sequence.

The contents of this subsection are based on the classical materials developed by Yarmukhamedov, Mittag-Leffler and Vekua.

1.3.1 Yarmukhamedov

The following fact is taken from the article [45].

Theorem 1.2. *Let $K(w)$ be an entire function such that: $K(w)$ is real for real w; $K(0) = 1$; for each $R > 0$ and $m = 0, 1, 2$ $\sup_{|Re w|<R} |K^{(m)}(w)| < \infty$.*
Define

$$-2\pi^2\Phi_K(x) = \int_0^\infty Im\left(\frac{K(w)}{w}\right)\frac{du}{\sqrt{|x'|^2 + u^2}},$$

where $w = x_3 + i\sqrt{|x'|^2 + u^2}$ and $x' = (x_1, x_2) \neq (0, 0)$. Then one has the expression $\Phi_K(x) = 1/(4\pi|x|) + H_K(x)$ where H_K satisfies $\Delta H_K(x) = 0$ in $\mathbf{R}^3$.

Note that Φ_K can be identified with a unique distribution in the whole space and satisfies $\Delta\Phi_K(x) + \delta(x) = 0$ in $\mathbf{R}^3$.
Example 1. $K(w) \equiv 1$. In this case we have $\Phi_K(x) = 1/(4\pi|x|)$. This is because of

$$\frac{1}{4\pi|x|} = \int_{-\infty}^\infty \frac{du}{4\pi^2(|x|^2 + u^2)}$$

and

$$\frac{1}{|x|^2 + u^2} = -\text{Im}\left(\frac{1}{x_3 + i\sqrt{|x'|^2 + u^2}}\right)\frac{1}{\sqrt{|x'|^2 + u^2}}.$$

Thus for general K we have

$$H_K(x) = -\frac{1}{2\pi^2}\int_0^\infty \text{Im}\left(\frac{K(w) - 1}{w}\right)\frac{du}{\sqrt{|x'|^2 + u^2}}.$$

Example 2. $K(w) \equiv e^{\tau w}$. $\tau > 0$ a parameter. In [12] the author pointed out that $\Phi_K(x)$ with this K coincides with the *Faddeev Green function* $G_z(x)$ with $z = \tau(e_3 + ie_1)$:

$$G_z(x) = \frac{e^{x \cdot z}}{(2\pi)^3} \int_{\mathbf{R}^3} \frac{e^{ix\cdot\eta}}{|\eta|^2 - i2z \cdot \eta} d\eta.$$

The Faddeev Green function has been applied to several inverse boundary value/scattering problems by Sylvester-Uhlmann [43], Novikov [36], Nachman [35], et al..

1.3.2 Mittag-Leffler

Let $0 < \alpha \leq 1$. The entire function of the complex variable w

$$E_\alpha(w) = 1 + \frac{w}{\Gamma(1+\alpha)} + \frac{w^2}{\Gamma(1+2\alpha)} + \frac{w^3}{\Gamma(1+3\alpha)} + \cdots,$$

is introduced in [34] and called the *Mittag-Leffler function.*

It is known that $K(w) = E_\alpha(\tau w)$ with $\tau > 0$ satisfies the condition in Theorem 1.2 (cf. [2]). In [46] Yarmukhamedov applied this function with a fixed α to the Cauchy problem for the Laplace equation in two dimensions.

1.3.3 Vekua

The *Vekua transform* $v \longmapsto T_k v$ in three dimensions [44] takes the form

$$T_k v(y) = v(y) - \frac{k|y|}{2} \int_0^1 v(ty) J_1(k|y|\sqrt{1-t}) \sqrt{\frac{t}{1-t}}\, dt$$

where J_1 stands for the Bessel function of order 1 of the first kind.

The important property of this transform is: if v is harmonic in the whole space, then $T_k v$ is a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ in the whole space.

1.3.4 Generator of needle sequence

Using materials introduced by Yarmukhamedov, Mittag-Leffler and Vekua, the author found an explicit needle sequence when the needle is given by a *segment.*

Given $0 < \alpha \leq 1$ and $\tau > 0$ define $v(y; \alpha, \tau) = -H_K(y)$, $y \in \mathbf{R}^3$, where $K(w) \equiv E_\alpha(\tau w)$. This v is harmonic in the whole space and thus the function $v^k(y; \alpha, \tau) = T_k v(y; \alpha, \tau)$, $y \in \mathbf{R}^3$ satisfies the Helmholtz equation in the whole space.

Theorem 1.3([21]). *Let $x \in \Omega$ and σ be a straight needle with tip at x directed to $\omega = (0,0,1)^T$, that means: σ has the expression $\sigma = \{x + s\omega \,|\, 0 \leq s \leq l\}$ with $l > 0$. Then the function $v^k(\,\cdot\, - x; \alpha, \tau)|_\Omega$ as $\alpha \longrightarrow 0$ and $\tau \longrightarrow \infty$ generates a needle sequence for (x, σ) with $G = G_k$ given by*

$$G_k(y) = Re\left(\frac{e^{ik|y|}}{4\pi|y|}\right).$$

Note that since the function

$$\frac{\sin k|y|}{4\pi|y|}, \; y \in \mathbf{R}^3$$

satisfies the Helmholtz equation in the whole space, the function

$$v^k(y-x;\alpha,\tau)+i\,\frac{\sin k|y-x|}{4\pi|y-x|},\ y\in\Omega$$

generates also a needle sequence for (x,σ) with

$$G(y)=\frac{e^{ik|y|}}{4\pi|y|}. \tag{1.6}$$

Thus now we have an explicit generator of a needle sequence for a straight needle with (1.6). This makes the probe method completely *explicit* in the case when one uses only such a needle. Everything is reduced to the choice of small α and large τ.

This is very important also in the *singular sources method* by Potthast [38] since in his method one has to construct the density of the Herglotz wave function (cf. [3]) that approximates locally the fundamental solution of the Helmholtz equation in a domain like $\Omega\setminus\sigma$. However, Theorem 1.3 shows that instead one can consider only a simpler problem: construct the density of the Herglotz wave function that approximates $v^k(y-x;\alpha,\tau)$ on the whole *boundary* of a geometrically simpler domain like a *ball.*

Open problem 1.2. It would be interesting: do the numerical testing of the probe and singular sources methods in three dimensions with this *explicit* needle sequence.

Open problem 1.3. A mathematically interesting question is: find a generator of a needle sequence for a general needle.

Note that Yarmukhamedov [47] made use of $\Phi_K(y-x)$ itself not its regular part $H_K(y-x)$ to give a Carleman function which yields a representation of the solution of the Cauchy problem for the Laplace equation in three dimensions.

Finally we give a remark that is closely related to Open problem 1.1. In [21] an explicit formula of the precise values of $v^k(y-x;\alpha,\tau)$ on the line $y=x+s\,\omega(-\infty<s<\infty)$ is given. They are:

- if $y=x+s\,\omega$ with $s\neq 0$, then

$$v^k(y-x;\alpha,\tau)=\frac{1}{4\pi}\frac{E_\alpha(\tau s)-\cos ks}{s}-\frac{k}{4\pi}\int_0^1(1-w^2)^{-1/2}E_\alpha(\tau(1-w^2)s)J_1(ksw)dw;$$

- if $y=x$, then

$$v^k(y-x;\alpha,\tau)|_{y=x}=\frac{\tau}{4\pi\Gamma(1+\alpha)}.$$

Moreover, we see that $\nabla v^k(y-x;\alpha,\tau)$ on the line $y=x+s\,\omega\,(-\infty<s<\infty)$ is parallel to ω. In particular, we have

$$\nabla v^k(y-x;\alpha,\tau)|_{y=x}=\frac{\tau^2}{4\pi\,\Gamma(1+2\alpha)}\,\omega.$$

It seems that the behaviuor of $v^k(y-x;\alpha,\tau)$ and its gradient at $y=x$ suggest the validity of (1.5).

2 The enclosure method for inverse obstacle scattering problems at a fixed wave number

The *enclosure method* was introduced by the author in [10] and has been applied to several inverse problems for partial differential equations. In this section we present its applications to inverse obstacle scattering problems at a fixed wave number.

2.1 The enclosure method with infinitely many data

The method applied to inverse obstacle scattering problems is based on the asymptotic behaviour of the function (we call the indicator function again)

$$\tau \longmapsto \int_{\partial\Omega}(\Lambda_0 - \Lambda_D)(v|_{\partial\Omega})\cdot\overline{v}dS,$$

where $v = e^{x\cdot(\tau\omega + i\sqrt{\tau^2+k^2}\,\omega^\perp)}$ having large parameter τ; both ω and $\omega^\perp$ are unit vectors and perpendicular to each other.

This v satisfies the Helmholtz equation $\Delta v + k^2 v = 0$ in the whole space and divides the whole space into two parts: if $x\cdot\omega > t$, then $e^{-\tau t}|v| \longrightarrow \infty$ as $\tau \longrightarrow \infty$; if $x\cdot\omega < t$, then $e^{-\tau t}|v| \longrightarrow 0$ as $\tau \longrightarrow \infty$.

The method yielded the convex hull of unknown *sound-soft* obstacles by checking the behaviour of the indicator function. It virtually checks whether given t the half space $x\cdot\omega > t$ touches unknown obstacles.

In [9] an extraction formula of an *sound-hard obstacle* $D \subset \mathbf{R}^3$ with a constrained on the *Gaussian curvature* of ∂D from Dirichlet-to-Neumann map Λ_D has been established. Its precise statement rewritten with the present style is the following.

Let us recall the *support function* of D: $h_D(\omega) = \sup_{x\in D} x\cdot\omega$, $\omega \in S^2$. The convex hull of D is given by the set $\cap_{\omega\in S^2}\{x \in \mathbf{R}^3\,|\,x\cdot\omega < h_D(\omega)\}$. Therefore, knowing $h_D(\omega)$ for a ω yields an estimation of the convex hull of D from above.

Theorem 2.1. *Assume that the set $\{x \in \partial D\,|\,x\cdot\omega = h_D(\omega)\}$ consists of only one point and the Gaussian curvature of ∂D doesn't vanish at the point. Then the formula*

$$\lim_{\tau\longrightarrow\infty}\frac{1}{2\tau}\log\left|\int_{\partial\Omega}(\Lambda_0 - \Lambda_D)(v|_{\partial\Omega})\cdot\overline{v}dS\right| = h_D(\omega),$$

is valid. Moreover, we have:

if $t > h_D(\omega)$, then

$$\lim_{\tau\longrightarrow\infty}\int_{\partial\Omega}(\Lambda_0 - \Lambda_D)(e^{-\tau t}v|_{\partial\Omega})\cdot\overline{e^{-\tau t}v}dS = 0;$$

if $t < h_D(\omega)$, then

$$\lim_{\tau\longrightarrow\infty}\int_{\partial\Omega}(\Lambda_0 - \Lambda_D)(e^{-\tau t}v|_{\partial\Omega})\cdot\overline{e^{-\tau t}v}dS = \infty;$$

if $t = h_D(\omega)$, then

$$\liminf_{\tau\longrightarrow\infty}\int_{\partial\Omega}(\Lambda_0 - \Lambda_D)(e^{-\tau t}v|_{\partial\Omega})\cdot\overline{e^{-\tau t}v}dS > 0.$$

Note that: if one considers the Dirichlet boundary condition $u = 0$ on ∂D instead of the Neumann boundary condition $\partial u/\partial \nu = 0$ on ∂D, one can *drop* the assumption on ω and the Gaussian curvature of ∂D. See [10] for this result. Thus we propose

Open problem 2.1. Remove the curvature condition in Theorem 2.1.

A sketch of the proof of Theorem 2.1. Let u solve (1.1) with $f = v|_{\partial\Omega}$ and set $w = u - v$ in $\Omega \setminus \overline{D}$. The w satisfies (1.2). We have three lemmas.

Lemma 2.1. *There exists a positive constat $C(k)$ such that for all $\omega \in S^2$, $\tau > 0$*

$$2\tau^2 \int_D e^{2\tau x\cdot\omega}dx - k^2 \int_{\Omega\setminus\overline{D}} |w|^2 dx \le \int_{\partial\Omega} (\Lambda_0 - \Lambda_D)(v|_{\partial\Omega}) \cdot \overline{v} dS \le C(k)(\tau^2 + k^2) \int_D e^{2\tau x\cdot\omega}dx.$$

This is a consequence of the representation formula (1.3) and the estimate $\|w\|_{H^1(\Omega\setminus\overline{D})} \le C(k)\|v\|_{H^1(D)}$.

Lemma 2.2.

$$\liminf_{\tau\longrightarrow\infty} e^{-2\tau h_D(\omega)}\tau^2 \int_D e^{2\tau x\cdot\omega}dx > 0.$$

The proof of this lemma can be done by slicing D with the planes $x\cdot\omega = h_D(\omega) - s$ with $0 < s << 1$.

Lemma 2.3. *Assume that the set $\{x \in \partial D \,|\, x\cdot\omega = h_D(\omega)\}$ consists of the only one point and the Gaussian curvature of ∂D doesn't vanish at the point. Then*

$$\lim_{\tau\longrightarrow\infty} \frac{\int_{\Omega\setminus\overline{D}} |w|^2 dx}{2\tau^2 \int_D e^{2\tau x\cdot\omega}dx} = 0.$$

From Lemmas 2.1, 2.2 and 2.3 one knows that there exist positive constants C_1, C_2 and $\tau_0 > 0$ such that for all $\tau \ge \tau_0$

$$C_1 e^{2\tau h_D(\omega)} \le \int_{\partial\Omega} (\Lambda_0 - \Lambda_D)(v|_{\partial\Omega}) \cdot \overline{v} dS \le C_2 \tau^2 e^{2\tau h_D(\omega)}.$$

All the statements in Theorem 2.1 now follows from these estimates.

Finally we describe the outline of the proof of Lemma 2.3. One can find $p \in H^2(\Omega\setminus\overline{D})$ such that $(\Delta + k^2)p = \overline{w}$ in $\Omega\setminus\overline{D}$, $p = 0$ on $\partial\Omega$ and $\partial p/\partial\nu = 0$ on ∂D. From the Sobolev imbedding and the estimate $\|p\|_{H^2(\Omega\setminus\overline{D})} \le C(k)\|w\|_{L^2(\Omega\setminus\overline{D})}$ we have: $|p(x) - p(y)| \le C(k)|x - y|^{1/2}\|w\|_{L^2(\Omega\setminus\overline{D})}$ and $\sup_{x\in\overline{\Omega}\setminus D} |p(x)| \le C(k)\|w\|_{L^2(\Omega\setminus\overline{D})}$.

Let x_0 be the point in the set $\{x \in \partial D \,|\, x\cdot\omega = h_D(\omega)\}$. Since $\int_{\partial D} (\partial v/\partial\nu) dS(x) = -k^2 \int_D v dx$, one can write

$$\int_{\Omega\setminus\overline{D}} |w|^2 dx = -\int_{\partial D} p\frac{\partial v}{\partial\nu} dS(x) = \int_{\partial D} \{p(x_0) - p(x)\}\frac{\partial v}{\partial\nu} dS(x) + k^2 p(x_0)\int_D v dx.$$

From these one gets

$$\int_{\Omega\setminus\overline{D}} |w|^2 dx \le C(k)\left(\sqrt{2\tau^2 + k^2}\int_{\partial D} |x_0 - x|^{1/2} e^{\tau x\cdot\omega} dS(x) + \int_D e^{\tau x\cdot\omega}dx\right)\|w\|_{L^2(\Omega\setminus\overline{D})}$$

and this thus yields

$$\int_{\Omega\setminus\overline{D}} |w|^2 dx \le C(k)\left\{\left(\tau\int_{\partial D}|x_0-x|^{1/2}e^{\tau x\cdot\omega}dS(x)\right)^2+\left(\int_D e^{\tau x\cdot\omega}dx\right)^2\right\}.$$

The Schwarz inequality yields

$$\left(\int_D e^{\tau x\cdot\omega}dx\right)^2 \le |D|\int_D e^{2\tau x\cdot\omega}dx.$$

Thus from this and Lemma 2.2 one knows that it suffices to prove

$$\lim_{\tau\longrightarrow\infty}\tau e^{-\tau h_D(\omega)}\int_{\partial D}|x_0-x|^{1/2}e^{\tau x\cdot\omega}dS(x)=0.$$

In fact, one gets

$$\tau e^{-\tau h_D(\omega)}\int_{\partial D}|x_0-x|^{1/2}e^{\tau x\cdot\omega}dS(x)=O(\tau^{-1/4}).$$

This is proved by using a localization at x_0 and a local coordinates at the point.

2.2 The enclosure method with a single incident plane wave

The idea started with considering an inverse boundary value problem for the Laplace equation in two dimensions in [8]. Five years later in [15] the idea was applied to an inverse obstacle scattering problem in two dimensions. The problem is to reconstruct a two dimensional obstacle from the Cauchy data on a circle surrounding the obstacle of the total wave field for a *single* incident plane wave with a *fixed* wave number.

In this subsection we assume that D is *polygonal*, that is, D takes the form $D_1\cup\cdots\cup D_m$ with $1\le m<\infty$ where each D_j is open and a polygon; $\overline{D}_j\cap\overline{D}_{j'}=\emptyset$ if $j\ne j'$.

The total wave field u outside obstacle D satisfies

$$\Delta u+k^2u=0 \text{ in } \mathbf{R}^2\setminus\overline{D},\ \frac{\partial u}{\partial\nu}=0 \text{ on } \partial D$$

and the scattered wave $w=u-e^{ikx\cdot d}$ with $k>0$ and $d\in S^1$ satisfies the outgoing Sommerefeld radiation condition $\lim_{r\longrightarrow\infty}\sqrt{r}(\partial w/\partial r-ikw)=0$, where $r=|x|$.

Let B_R be an open disc with radius R satisfying $\overline{D}\subset B_R$. We assume that B_R is *known*. Our data are u and $\partial u/\partial\nu$ on ∂B_R. Let ω and $\omega^\perp$ be two unit vectors perpendicular to each other. Set $z=\tau\omega+i\sqrt{\tau^2+k^2}\omega^\perp$ with $\tau>0$ and $v(x;z)=e^{x\cdot z}$. Recall $h_D(\omega)=\sup_{x\in D}x\cdot\omega$.

Theorem 2.2. *Assume that the set $\partial D\cap\{x\in\mathbf{R}^2\,|\,x\cdot\omega=h_D(\omega)\}$ consists of only one point. Then the formula*

$$\lim_{\tau\longrightarrow\infty}\frac{1}{\tau}\log\left|\int_{\partial B_R}\left(\frac{\partial u}{\partial\nu}v(x;z)-\frac{\partial v}{\partial\nu}(x;z)u\right)dS(x)\right|=h_D(\omega),$$

is valid. Moreover, we have:

if $t\ge h_D(\omega)$, then

$$\lim_{\tau\longrightarrow\infty}\left|\int_{\partial B_R}\left(\frac{\partial u}{\partial\nu}e^{-\tau t}v(x;z)-e^{-\tau t}\frac{\partial v}{\partial\nu}(x;z)u\right)dS(x)\right|=0;$$

if $t < h_D(\omega)$, *then*

$$\lim_{\tau \longrightarrow \infty} \left| \int_{\partial B_R} \left(\frac{\partial u}{\partial \nu} e^{-\tau t} v(x;z) - e^{-\tau t} \frac{\partial v}{\partial \nu}(x;z) u \right) dS(x) \right| = \infty.$$

Sketch of the proof. The one of key points is: *introducing a new parameter s* instead of τ by the equation $s = \sqrt{\tau^2 + k^2} + \tau$, we obtain, as $s \longrightarrow \infty$ the complete asymptotic expansion

$$\int_{\partial B_R} \left(\frac{\partial u}{\partial \nu} v(x;z) - \frac{\partial v}{\partial \nu}(x;z) u \right) dS(x) e^{-i\sqrt{\tau^2+k^2} x_0 \cdot \omega^{\perp} - \tau h_D(\omega)} \sim -i \sum_{n=2}^{\infty} \frac{e^{i\frac{\pi}{2}\lambda_n} k^{\lambda_n} \alpha_n K_n}{s^{\lambda_n}}. \quad (2.1)$$

Here the λ_n describes the *singularity* of u at a *corner* and in this case explicitly given by the formula $\lambda_n = (n-1)\pi/\Theta$, where Θ denotes the outside angle of D at $x_0 \in \partial D \cap \{x \in \mathbf{R}^2 \,|\, x \cdot \omega = h_D(\omega)\}$ and thus satisfies $\pi < \Theta < 2\pi$; K_n are constants depending on λ_n, ω and shape of D around x_0; α_2, α_3, $\cdots$ are the coefficients of the convergent series expansion of u with polar coordinates at a corner:

$$u(r,\theta) = \alpha_1 J_0(kr) + \sum_{n=2}^{\infty} \alpha_n J_{\lambda_n}(kr) \cos \lambda_n \theta, \;\; 0 < r << 1, 0 < \theta < \Theta.$$

Now all the statements in Theorem 2.2 follow from (2.1) and another key point: $\exists n \geq 2$ $\alpha_n K_n \neq 0$. This is due to a contradiction argument. Assume that the assertion is not true, that is, $\forall n \geq 2$ $\alpha_n K_n = 0$.

First we consider the case when Θ/π is *irrational.* In this case we see that $\forall n \geq 2$ $K_n \neq 0$. Thus $\alpha_n = 0$ and this yields $u(r,\theta) = \alpha_1 J_0(kr)$ near a corner. Since this right-hand side is an entire solution of the Helmholtz equation, the unique continuation property of the solution of the Helmholtz equation yields $u(x) = \alpha_1 J_0(k|x - x_0|)$ in $\mathbf{R}^2 \setminus \overline{D}$. However, we see that the asymptotic behaviour of this right-hand and left-hand sides are completely different. Contradiction.

Next consider the case when Θ/π is a *rational.* By carefully checking the constant K_n we know that for each $n \geq 2$ with $K_n = 0$ the λ_n becomes an *integer.* From the assumption of the contradiction argument one knows if n satisfies $K_n \neq 0$, then $C_n = 0$. Thus we have the expansion

$$u(r,\theta) = \sum_{n_j} C_{n_j} J_{\lambda_{n_j}}(kr) \cos \lambda_{n_j} \theta,$$

where $n_j \geq 2$ satisfy $K_{n_j} = 0$. Since λ_{n_j} is an integer and $\lambda_{n_j} \Theta = (n_j - 1)\pi$, from this right-hand side one gets: for all r with $0 < r << 1$ $\partial u/\partial\theta(r,\pi) = \partial u/\partial\theta(r, \Theta - \pi) = 0$. Then a reflection argument ([1]) yields that this is true for all $r > 0$. However, from this together with the asymptotic behaviour of ∇u one can conclude that incident direction d has to be parallel to two linearly independent vectors which are directed along the lines $\theta = \pi$ and $\theta = \Theta - \pi$. Contradiction.

Remarks are in order.

- In Theorem 2.2 one uses the Cauchy data on the circle surrounding the obstacle as the observation data. However, $\partial u/\partial \nu$ on B_R can be calculated from u on ∂B_R by solving an exterior Dirichlet problem for the Helmholz equation.

• In [18] a similar formula has been established by using the *far field pattern* $F_D(\varphi, d; k)$, $\varphi \in S^1$ of scattered wave $w = u - e^{ikx\cdot d}$ for fixed d and k which determines the leading term of the asymptotic expansion of w at infinity in the following sense

$$w(r\varphi) \sim \frac{e^{ikr}}{\sqrt{r}} F_D(\varphi, d; k)\ r \longrightarrow \infty.$$

Moreover, therein instead of volumetric obstacle, similar formulae for *thin* sound-hard obstacle (or *screen*) also have been established with *two* incident plane waves.
• In [37] the numerical testing of a method based on results in [14, 15, 18] has been reported.
• It would be interesting to consider the case when the total wave u satisfies the equation $\nabla \cdot \gamma \nabla u + k^2 u = 0$ in $\mathbf{R}^2$ where $\gamma(x) = 1$ for $x \in \mathbf{R}^2 \setminus D$ and $\gamma(x) = A_j$ for $x \in D_j$, $j = 1, \cdots, m$; each A_j are positive constants and $A_j \neq 1$. The author thinks that this case becomes extremely difficult because of the complicated behaviour of u at a corner. However we propose
Open problem 2.2. Establish Theorem 2.2 for u above.
See [11] for $k = 0$ and [17] for the equation $\nabla \cdot \gamma \nabla u + k^2 \gamma u = 0$.
• For recent applications of the enclosure method with a single measurement for a system arising in linear theory of elasticity we have [24, 25, 26]. However, their extension to the elastic wave with a single incident plane wave remains open. It is a challenging problem to be solved.

3 Inverse obstacle scattering problems with dynamical data over a finite time interval

Previously we considered only the stationary or time harmonic problem. In this section we consider how one can use the data over a *finite time interval* to extract information about the location and shape of unknown obstacles. In [28, 39, 40] some uniqueness results have been established, however, it seems that mathematically rigorous study of the reconstruction issue in this type of problem has not been paid much attention. Note that: there are some results [31, 32, 33] in the context of the Lax-Phillips scattering theory, which give the convex hull of an unknown obstacle, however, the data are taken from $t = 0$ to $t = \infty$.

The purpose of this section is to introduce a new and simple method in [23] which is an application of the idea developed in [22, 27] and employs the data over a finite time interval on a known surface surrounding unknown obstacles.

3.1 New development of the enclosure method

In order to explain the basic idea, in this subsection we present an application to the one-space dimensional wave equation which is taken from Appendix B in [22]. Let $a > 0$ and $c > 0$. Let $u = u(x, t)$ be a solution of the problem:

$$\frac{1}{c^2} u_{tt} = u_{xx} \text{ in }]0,\ a[\times]0,\ T[,\ \ c u_x(a, t) = 0 \text{ for } t \in]0,\ T[,$$

$$u(x, 0) = 0,\ \ u_t(x, 0) = 0 \text{ in }]0,\ a[.$$

The quantity c denotes the propagation speed of the signal governed by the equation.
Inverse Problem 3.1. Assume that a is *unknown*. Extract a from $u(0,t)$ and $u_x(0,t)$ for $0<t<T$.

Theorem 3.1. *Let $u_x(0,t)\in L^2(0,T)$ satisfy the condition: there exists a real number μ such that*

$$\liminf_{\tau\longrightarrow\infty}\tau^{\mu}\left|\int_0^T u_x(0,t)e^{-\tau t}dt\right|>0. \tag{3.1}$$

Let $T>2a/c$ and $v(x,t)=v(x,t;\tau)=e^{-\tau(x/c+t)}$. Then the formula

$$\lim_{\tau\longrightarrow\infty}\frac{1}{\tau}\log\left|\int_0^T(-cv_x(0,t)u(0,t)+cu_x(0,t)v(0,t))\,dt\right|=-2a/c, \tag{3.2}$$

is valid.

Some remarks are in order.

• The v satisfies the wave equation $(1/c)^2v_{tt}=v_{xx}$ and satisfies: if $x+ct>0$, then $v(x,t)\longrightarrow 0$ as $\tau\longrightarrow\infty$; if $x+ct<0$, then $v(x,t)\longrightarrow+\infty$ as $\tau\longrightarrow\infty$.

• The quantity $2a/c$ coincides with the travel time of a signal governed by the wave equation with propagation speed c which starts at the boundary $x=0$ and initial time $t=0$, reflects another boundary $x=a$ and returns to $x=0$. Thus the restriction $T>2a/c$ is quite reasonable and does not against the well known fact: the wave equation has the *finite propagation property.*

• The condition (3.1) ensures that $u_x(0,t)$ can not be identically zero in an interval $]0,T'[\subset]0,T[$. Therefore surely a signal occurs at the initial time. However, it should be emphasized that the formula (3.2) makes use of the *averaged value* of the measured data with an *exponential weight* over the observation time. This is a completely different idea from the well known approach in nondestructive evaluation by sound wave: monitoring of the first *arrival time* of the *echo*, one knows the travel time.

A sketch of the proof of Theorem 3.1. Introduce the function w by the formula

$$w(x)=w(x;\tau)=\int_0^T u(x,t)e^{-\tau t}dt,\ \ 0<x<a.$$

It holds that

$$c^2w''-\tau^2w=e^{-\tau T}(u_t(x,T)+\tau u(x,T))\text{ in }]0,a[,\ cw'(a)=0.$$

Then, this together with integration by parts gives the expression

$$e^{2a\tau/c}\int_0^T(-cv_x(0,t)u(0,t)+cu_x(0,t)v(0,t))\,dt$$

$$=\tau w(a)e^{a\tau/c}-c^{-1}e^{-\tau(T-(2a/c))}\int_0^a(u_t(\xi,T)+\tau u(\xi,T))e^{-\xi\tau/c}d\xi.$$

Now (3.2) can be checked by studying the asymptotic behaviour of this right-hand side with the help of the expression

$$w(a)=-\frac{2cw'(0)}{\tau\left(e^{a\tau/c}-e^{-a\tau/c}\right)}-\frac{e^{-\tau T}}{\tau\left(e^{a\tau/c}-e^{-a\tau/c}\right)}$$

$$\times\left\{\int_0^a(u_t(\xi,T)+\tau u(\xi,T))e^{-\xi\tau/c}d\xi+\int_0^a(u_t(\xi,T)+\tau u(\xi,T))e^{\xi\tau/c}d\xi\right\}$$

together with (3.1).

The proof presented here heavily relies on the spaciality of one-space dimension. In [27] we found another method for the proof which works also for higher space dimensions and applied it to a similar problem for the heat equation. In the following two subsections we present further applications of the method to the wave equations.

3.2 Sound-hard obstacle

Let $D \subset \mathbf{R}^3$ be a bounded open set with smooth boundary such that $\mathbf{R}^3 \setminus \overline{D}$ is connected. Denote by ν the unit outward normal to ∂D. Let $0 < T < \infty$.

Given $f \in L^2(\mathbf{R}^3)$ with compact support satisfying $\text{supp}\, f \cap \overline{D} = \emptyset$ let $u = u(x,t)$ satisfy the initial boundary value problem:

$$\partial_t^2 u - \Delta u = 0 \text{ in} (\mathbf{R}^3 \setminus \overline{D}) \times]0,\, T[,\ \frac{\partial u}{\partial \nu} = 0 \text{ on} \,\partial D \times]0,\, T[,$$

$$u(x,0) = 0,\ \partial_t u(x,0) = f(x) \text{ in} \,\mathbf{R}^3 \setminus \overline{D}.$$

Let Ω be a bounded domain with smooth boundary such that $\overline{D} \subset \Omega$ and $\mathbf{R}^3 \setminus \overline{\Omega}$ is connected. Denote by the same symbol ν the unit outward normal to $\partial\Omega$.

The $\partial\Omega$ is considered as the location of the receivers of the acoustic wave produced by an emitter located at the support of f. In this section we consider the following problem.
Inverse Problem 3.2. Assume that D is *unknown*. Extract information about the location and shape of D from u on $\partial\Omega \times]0,\, T[$ for some fixed *known* f satisfying $\text{supp}\, f \cap \overline{\Omega} = \emptyset$ and $T < \infty$.
Note that u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0,\, T[$ can be computed from u on $\partial\Omega \times]0,\, T[$ by the formula

$$u = z \text{ in} (\mathbf{R}^3 \setminus \overline{\Omega}) \times]0,\, T[\tag{3.3}$$

where z solves the initial boundary value problem in $\mathbf{R}^3 \setminus \overline{\Omega}$:

$$\partial_t^2 z - \Delta z = 0 \text{ in} (\mathbf{R}^3 \setminus \overline{\Omega}) \times]0,\, T[,\ z = u \text{ on} \,\partial\Omega \times]0,\, T[,$$

$$z(x,0) = 0,\ \partial_t z(x,0) = f(x) \text{ in} \,\mathbf{R}^3 \setminus \overline{\Omega}.$$

Thus the problem can be reformulated as
Inverse Problem 3.2'. Extract information about the location and shape of D from u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0,\, T[$ for some known f satisfying $\text{supp}\, f \cap \overline{\Omega} = \emptyset$ and $T < \infty$.

Now we state the result. Let B be an open ball with $\overline{B} \cap \overline{\Omega} = \emptyset$. Choose the initial data $f \in L^2(\mathbf{R}^3)$ in such a way that:
(I1) $f(x) = 0$ a.e. $x \in \mathbf{R}^3 \setminus B$;
(I2) there exists a positive constant C such that $f(x) \ge C$ a.e. $x \in B$ or $-f(x) \ge C$ a.e. $x \in B$.

Set

$$w(x;\tau) = \int_0^T e^{-\tau t} u(x,t)\,dt,\ x \in \mathbf{R}^3 \setminus \overline{D},\ \tau > 0.$$

Our result is the following extraction formula from w and $\partial w/\partial \nu$ on $\partial\Omega \times]0\ T[$ which can be computed from the data u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0,\, T[$.

Theorem 3.2. *Let* $\tau > 0$ *and* $v \in H^1(\mathbf{R}^3)$ *be the weak solution of*

$$(\Delta - \tau^2)v + f(x) = 0 \ in \mathbf{R}^3. \tag{3.4}$$

If the observation time T *satisfies*

$$T > 2dist(D, B) - dist(\Omega, B), \tag{3.5}$$

then there exists a $\tau_0 > 0$ *such that, for all* $\tau \geq \tau_0$

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial\nu}w - \frac{\partial w}{\partial\nu}v\right) dS > 0$$

and the formula

$$\lim_{\tau\longrightarrow\infty} \frac{1}{2\tau} \log \int_{\partial\Omega} \left(\frac{\partial v}{\partial\nu}w - \frac{\partial w}{\partial\nu}v\right) dS = -dist(D, B), \tag{3.6}$$

is valid.

Some remarks are in order.

• The v is unique and is given by the explicit form

$$v(x;\tau) = \frac{1}{4\pi}\int_B \frac{e^{-\tau|x-y|}}{|x-y|}f(y)dy, \ x \in \mathbf{R}^3.$$

• The quantity dist $(D, B) + \sqrt{|\partial B|/4\pi}$ coincides with the distance from the center of B to D and thus (3.6) yields the information about $d_D(p)$ for a given point p in $\mathbf{R}^3 \setminus \overline{\Omega}$.

• It is easy to see that 2dist $(D, B) -$ dist $(\Omega, B) \geq l(\partial B, \partial D, \partial\Omega)$, where $l(\partial B, \partial D, \partial\Omega) = \inf\{|x-y|+|y-z| \,|\, x \in \partial B, y \in \partial D, z \in \partial\Omega\}$. This is the *minimum length of the broken paths* that start at $x \in \partial B$ and reflect at $y \in \partial D$ and return to $z \in \partial\Omega$. Therefore (3.5) ensures that T is greater than the *first arrival time* of a signal with the unit propagation speed that starts at a point on ∂B at $t = 0$, reflects at a point on ∂D and goes to a point on $\partial\Omega$.

The main part of the proof of Theorem 3.2 is to show that

$$\liminf_{\tau\longrightarrow\infty} \tau^4 e^{2\tau \,\text{dist}\,(D,B)} \int_{\partial\Omega} \left(\frac{\partial v}{\partial\nu}w - \frac{\partial w}{\partial\nu}v\right) dS > 0. \tag{3.7}$$

It is a consequence of the following representation formula which corresponds to (1.3) and the estimate for v:

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial\nu}w - \frac{\partial w}{\partial\nu}v\right) dS$$

$$= \int_D |\nabla v|^2 dx + \tau^2 \int_D |v|^2 dx + \int_{\mathbf{R}^3\setminus\overline{D}} |\nabla(w-v)|^2 dx + \tau^2 \int_{\mathbf{R}^3\setminus\overline{D}} |w-v|^2 dx$$

$$+e^{-\tau T}\int_{\mathbf{R}^3\setminus\overline{D}} (w-v)(\partial_t u(x,T) + \tau u(x,T))dx - e^{-\tau T}\int_{\Omega\setminus\overline{D}} (\partial_t u(x,T) + \tau u(x,T))v dx;$$

$$\liminf_{\tau\longrightarrow\infty} \tau^6 e^{2\tau \text{dist}\,(D,B)} \int_D |v|^2 dx > 0. \tag{3.8}$$

Note that the precise values of 4 and 6 of τ^4 in (3.7) and τ^6 in (3.8), respectively are not essential.

3.3 Penetrable obstacle

The method in the former subsection can be applied to a more general case. Given $f \in L^2(\mathbf{R}^3)$ with compact support let $u = u(x,t)$ satisfy the initial value problem:

$$\begin{aligned} &\partial_t^2 u - \nabla \cdot \gamma \nabla u = 0 \text{ in } \mathbf{R}^3 \times]0,\ T[, \\ &u(x,0) = 0,\ \partial_t u(x,0) = f(x) \text{ in } \mathbf{R}^3, \end{aligned} \tag{3.9}$$

where $\gamma = \gamma(x) = (\gamma_{ij}(x))$ satisfies: for each $i, j = 1, 2, 3$ $\gamma_{ij}(x) = \gamma_{ji}(x) \in L^\infty(\mathbf{R}^3)$; there exists a positive constant C such that $\gamma(x)\xi \cdot \xi \ge C|\xi|^2$ for all $\xi \in \mathbf{R}^3$ and a. e. $x \in \mathbf{R}^3$.

We assume: there exists a bounded open set D with a smooth boundary such that $\gamma(x)$ a.e. $x \in \mathbf{R}^3 \setminus D$ coincides with the 3×3 identity matrix I_3. Write $h(x) = \gamma(x) - I_3$ a.e. $x \in D$.

Our second inverse problem is the following.

Inverse Problem 3.3. Assume that both D and h are *unknown* and that one of the following two conditions is satisfied:

(A1) there exists a positive constant C such that $-h(x)\xi \cdot \xi \ge |\xi|^2$ for all $\xi \in \mathbf{R}^3$ and a.e. $x \in D$;

(A2) there exists a positive constant C such that $h(x)\xi \cdot \xi \ge |\xi|^2$ for all $\xi \in \mathbf{R}^3$ and a.e. $x \in D$.

Let Ω be a bounded domain with smooth boundary such that $\overline{D} \subset \Omega$. Extract information about the location and shape of D from u on $\partial\Omega \times]0,\ T[$ for some fixed *known* f satisfying $\text{supp} f \cap \overline{\Omega} = \emptyset$ and $T < \infty$.

Note that u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0,\ T[$ can be computed from u on $\partial\Omega \times]0,\ T[$ by the exactly same formula as (3.3) and thus the problem can be reformulated again as

Inverse Problem 3.3'. Extract information about the location and shape of D from u in $(\mathbf{R}^3 \setminus \overline{\Omega}) \times]0,\ T[$ for some known f satisfying $\text{supp} f \cap \overline{\Omega} = \emptyset$ and $T < \infty$.

Now we state our second result.

Theorem 3.3. *Assume that γ satisfies (A1) or (A2). Let f satisfy (I1) and (I2) in subsection 3.2 and v be the weak solution of (3.4). Let T satisfies (3.5) and w be given by*

$$w(x;\tau) = \int_0^T e^{-\tau t} u(x,t)\,dt,\ x \in \mathbf{R}^3,\ \tau > 0$$

with solution u of (3.9). If (A1) is satisfied, then there exists a $\tau_0 > 0$ such that, for all $\tau \ge \tau_0$

$$\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS > 0;$$

if (A2) is satisfied, then there exists a $\tau_0 > 0$ such that, for all $\tau \ge \tau_0$

$$-\int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS > 0.$$

In both cases we have

$$\lim_{\tau \longrightarrow \infty} \frac{1}{2\tau} \log \left| \int_{\partial\Omega} \left(\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS \right| = -dist(D, B).$$

The key points for the proof are an estimate for ∇v similar to (3.8) and the following two representation formula:

$$\int_{\partial\Omega}\{(\nabla v\cdot\nu)w-(\gamma\nabla w\cdot\nu)v\}dS=-\int_D h\nabla v\cdot\nabla v dx$$

$$+\int_{\mathbf{R}^3}\gamma\nabla(w-v)\cdot\nabla(w-v)dx+\tau^2\int_{\mathbf{R}^3}|w-v|^2dx$$

$$+e^{-\tau T}\int_{\mathbf{R}^3}(\partial_t u(x,T)+\tau u(x,T))(w-v)dx-e^{-\tau T}\int_{\Omega}(\partial_t u(x,T)+\tau u(x,T))vdx;$$

$$-\int_{\partial\Omega}\{(\nabla v\cdot\nu)w-(\gamma\nabla w\cdot\nu)v\}dS=\int_D h\nabla w\cdot\nabla w dx$$

$$+\int_{\mathbf{R}^3}\nabla(v-w)\cdot\nabla(v-w)dx+\tau^2\int_{\mathbf{R}^3}|v-w|^2dx$$

$$-e^{-\tau T}\int_{\mathbf{R}^3}(\partial_t u(x,T)+\tau u(x,T))(v-w)dx+e^{-\tau T}\int_{\Omega}(\partial_t u(x,T)+\tau u(x,T))vdx.$$

4 Summary and further research direction

In this paper we presented: past applications of the probe and enclosure methods to inverse obstacle scattering problems with a fixed wave number and related open problems; recent applications of the enclosure method to *inverse obstacle scattering problems with dynamical data over a finite time interval.*

In particular, in Section 3 we presented a new and simple method in [23] for a typical class of inverse obstacle scattering problems that employs the values of the wave field over a *finite* time interval on a known surface surrounding unknown obstacles as the observation data. The wave field is generated by an initial data localized outside the surface and its form is not specified except for the condition on the support. The method *explicitly* yields information about the location and shape of the obstacles *more than* the convex hull.

It would be interesting to apply the method presented in Section 3 to other time dependent problems in electromagnetism(e.g., *subsurface radar* [4], *microwave tomography* [41]), linear elasticity, classical fluids etc.. Those applications belong to our future plan.

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