数理解析研究所講究録
第 1702 巻 2010 年 51-64

Commutative gain matrices in delayed feedback controls

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1 Introduction

The delayed feedback control (DFC) is proposed by Pyragas [5] as a method of chaos controls. To stabilize unstable periodic orbits of a differential equation

(1) $$x'(t) = f(x) \quad (x \in \mathbb{R}^d),$$

DFC uses the feedback control input given by the difference between the delayed state and the current state:

(2) $$y'(t) = f(y) + K(y(t-\omega) - y(t)).$$

Here, K is a $d \times d$ matrix and we call it the **gain matrix**, and ω is a time delay. If this time delay coincides with the period of one of the unstable periodic orbits of Eq.(1), then the solution of Eq.(1) is also a solution of Eq.(2). To achieve the stabilization of the desired unstable periodic orbit, the time of delay ω and the gain matrix K should be adjusted. In many works ω and K are adjusted in numerical experiment, by using the amplitude of the feedback control input as a criterion: If the amplitude tends to zero as t increases, then DFC succeeds. But there are few analytical results how to choose ω and K to achieve the stabilization.

Recently, in [3], we give an analytical result on this subject for a special case where the gain matrix is given by $K = kE$ ($k \in \mathbb{R}$, E is $d \times d$ identity matrix). To judge the stability of a periodic orbit we can use characteristic multipliers of the first variational equation around the orbit. But there are few information about characteristic multipliers or periodic operators for delay differential equations. In the paper, we have introduced newly a mapping named "C-map". C-map gives a relationship between characteristic multipliers of the first variational equations around the unstable periodic orbit of Eqs.(2)

and (1) and enables to judge the stability of periodic orbits of Eq.(2). This is a very useful and strong tool.

The aim of the present article is to establish C-map theorem for discrete systems as in the paper [3]. The discrete versions of DFC are also proposed to stabilize unstable periodic orbits with period $\omega \in \mathbb{N}$ of a difference equation

$$x(n+1) = f(x(n)), \qquad (x \in \mathbb{R}^d). \tag{3}$$

There are two methods. One of the methods is

$$y(n+1) = f(y(n)) + K(y(n-\omega) - y(n)), \tag{4}$$

which is proposed by Pyragas[5] and Ushio [6]. Another method is

$$y(n+1) = f(y(n)) + K(y(n+1-\omega) - f(y(n)), \tag{5}$$

which is proposed by Buchner and Żebrowski [1]. The latter case comes from the idea that the feedback term should be not $K(y(n-\omega) - y(n))$, but

$$K(y(n+1-\omega) - y(n+1)) \approx K(y(n+1-\omega) - f(y(n)).$$

The former case is more familiar as the discrete DFC than latter case. But it is hard to apply our analytical processes for the continuous case. On the other hand, we can apply it to the latter case very well. So we confirm that the latter case is more natural as the discrete version of DFC and in this paper we consider the latter case.

2 Summary of the results for continuous case

We will summarize the results in [3]. Let $\phi(t)$ be an unstable periodic orbit of Eq.(1) and ω the period of $\phi(t)$. Then $\phi(t)$ is also a solution of Eq.(2). Analytically, we could consider that the stabilization of $\phi(t)$ is successful when $\phi(t)$ is orbitally stable as a solution of Eq.(2). Consider the first variational equations around $\phi(t)$ such that

$$x'(t) = A(t)x(t) \tag{L$_\mathrm{c}$}$$

and

$$y'(t) = A(t)y(t) + K(y(t-\omega) - y(t)), \tag{CP$_\mathrm{c}$}$$

where $A(t) = Df(\phi(t))$ is a Jacobian matrix of f. Obviously, $A(t)$ is an ω-periodic matrix. The stability of $\phi(t)$ as a solution of Eqs.(1) and (2) are governed by the characteristic multipliers (which will be defined in the next paragraph) of Eqs.($\mathrm{L_c}$) and ($\mathrm{CP_c}$), respectively.

Let $T(t,s)$ be the solution operator of Eq.(L_c) defined on $\mathbb{C}^d$. The eigenvalue μ of the periodic operator $T(\omega,0)$, i.e., $\mu \in \sigma(T(\omega,0))$, is called a **characteristic multiplier of Eq.(L_c)**. Let $U(t,s)$ be the solution operator of Eq.(CP_c) defined on $C([-\omega,0],\mathbb{C}^d)$. Note that the periodic operator $U(\omega,0)$ is a compact operator. The point spectrum ν of the operator, i.e., $\nu \in P_\sigma(U(\omega,0)) = \sigma(U(\omega,0)) \setminus \{0\}$, is called a **characteristic multiplier of Eq.(CP_c)**. Since $f(\phi(t))$ is a periodic solution of both Eqs.(L_c) and (CP_c), $1 \in \sigma(T(\omega,0))$ and $1 \in P_\sigma(U(\omega,0))$. If any other points $\nu \in P_\sigma(U(\omega,0)) \setminus \{1\}$ have modulus less than one, i.e. $|\nu| < 1$, the periodic orbit $\phi(t)$ of the nonlinear Eq.(2) is orbitally stable; in this case we say that **the stabilization of $x = \phi(t)$ by DFC with feedback gain K succeeds.**

In [3] we had a relationship between characteristic multipliers of Eq.(L_c) and Eq.(CP_c), under the assumption that the commutative condition

$$A(t)K = KA(t) \quad (\forall t \in \mathbb{R}). \tag{6}$$

Theorem A ([3, Theorem 3.7]). Assume that the commutative condition (6) holds. Then $\nu \in P_\sigma(U(\omega,0))$ if and only if there exist $\kappa \in \sigma(K)$ and $\mu \in \sigma(T(\omega,0))$ such that

$$g_\kappa(\nu) = \mu, \quad W_K(\kappa) \cap W_{T(\omega,0)}(\mu) \neq \{0\}.$$

Here $g_\kappa(z) := ze^{\omega(1-z^{-1})\kappa}$ for $z \in \mathbb{C} \setminus \{0\}$.

We call the map g_κ **characteristic map, in short, C-map.** By using the above C-map theorem, we can obtain a criterion of the gain matrix for DFC to succeed.

Theorem B ([3, Theorem 3.9]). Assume $K = kE$.

(i) If there exists $\mu \in \sigma_U := \{\mu \in \sigma(T(\omega,0)) \colon |\mu| > 1\}$ such that $\mu > 1$, then there exists $\nu \in P_\sigma(U(\omega,0))$ such that $\nu > 1$.

(ii) Let $\sigma_U \subset (-e^2,-1)$ and $\alpha_0 = \max\{\log|\mu| : \mu \in \sigma_U\}$. For any k, if

$$\frac{\alpha_0}{2\omega} < k < \frac{\beta(\alpha_0)}{2\omega}$$

holds, then $|\nu| < 1$ or $\nu = 1$ for any $\nu \in P_\sigma(U(\omega,0))$.

Refer to [3] for the notation β. This theorem guarantees that DFC can be stabilize the unstable periodic orbit of Eq.(1) whose unstable characteristic multipliers are more are than $-e^2$ and less than -1. Moreover, it gives concretely the gain matrix to achieve the stabilization. This might be a new and very useful result. C-map theorem plays an important role in proving this result, because we have few information about the periodic operator $U(\omega,0)$.

3 C-map theorem for discrete system

Let $\phi(n)$ be an unstable periodic orbit of Eq.(3). We consider the first variational equations around $\phi(n)$ of Eqs.(3) and (5):

$$(\mathrm{L_d}) \qquad x(n+1) = A(n)x(n),$$

and

$$(\mathrm{CB_d}) \qquad y(n+1) = A(n)y(n) + K\{y(n+1-\omega) - A(n)y(n)\},$$

respectively. Here $A(n) = Df(\phi(n))$ which is an ω-periodic matrix. We assume that $A(n)$ is nonsingular for any $n \in \mathbb{Z}$. Let $T(m,n)$ be the solution operator of Eq.($\mathrm{L_d}$) defined on $\mathbb{R}^d$, and $U(m,n)$ the solution operator of Eq.($\mathrm{CB_d}$) defined on $\mathcal{C}_{\omega-1}$. Here we use the following notations: Define

$$[m:n] := \{m, m+1, \ldots, n-1, n\}, \quad (m < n, \quad m,n \in \mathbb{Z}).$$

If $n = \infty$, then we use a notation $[m:\infty)$. For a positive integer ω

$$\mathcal{C}_{\omega-1} := \{\phi\colon\ [-(\omega-1):0] \ \to \mathbb{R}^d\}.$$

Note that $\mathcal{C}_{\omega-1}$ is a Banach space with a norm $|\varphi|_{\mathcal{C}_{\omega-1}} = \max_{s\in[-\omega+1:0]} |\varphi(s)|$ and the dimension is ωd.

The eigenvalues of the periodic operators $T(\omega,0)$ and $U(\omega,0)$ are called **characteristic multipliers of Eqs.($\mathrm{L_d}$) and ($\mathrm{CB_d}$)**, respectively. If any $\nu \in \sigma(U(\omega,0))$ has modulus less than one, i.e. $|\nu| < 1$, the periodic orbit $y = \phi(n)$ of the nonlinear Eq.(5) is stable; in this case we say that **the stabilization of $x = \phi(n)$ by Buchner type DFC with feedback gain K succeeds.**

We will give C-map theorem for the discrete system. To prove C-map theorem, we need two more assumptions (KU) and (KI) in addition to commutative condition (KA).

Assumption H:

(KA): $KA(n) = A(n)K$, $(n \in \mathbb{Z})$;

(KU): $\sigma(U(\omega,0)) \cap \sigma(K) = \emptyset$;

(KI): $1 \notin \sigma(K)$.

C-map is given by the following function $g(k,z)$:

$$g(k,z) := z[z(z-k)^{-1}(1-k)]^{-\omega}, \quad (k,z \in \mathbb{C}).$$

In the following $W_A(\alpha)$ represents the eigenspace of A corresponding to the spectrum $\alpha \in \sigma(A)$.

Theorem 1. (C-map theorem) Let Assumption H be satisfied. Then $\nu \in \sigma(U(\omega,0))$ if and only if there exist $\kappa \in \sigma(K)$ and $\mu \in \sigma(T(\omega,0))$ such that

$$g(\kappa,\nu)=\mu, \quad W_K(\kappa)\cap W_{T(\omega,0)}(\mu)\neq\{0\}.$$

Corollary 1. Assume that $K=kE$ $(k\neq 0,1)$. Then

$$\nu\in\sigma(U(\omega,0)) \Longleftrightarrow g(k,\nu)\in\sigma(T(\omega,0)).$$

4 Proof of C-map theorem

To prove our C-map theorem (Theorem 1), we have to prepare some fundamental lemmas.

4.1 Solution operators of Eqs.($\mathrm{L_d}$) and ($\mathrm{CB_d}$).

The solution operator $T(n,m)$, $(m,n\in\mathbb{Z})$ of Eq.($\mathrm{L_d}$) and the periodic operator $T(\omega,0)$ are given as follows:

$$(7)\quad T(n,m)=\begin{cases}\prod_{i=m}^{n-1}A(i) & (n\geq m)\\ T^{-1}(m,n)=\left(\prod_{i=n}^{m-1}A(i)\right)^{-1} & (n<m),\end{cases}\qquad T(\omega,0)=\prod_{i=0}^{\omega-1}A(i).$$

Since $T(\omega,0)$ is nonsingular, it is clear that $\mu\in\sigma(T(\omega,0))\Longrightarrow\mu\neq 0$.

Lemma 1 ([2]). The solution operator $T(n,m)$, $(m,n\in\mathbb{Z})$ of Eq.($\mathrm{L_d}$) has the following properties:

1) $T(n,n)=E$.
2) $T(n,r)T(r,m)=T(n,m)$.
3) $T(n+\omega,m+\omega)=T(n,m)$.
4) $T(m+n\omega,m)=T^n(m+\omega,m)$.
5) $T(m+n\omega,r)=T(m,0)T^n(\omega,0)T(0,r)$.

The proofs of the following two lemmas are easy so that they are omitted.

Lemma 2. The following statements are equivalent:

1) $\mu\in\sigma(T(\omega,0))$.
2) There is a nontrivial solution $x(n)$ of Eq.($\mathrm{L_d}$) such that $x(n+\omega)=\mu x(n)$, $n\in\mathbb{Z}$.
3) There is a nontrivial solution $x(n)$ of Eq.($\mathrm{L_d}$) such that $x(\omega)=\mu x(0)$.
4) There is a nontrivial solution $x(n)$ of Eq.($\mathrm{L_d}$) such that $x(0)\in N(T(\omega,0)-\mu E)$.

We give several conditions equivalent to the commutative condition (KA). The proof is similar to [3]

Lemma 3. In Eq.(L_d), the following statements are equivalent:
1) $A(n)K = KA(n), \ n \in \mathbb{Z}$.
2) $T(n,m)K = KT(n,m), \ n,m \in \mathbb{Z}$.
3) $T(n,0)K = KT(n,0), \ n \in \mathbb{Z}$.

To discuss the properties with the solutions of (CB_d) in the function space $\mathcal{C}_{\omega-1}$, we introduce a new notation. For a function $y : [m+1-\omega : \infty) \to \mathbb{C}^d$ and any n, $(n \geq m)$, we define a function $y_n \in \mathcal{C}_{\omega-1}$ as follows :

$$y_n(s) = y(n+s), \quad s \in [-\omega+1 : 0].$$

We will denote the solution of Eq.(CB_d) with an initial condition $y_m = \varphi$ by $y(\cdot; m, \varphi)$. For any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, $(n \geq m)$, the solution operator $U(n,m) : \mathcal{C}_{\omega-1} \to \mathcal{C}_{\omega-1}$ of Eq.(CB_d) is given by

$$[U(n,m)\varphi](\theta) = y(n+\theta; m, \varphi), \quad \theta \in [-\omega+1 : 0], \ \varphi \in \mathcal{C}_{\omega-1}.$$

Then $y_n(m,\varphi) = U(n,m)\varphi$.

Theorem 2. For any $(m,\varphi) \in \mathbb{Z} \times \mathcal{C}_{\omega-1}$, there exists a unique solution of Eq.(CB_d) on $[m : \infty)$ satisfying an initial condition $y_m = \varphi$.

We can easily find that the same properties as Lemma 1 for Eq.(L_d) hold.

Lemma 4. The solution operator $U(n,m)$ of Eq.(CB_d) has the following properties:
1) $U(n,n) = I$ (I is the identity operator).
2) $U(n,m)U(m,r) = U(n,r)$, $r \leq m \leq n$.
3) $U(n+\omega, m+\omega) = U(n,m)$, $m \leq n$.
4) $U(n+\omega, m) = U(n,m)U(m+\omega, m)$, $m \leq n$.

4.2 An extended system of Eq.(CB_d)

By changing of

$$y(n-\omega+i) = z_i(n), \quad i = 1, 2, \cdots, \omega$$

in Eq.(CB_d), we can obtain

$$z_i(n+1) = z_{i+1}(n), \ (i = 1, \cdots, \omega-1), \ z_\omega(n+1) = A(n)z_\omega(n) + K(z_1(n) - A(n)z_\omega(n)).$$

Put

$$z(n) = {}^t(z_1(n), z_2(n), \cdots, z_\omega(n)),$$

$$B_K(n) = \begin{pmatrix} 0 & E & 0 & \cdots & 0 & 0 \\ 0 & 0 & E & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & E \\ K & 0 & 0 & \cdots & 0 & (E-K)A(n) \end{pmatrix}. \tag{8}$$

Then Eq.($\mathrm{CB_d}$) becomes

$$z(n+1) = B_K(n)z(n). \tag{9}$$

This equation is called an extended system of Eq.($\mathrm{CB_d}$). Under Assumption (KI), solution operator $T_B(n,m)$ and periodic operator $T_B(n)$ of Eq.(9) are given by

$$T_B(n,m) = \begin{cases} \prod_{i=m}^{n-1} B_K(i) & (n \geq m) \\ T_B^{-1}(m,n) = \left(\prod_{i=n}^{m-1} B_K(i)\right)^{-1} & (n < m), \end{cases} \quad T_B(n) = \prod_{i=n}^{n+\omega-1} B_K(i).$$

Define a mapping $S_{\omega-1}$ from $\mathcal{C}_{\omega-1}$ into $\mathbb{C}^{\omega d}$ by

$$\varphi \in \mathcal{C}_{\omega-1} \mapsto {}^t(\varphi(-\omega+1), \varphi(-\omega+2), \cdots, \varphi(-1), \varphi(0)) \in \mathbb{C}^{\omega d}.$$

Then $S_{\omega-1}$ is bijective. For the solution $y_n(m,\varphi) = U(n,m)\varphi$ of Eq.($\mathrm{CB_d}$) with the initial condition $y_m = \varphi \in \mathcal{C}_{\omega-1}$ and for the solution $z(n) = T_B(n,m){}^t(\varphi(-\omega+1), \varphi(-\omega+2), \cdots, \varphi(-1), \varphi(0))$ of Eq.(9) with the initial condition $z(m) = {}^t(\varphi(-\omega+1), \varphi(-\omega+2), \cdots, \varphi(-1), \varphi(0))$, we have

$$\begin{aligned} S_{\omega-1}U(n,m)\varphi &= S_{\omega-1}y_n(m,\varphi) \\ &= \begin{pmatrix} y(n-(\omega-1);m,\varphi) \\ y(n-(\omega-2);m,\varphi) \\ \cdots \\ y(n-1;m,\varphi) \\ y(n;m,\varphi) \end{pmatrix} = \begin{pmatrix} z_1(n) \\ z_2(n) \\ \cdots \\ z_{\omega-1}(n) \\ z_\omega(n) \end{pmatrix} \\ &= T_B(n,m) \begin{pmatrix} \varphi(-\omega+1) \\ \varphi(-\omega+2) \\ \cdots \\ \varphi(-1) \\ \varphi(0) \end{pmatrix} = T_B(n,m)S_{\omega-1}\varphi. \end{aligned}$$

Summarizing the above facts we obtain the following result.

Theorem 3. Under Assumptions (KA),(KI), the following relations hold true :

$$S_{\omega-1}U(n,m) = T_B(n,m)S_{\omega-1}, \quad (n \geq m).$$

Corollary 2. Under Assumptions (KA),(KI), $S_{\omega-1}U(\omega,0)S_{\omega-1}^{-1} = T_B(0)$.

Corollary 3. Under Assumptions (KA),(KI), if $\nu \in \sigma(U(\omega,0))$, $\nu \neq 0$ holds.

Since Theorem 3 implies

$$U(\omega,0)\varphi = \nu\varphi \Longleftrightarrow S_{\omega-1}U(\omega,0)\varphi = \nu S_{\omega-1}\varphi \Longleftrightarrow T_B(0)S_{\omega-1}\varphi = \nu S_{\omega-1}\varphi,$$

we obtain the following result.

Theorem 4. Under Assumptions (KA),(KI), the following statements are equivalent :

1) $\nu \in \sigma(U(\omega,0))$.
2) $\nu \in \sigma(T_B(0))$.
3)

$$|T_B(0) - \nu E| = \begin{vmatrix} K-\nu E & 0 & \cdots & 0 & H_0^1 \\ KH_1^2 & K-\nu E & \cdots & 0 & H_0^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ KH_1^{\omega-1} & KH_2^{\omega-1} & \cdots & K-\nu E & H_0^{\omega-1} \\ KH_1^{\omega} & KH_2^{\omega} & \cdots & KH_{\omega-1}^{\omega} & H_0^{\omega}+K-\nu E \end{vmatrix} = 0.$$

From the above theorem we get the following result.

Theorem 5. The number of all characteristic multipliers of Eq.($\mathrm{CB_d}$) is ωd.

Theorem 6. Assume that $A(n)$ and $K \neq 1$ in Eq.($\mathrm{CB_d}$) are one dimensional. Then

$$|T_B(0) - \nu E| = 0 \Longleftrightarrow (\nu - K)^{\omega} - \nu^{\omega-1}(1-K)^{\omega}\prod_{i=0}^{\omega-1} A(i) = 0.$$

4.3 Properties of characteristic multipliers of Eq.($\mathrm{CB_d}$)

Lemma 5. Let $\nu \neq 0$. If $y(n)$ is a solution of Eq.($\mathrm{CB_d}$) defined on $[0:\infty)$ and satisfies $y_{n+\omega} = \nu y_n$ for $n \geq 0$, then there exists a solution $y(n)$ of Eq.($\mathrm{CB_d}$) defined on $\mathbb{Z}$ uniquely and satisfies $y_{n+\omega} = \nu y_n$ for $n \in \mathbb{Z}$.

Proof. Let $y_n = U(n,0)y_0$ $(n \geq 0)$. Since $y_{n+\omega} = \nu y_n$ holds for $n \geq 0$, substituting $n = 0$, we have $U(\omega,0)y_0 = \nu y_0$. From 3) in Lemma 4, we can find $U(\omega,0) = U(0,-\omega)$. Thus $U(0,-\omega)\nu^{-1}y_0 = y_0$ holds. If we put $y_{-\omega} = \nu^{-1}y_0$, then $y(n)$ is defined on $[-2\omega+1:-\omega]$ and $U(0,-\omega)y_{-\omega} = y_0$. The solution z_n of Eq.($\mathrm{CB_d}$) with the initial condition $z_{-\omega} = y_{-\omega}$ is defined for $n \geq -\omega$ and given by $z_n = U(n,-\omega)y_{-\omega}$. For $n \geq 0$, $z_n = y_n$ holds. In fact,

$$z_n = U(n,-\omega)y_{-\omega} = U(n,0)U(0,-\omega)y_{-\omega} = U(n,0)y_0 = y_n.$$

Consider the case where $0 > n \geq -\omega$. From 3) in Lemma 4,

$$z_n = U(n,-\omega)y_{-\omega} = U(n+\omega,0)y_{-\omega},$$

which implies

$$\nu z_n = U(n+\omega,0)\nu y_{-\omega} = U(n+\omega,0)y_0 = y_{n+\omega} = z_{n+\omega},$$

because $n+\omega \geq 0$. Denote z_n by y_n. Then $y_n = U(n,-\omega)y_{-\omega}$ is defined for $n \geq -\omega$ and $y_{n+\omega} = \nu y_n, n \geq -\omega$ holds.

Next, we will extend the similar argument as the above to $n \geq -2\omega$. By putting $y_{-2\omega} = \nu^{-1}y_{-\omega}$, $y(n)$ is defined on $[-3\omega+1 : -2\omega]$ and $U(-\omega,-2\omega)y_{-2\omega} = y_{-\omega}$ holds. Consider the solution z_n of Eq.(CB$_d$) given by $z_n = U(n,-2\omega)y_{-2\omega}$ for $n \geq -2\omega$. For $n \geq -\omega$, $z_n = y_n$ holds. If $-\omega > n \geq -2\omega$, then

$$z_n = U(n,-2\omega)y_{-2\omega} = U(n+\omega,-\omega)y_{-2\omega}.$$

Because of $n+\omega \geq -\omega$,

$$\nu z_n = U(n+\omega,-\omega)\nu y_{-2\omega} = U(n+\omega,-\omega)y_{-\omega} = y_{n+\omega} = z_{n+\omega}.$$

Denote z_n by y_n. Then $y_n = U(n,-2\omega)y_{-2\omega}$ is defined for $n \geq -2\omega$ and $y_{n+\omega} = \nu y_n, n \geq -2\omega$ holds.

In the same way, the solution y_n of Eq.(CB$_d$) can be defined on the whole $\mathbb{Z}$ and $y_{n+\omega} = \nu y_n$ for $n \in \mathbb{Z}$ holds. □

Lemma 6. Under Assumptions (KA),(KI), the following statements are equivalent:

1) $\nu \in \sigma(U(\omega,0))$.

2) There exists a nontrivial solution $y(n)$ of Eq.(CB$_d$) such that $y(n+\omega) = \nu y(n), n \in \mathbb{Z}$.

3) There exists a nontrivial solution $y(n)$ of Eq.(CB$_d$) for $n \geq 0$ such that $y_0 \in N(U(\omega,0)-\nu I)$.

Proof. 1) $\Longrightarrow$ 2). It follows from Corollary 3 that $0 \notin \sigma(U(\omega,0))$. If $\nu \in \sigma(U(\omega,0))$, then there exists ϕ such that $U(\omega,0)\phi = \nu\phi$ $(\phi \neq 0)$. By multiplying $U(n+\omega,\omega)$ the both sides of this equation, we obtain $U(n+\omega,\omega)U(\omega,0)\phi = \nu U(n+\omega,\omega)\phi$. Since $U(n+\omega,\omega)U(\omega,0) = U(n+\omega,0)$ holds from 2) in Lemma 4, we have

$$U(n+\omega,0)\phi = \nu U(n,0)\phi.$$

Letting $y(n)$ be a solution of Eq.(CB$_d$) such that $y_0 = \phi$, y_n is given by $y_n = U(n,0)\phi$. Then $y_{n+\omega} = \nu y_n, n \geq 0$ holds. From Lemma 5, the solution y_n of Eq.(CB$_d$) is defined on the whole $\mathbb{Z}$ and $y_{n+\omega} = \nu y_n, n \in \mathbb{Z}$ holds.

2) $\Longrightarrow$ 1). If $\nu = 0$, then $y(n) = 0, n \in \mathbb{Z}$, which contradicts that $y(n)$ is a nontrivial solution. Hence $\nu \neq 0$. Since $y(n)$ is nontrivial, there exists $\phi \in C_{\omega-1}$ such that $\phi \neq 0$ and $y_n = U(n,0)\phi$. From the assumption $y(n+\omega) = \nu y(n)$, $U(n+\omega,0)\phi = \nu U(n,0)\phi$. Since $U(n+\omega,0) = U(n,0)U(\omega,0)$ holds from 4) in Lemma 4, we have $U(n,0)U(\omega,0)\phi =$

$\nu U(n,0)\phi$, that is, $U(n,0)[U(\omega,0)\phi - \nu\phi] = 0, \quad \forall n \geq 0$. By putting $n = 0$, we have $U(\omega,0)\phi - \nu\phi = 0$, that is, $U(\omega,0)\phi = \nu\phi$. This implies that $\nu \in \sigma(U(\omega,0))$.

2) $\Longrightarrow$ 3). From the assumption, $y_{n+\omega} = \nu y_n$ holds. Then, by putting $n = 0$, we obtain $y_\omega = \nu y_0$, that is, $U(\omega,0)y_0 = \nu y_0$. Hence the statement 3) holds.

3) $\Longrightarrow$ 2). 3) implies $U(\omega,0)y_0 = \nu y_0$ $(y_0 \neq 0)$. By multiplying $U(n+\omega,\omega)$, we can obtain that $U(n+\omega,0)y_0 = \nu U(n,0)y_0$ for $n \geq 0$. That is, $y_{n+\omega} = \nu y_n$ holds for $n \geq 0$. From Lemma 5, we also obtain $y_{n+\omega} = \nu y_n$ for $n \in \mathbb{Z}$. $\square$

4.4 A reduced equation of Eq.(CB_d).

For $\nu \in \mathbb{C}$ and $\nu \notin \sigma(K) \cup \{0\}$, we call the following equation the reduced equation of Eq.(CB_d):

$$y(n+1) = K(\nu)A(n)y(n), \tag{10}$$

where $K(\nu) = \nu(\nu E - K)^{-1}(E - K)$. We note that under Assumption (KI) and $\nu \notin \sigma(K) \cup \{0\}$, $K(\nu)A(n)$ is nonsingular and ω-periodic. Then Lemmas 1 and 2 hold for the solution operator $T_{(10)}(m,n;\nu^{-1})$ of Eq.(10) instead of the solution operator of Eq.(L_d).

Lemma 7. Let Assumption H hold. The solution operator $T_{(10)}(n,m;\nu^{-1})$ of the reduced equation (10) is given by

$$T_{(10)}(n,m;\nu^{-1}) = K(\nu)^{n-m}T(n,m). \tag{11}$$

Proof. From Assumption (KI) and $\nu \notin \sigma(K) \cup \{0\}$, $K(\nu)A(n)$ is nonsingular and ω-periodic. By using the commutative assumption (KA), the formulas (11) are obtained as follows:

If $n \geq m$, then

$$T_{(10)}(n,m;\nu^{-1}) = \prod_{i=m}^{n-1}(K(\nu)A(i)) = K(\nu)^{n-m}T(n,m).$$

If $n < m$, then

$$\begin{aligned} T_{(10)}^{-1}(m,n;\nu^{-1}) &= K(\nu)^{n-m}\left(\prod_{i=n}^{m-1}A(i)\right)^{-1} \\ &= K(\nu)^{n-m}T^{-1}(m,n) = K(\nu)^{n-m}T(n,m) \end{aligned}$$

holds. $\square$

Lemma 8. Let Assumption H hold and $y(n)$ be a function such that

$$y(n+\omega) = \nu y(n), \quad n \in \mathbb{Z}. \tag{12}$$

Then $y(n)$ is a nontrivial solution of Eq.(CB_d) if and only if it is a nontrivial solution of Eq.(10).

Proof. If $y(n)$ is a nontrivial solution of Eq.($\mathrm{CB_d}$), from Lemma 6, then $\nu \in \sigma(U(\omega, 0))$. Hence $y(n+1-\omega) = \nu^{-1}y(n+1), \quad n \in \mathbb{Z}$. Substituting this relation to Eq.($\mathrm{CB_d}$), we have

$$y(n+1) = A(n)y(n) + K\{\nu^{-1}y(n+1) - A(n)y(n)\},$$

which implies

$$(\nu E - K)y(n+1) = \nu(E-K)A(n)y(n).$$

From Assumption (KU) and $\nu \in \sigma(U(\omega,0))$, $\nu \notin \sigma(K)$. Then we can find that $\nu E - K$ is nonsingular and $y(n)$ satisfies Eq.(10). If $y(n)$ is a nontrivial solution of Eq.(10), by following the above argument backward, we can easily see that $y(n)$ is a solution of Eq.($\mathrm{CB_d}$). □

Theorem 7. Under Assumption H, the following statements are equivalent:
1) $\nu \in \sigma(U(\omega, 0))$.
2) $\nu \in \sigma(T_{(10)}(\omega, 0; \nu^{-1}))$.

Proof. From Lemma 6, 1) holds if and only if there exists a nontrivial solution of Eq.($\mathrm{CB_d}$) satisfying the relation (12). From Assumption H and Lemma 8, $y(n)$ is a nontrivial solution of Eq.(10). Moreover, from Lemma 2, this is equivalent to $\nu \in \sigma(T_{(10)}(\omega, 0; \nu^{-1}))$. □

Set

$$\Delta(\gamma) = \gamma E - K(\gamma)^{\omega} T(\omega, 0).$$

Then Theorem 7 is refined as follows.

Theorem 8. Under Assumption H, the following statements are equivalent :
1) $\nu \in \sigma(U(\omega, 0))$.
2) $\nu \in \sigma(K(\nu)^{\omega} T(\omega, 0))$.
3) $\det \Delta(\nu) = 0$.
4) $\det(\nu K(\nu)^{-\omega} - T(\omega, 0)) = 0$.
5) $0 \in \sigma(\nu K(\nu)^{-\omega} - T(\omega, 0))$.

Proof. By the formula (11), Theorem 7 implies that the statements 1) and 2) are equivalent. The equivalence of 2) and 3) is obvious. The matrix $\Delta(\nu)$ is rewritten as

$$\Delta(\nu) = K(\nu)^{\omega}\left(\nu K(\nu)^{-\omega} - T(\omega, 0)\right).$$

Since $K(\nu)^{\omega}$ is nonsingular, the conditions 3) and 4) are equivalent. The equivalence of 4) and 5) is obvious. □

4.5 Proof of C-map theorem.

We are now in a position to prove C-map theorem. The following two lemmas on relations between two commutative matrices play an important role. In the following $G_A(\alpha)$ represents the generalized eigenspace of A corresponding to the spectrum $\alpha \in \sigma(A)$.

Lemma 9 ([3], Lemma 2.1). If two matrices A and B are commutative, then

$$\sigma(A - B) = \{\alpha - \beta \mid \alpha \in \sigma(A), \beta \in \sigma(B), G_A(\alpha) \cap G_B(\beta) \neq \{0\}\}.$$

Lemma 10 ([3], Lemma 2.2). Let two matrices A and B be commutative and $\alpha \in \sigma(A), \beta \in \sigma(B)$. Then $G_A(\alpha) \cap G_B(\beta) \neq \{0\}$ if and only if $W_A(\alpha) \cap W_B(\beta) \neq \{0\}$.

Proof of Theorem 1. It is follows from Theorem 8 that $\nu \in \sigma(U(\omega, 0))$ if and only if

$$0 \in \sigma(g(K, \nu) - T(\omega, 0)). \tag{13}$$

Since $g(k, \nu) = \nu \left(\frac{\nu-k}{\nu(1-k)}\right)^\omega$ is holomorphic on $\mathbb{C} \setminus \{1\}$, by the spectrum mapping theorem we have

$$\sigma(g(K, \nu)) = \{g(k, \nu) \mid k \in \sigma(K)\}.$$

From Assumption (KA) and Lemma 3, we can find that $g(K, \nu)$ and $T(\omega, 0)$ are commutative. Therefore, by Lemma 9, the condition (13) shows that $\nu \in \sigma(U(\omega, 0)) \setminus \{0\}$ if and only if there exist $k_0 \in \sigma(K)$ and $\mu \in \sigma(T(\omega, 0))$ such that

$$g(k_0, \nu) = \mu, \quad G_{g(K,\nu)}(g(k_0, \nu)) \cap G_{T(\omega,0)}(\mu) \neq \{0\}. \tag{14}$$

For such a $k_0 \in \sigma(K)$, we denote by $\{k_0, k_1, \cdots, k_p\}$, $p \leq d + 1$ the set of $k \in \sigma(K)$ such that $g(k, \nu) = g(k_0, \nu)$. Using the spectrum mapping theorem again, we have

$$G_{g(K,\nu)}(g(k_0, \nu)) = \bigoplus_{i=0}^{p} G_K(k_i).$$

Therefore we see that $G_{g(K,\nu)}(g(k_0, \nu)) \cap G_{T(\omega,0)}(\mu) \neq \{0\}$ if and only if $G_{T(\omega,0)}(\mu) \cap \bigoplus_{i=0}^p G_K(k_i) \neq \{0\}$. Then $x \in G_{T(\omega,0)}(\mu) \cap \bigoplus_{i=1}^p G_K(k_i), x \neq 0$, is expressed as $x = \sum_{i=0}^p P_i x$, $P_i x \in G_K(k_i)$, where $P_i : \mathbb{C}^n \to G_K(k_i)$ is the projection. Since $T(\omega, 0)$ and K are commutative, we have

$$T(\omega, 0)P_i x = P_i T(\omega, 0)x = P_i \mu x = \mu P_i x, \quad i = 0, \cdots, p.$$

Since there is at least one i such that $P_i x \neq 0$, there exists an i such that

$$G_{T(\omega,0)}(\mu) \cap G_K(k_i) \neq \{0\}. \tag{15}$$

Lemma 10 asserts that the condition (15) is reduced to the condition

$$W_K(k_i) \cap W_{T(\omega,0)}(\mu) \neq \{0\}.$$

Hence the condition (14) is replaced by the condition

$$g(k_i,\nu)=\mu,\quad W_K(k_i)\cap W_{T(\omega,0)}(\mu)\neq\{0\}.$$

This proves the theorem. $\square$

Proof of Corollary 1. If $K = kE$, then $\sigma(K) = \{k\}$ and $W_K(k) = \mathbb{C}^d$. Assumptions (KA) and (KI) are clearly satisfied. It suffices to show that Assumption (KU) is satisfied. Suppose $k \in \sigma(U(\omega,0))$. It follows from Lemma 6 that there exists a nontrivial solution $y(n)$ of Eq.(CB$_\mathrm{d}$) such that $y(n+\omega)=ky(n), n\in\mathbb{Z}$. Hence $y(n+1-\omega)=k^{-1}y(n+1), n\in\mathbb{Z}$. Substituting this relation into Eq.(CB$_\mathrm{d}$), we have

$$y(n+1)=A(n)y(n)+k\{k^{-1}y(n+1)-A(n)y(n)\},$$

which implies $(1-k)A(n)y(n)=0$. $k\neq 1$ and $A(n)$ is nonsingular, then we have $y(n)=0$. This is contradicts to that $y(n)$ is a nontrivial solution. Therefore Assumption (KU) is satisfied. $\square$

5 Application

Let $\mu_1,\mu_2,\ldots,\mu_p$ $(p\leq d)$ be the characteristic multipliers of Eq.(L$_\mathrm{d}$). If $K = kE$ $(k\notin\{0,1\})$, from Corollary 1, then the characteristic multipliers of Eq.(CB$_\mathrm{d}$) except for 0 are given by the solutions ν of $\mu_i = g(k,\nu)$, $\quad i=1,2,\ldots,p$. This equation is equivalent to the following ω-th polynomial equation of ν

$$(16)\qquad (\nu-k)^\omega-\mu_i(1-k)^\omega\nu^{\omega-1}=0,\quad i=1,2,\ldots,p.$$

The criterion for success of the stabilization by Buchner type DFC can be obtained from this equations. The following theorems given in [4] are obtained by using Eqs.(16).

Theorem 9 ([4], Theorem 1). Assume $d=1$ and $\omega=1$. Let $\mu=f'(\phi(0))$. Then $\sigma(T(\omega,0))=\{\mu\}$ and

(i) If $\mu>1$ and
$$1<k<\frac{\mu+1}{\mu-1}$$
hold, then $|\nu|<1$ for any $\nu\in\sigma(U(\omega,0))$. That is, the stabilization succeeds.

(ii) If $\mu<-1$ and
$$\frac{\mu+1}{\mu-1}<k<1$$
hold, then $|\nu|<1$ for any $\nu\in\sigma(U(\omega,0))$. That is, the stabilization succeeds.

Theorem 10 ([4], Theorem 2). Assume $d = 1$ and $\omega = 3$. Let $\mu = f'(\phi(0))f'(\phi(1)$. Then $\sigma(T(\omega, 0)) = \{\mu\}$ and

(i) If $\mu > 1$, then there exists $\nu \in \sigma(U(\omega, 0))$ such that $\nu > 1$. That is, the stabilization does not succeed.

(ii) If $\mu < -1$ and

$$\frac{\sqrt{-\mu}-1}{\sqrt{-\mu}+1} < k < 1$$

hold, then $|\nu| < 1$ for any $\nu \in \sigma(U(\omega, 0))$. That is, the stabilization succeeds.

Theorem 11 ([4], Theorem 3). Assume $d = 1$ and $\omega = 3$. Let $\mu = f'(\phi(0))f'(\phi(1)f'(\phi(2))$. Then $\sigma(T(\omega, 0)) = \{\mu\}$ and

(i) If $\mu > 1$, then there exists $\nu \in \sigma(U(\omega, 0))$ such that $\nu > 1$. That is, the stabilization does not succeed.

(ii) If $-27 < \mu < -1$ and

$$\frac{\sqrt[3]{-\mu}-1}{\sqrt[3]{-\mu}+1} < k < \min\left\{1, \frac{1}{\sqrt[3]{-\mu}-1}\right\},$$

hold, then $|\nu| < 1$ for any $\nu \in \sigma(U(\omega, 0))$. That is, the stabilization succeeds.

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