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Conditional oscillation for second order linear differential equations

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We are concerned with the second order linear differential equation

$$u'' + \lambda p(t)u = 0, \quad t > a, \tag{1.1}$$

where $p(t)$ is a positive continuous function on $[a, \infty)$, $a \geq 0$, and λ is a positive parameter.

We call equation (1.1) oscillatory if all solutions of (1.1) have arbitrarily large zeros on (a, ∞), otherwise, we say equation (1.1) is nonoscillatory. As a consequence of Sturm's Separation Theorem, if one of the solutions of (1.1) is oscillatory, then all of them are. The same is true for the nonoscillation of (1.1). Motivated by Nehari [10], the equation (1.1) is said to be *conditionally oscillatory* if (1.1) is oscillatory for some $\lambda > 0$ and nonoscillatory for some other $\lambda > 0$. By the comparison argument, if (1.1) is conditionally oscillatory, there exists a constant $\overline{\lambda} \in (0, \infty)$ such that (1.1) is nonoscillatory for $\lambda < \overline{\lambda}$ and oscillatory for $\lambda > \overline{\lambda}$. The number $\overline{\lambda}$ is called the *oscillation constant.*

For the second order ordinary differential equation

$$u'' + p(t)u = 0, \quad t > a, \tag{1.2}$$

there is an extensive literature on oscillation criteria (see, e.g., Swanson [13]). Hille [5] stated his results in terms of the numbers p_* and p^* defined by

$$p_* = \liminf_{t\to\infty} t \int_t^\infty p(s)ds \quad \text{and} \quad p^* = \limsup_{t\to\infty} t \int_t^\infty p(s)ds,$$

respectively. It was shown in [5] that $p_* \leq 1/4$ and $p^* \leq 1$ if (1.2) is nonoscillatory, and $p^* \geq 1/4$ if (1.2) is oscillatory, where all the inequalities are sharp.

By applying the oscillation and nonoscillation criteria for (1.2), it is easy to see that (1.1) is conditionally oscillatory if and only if

$$0 < \limsup_{t\to\infty} \ t \int_t^\infty p(s)ds < \infty, \tag{1.3}$$

and that, if

$$\lim_{t\to\infty} t\int_t^\infty p(s)ds = \alpha, \quad \alpha \in (0,\infty), \tag{1.4}$$

then the oscillation constant is $\overline{\lambda} = 1/4\alpha$.

In this paper we are interested in the situation where (1.1) is conditionally oscillatory, and are concerned with the problem of counting the number of zeros of solutions of (1.1) on (a,∞). In particular, we will investigate the precise behavior of solutions to (1.1) with λ which lies on the neighborhood of the oscillation constant, and we will show some applications to singular elliptic eigenvalue problems on a ball in $\mathbf{R}^N$.

In the case where (1.1) is nonoscillatory, there exists a unique (neglecting a constant factor) solution $u(t)$ of (1.1) satisfying the condition

$$\int^\infty \frac{dt}{u(t)^2} = \infty, \tag{1.5}$$

and any solutions $\tilde{u}(t)$ linearly independent of $u(t)$ has the property

$$\int^\infty \frac{dt}{\tilde{u}(t)^2} < \infty. \tag{1.6}$$

A solution $u(t)$ satisfying condition (1.5) is said to be *principal* (for $t=\infty$), and a solution $\tilde{u}(t)$ satisfying condition (1.6) is said to be *nonprincipal* (for $t=\infty$).

We denote by $u_\lambda(t)$ the solution of (1.1) satisfying $u(a)=0$ and $u'(a)=1$. When (1.1) is conditionally oscillatory, we denote by $\overline{\lambda}$ the oscillation constant of (1.1).

Our main results are the following.

Theorem 1. *Assume that* (1.3) *holds. Then one of the following statements* (I) *or* (II) *holds:*

(I) *For* $\lambda = \overline{\lambda}$, (1.1) *is nonoscillatory, and there exists a sequence* $\{\lambda_i\}_{i=0}^k$, $0 \le k < \infty$, *with*

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k \le \overline{\lambda}.$$

(II) *For* $\lambda = \overline{\lambda}$, (1.1) *is oscillatory, and there exists a sequence* $\{\lambda_i\}_{i=0}^\infty$ *with*

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{i-1} < \lambda_i < \cdots, \quad \lim_{i\to\infty}\lambda_i = \overline{\lambda}.$$

In both cases (I) *and* (II) *the following properties* (i) –(iii) *hold:*

(i) *if* $\lambda = \lambda_i$ *for some* $i \in \mathbf{N} = \{1,2,\ldots,\}$ *then* $u_\lambda(t)$ *is principal;*

(ii) *if* (1.1) *is nonoscillatory and* $\lambda \ne \lambda_i$, $i = 1,2,\ldots$, *then* $u_\lambda(t)$ *is nonprincipal;*

(iii) *if* $\lambda \in (\lambda_{i-1}, \lambda_i]$ *then* $u_\lambda(t)$ *has exactly* $i-1$ *zeros on* (a,∞).

In the case (I), *if* $\lambda_k < \overline{\lambda}$ *and if* $\lambda \in (\lambda_k, \overline{\lambda}]$ *then* $u_\lambda(t)$ *has exactly* $k-1$ *zeros on* (a,∞).

Assume, furthermore, that (1.4) *holds. Then, in both cases* (I) *and* (II), $\overline{\lambda} = 1/4\alpha$ *and*

$$\int^{\infty} |u_\lambda'(t)|^2 dt < \infty \tag{1.7}$$

if $\lambda < 1/4\alpha$ *and* u_λ *is principal, and*

$$\int^{\infty} |u_\lambda'(t)|^2 dt = \infty. \tag{1.8}$$

if u_λ *is nonprincipal.*

Remark. (i) By Nehari [10] it is known that, if

$$\lim_{t\to\infty} t\int_t^{\infty} p(s)ds = 0,$$

then (1.1) is nonoscillatory for all $\lambda > 0$. In this case, Kusano-M.Naito [7] have shown that, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ with

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \ \lim_{n\to\infty} \lambda_n = \infty,$$

such that the properties (i)–(iii) in Theorem 1 hold. For the extension of the results to the half-linear differential equations, we refer to [8].

(ii) Generally, principal solutions do not satisfy the condition (1.7). For example, the equation $u'' + u/(4t^2) = 0$ has a principal solution $u(t) = t^{1/2}$, which does not satisfies (1.7).

Theorem 2. *Assume that* (1.4) *holds and*

$$t\int_t^{\infty} p(s)ds \le \alpha \quad \textit{for} \ \ t \ge a.$$

Then (I) *with* $k = 0$ *holds in Theorem* 1, *that is, for* $\lambda \in (0, 1/4\alpha]$, *the solution* u_λ *is nonprincipal and has no zeros on* (a,∞), *and* $u_\lambda(t)$ *has infinity zeros on* (a,∞) *for* $\lambda \in (1/4\alpha, \infty)$.

Theorem 3. *Assume that* (1.4) *holds and*

$$t\int_t^{\infty} p(s)ds \ \textit{is strictly decreasing for sufficiently large } t.$$

(i) *If*

$$\limsup_{t\to\infty} \left(t\int_t^{\infty} p(s)ds - \alpha\right)^{1/2} \log t > k\pi\alpha^{1/2}$$

with some $k \in \mathbf{N}$, *then there exists* $\lambda_1 < \lambda_2 < \cdots < \lambda_k < 1/4\alpha$ *such that* u_λ *is principal if* $\lambda = \lambda_i$ *with some* $i \in \{1, 2, \ldots, k\}$.

(ii) *If*

$$\limsup_{t\to\infty}\left(t\int_t^\infty p(s)ds-\alpha\right)^{1/2}\log t=\infty$$

then (II) *holds in Theorem* 1.

As an application, let us consider the existence of principal eigenvalues for linear elliptic equations of the form

$$-\Delta v=\lambda V(x)v,\quad v\in\mathcal{D}_0^{1,2}(\Omega), \tag{1.9}$$

where $\Omega\subset\mathbf{R}^N$, $N\geq 3$, is a open domain, $V\in L^1_{\rm loc}(\Omega)$, and $\mathcal{D}_0^{1,2}(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ for the norm $(\int_\Omega|\nabla v|^2dx)^{1/2}$. We are interested in the potentials behaving like $1/|x|^2$, which appears in Hardy's inequality.

The existence of principal eigenvalue for the problem (1.9) has been studied previously by Allegret [1], Brown, Cosner and Fleckinger [2], Dancer [3], Jin [6], Murata [9], Pinchover [11], Rozenblum and Solomyak [12], Smets [13] and Tertikas[15]. In [1, 2, 12], the approach to the problem is by studying an appropriate self-adjoint operator in a suitable Hilbert spaces. In [9, 11], the approach is by a very careful analysis of the asymptotics of the Green function of a suitable linear operator.

In [13, 15], the existence of eigenvalues for (1.9) was shown in terms of the quantities of Rayleigh quotient. Define the following quantities for $x\in\overline{\Omega}$ and $r>0$,

$$\begin{aligned}
S^\infty_{r,V}&=\inf\left\{\int_\Omega|\nabla v|^2dx:v\in\mathcal{D}_0^{1,2}(\Omega\setminus B_r(x)),\ \int_\Omega Vv^2dx=1\right\},\\
S^\infty_V&=\sup_{r>0}S^\infty_{r,V}=\lim_{r\to\infty}S^\infty_{r,V},\\
S^x_{r,V}&=\inf\left\{\int_\Omega|\nabla v|^2dx:v\in\mathcal{D}_0^{1,2}(\Omega\cap B_r(x)),\ \int_\Omega Vv^2dx=1\right\},\\
S^x_V&=\sup_{r>0}S^x_{r,V}=\lim_{r\to 0}S^x_{r,V},\quad\text{and}\quad S_V=\inf_{x\in\overline{\Omega}}S^x_V,
\end{aligned}$$

where $B_r(x)$ denote the closed ball of radius r centered at x. Assume that $\Sigma_V=\{x\in\overline{\Omega}:S^x_V<\infty\}\neq\emptyset$, and that the closure of Σ_V is at most countable. Put

$$\lambda_1(\Omega)=\inf\left\{\int_\Omega|\nabla v|^2dx:v\in\mathcal{D}_0^{1,2}(\Omega),\ \int_\Omega Vv^2dx=1\right\},$$

and $\lambda_k(\Omega)$ is the k-th eigenvalue. It has been shown by Smets [13] that, if $\lambda_1(\Omega)<S_V$ and $\lambda_1(\Omega)<S^\infty_V$, then there exists a principal eigenvalue for (1.9), and, if $\lambda_k(\Omega)<\min\{S_V,S^\infty_V\}$ for some $k\geq 1$, then there exist eigenvalues $\lambda_1,\dots,\lambda_k$ for (1.9).

Our results for (1.1) can easily be applied to the existence and nonexistence of the eigenvalues for singular elliptic problems of the form

$$-\Delta v=\lambda V(|x|)v,\quad v\in H^1_0(\Omega), \tag{1.10}$$

where $\Omega = \{x \in \mathbf{R}^N : |x| < R\}$ with $N \geq 3$, $R > 0$, and $V \in C(\Omega\backslash\{0\})$, $V \geq 0$, $\not\equiv 0$. The problem of finding radially symmetric solutions $v = v(r)$, $r = |x|$, of (1.10) is converted to the following problem

$$r^{1-N}(r^{N-1}v')' + \lambda V(r)v = 0 \quad \text{for } 0 < r < R$$

with $v(R) = 0$. By the change of variables $u(t) = v(r)$ and $t = r^{2-N}$, the problem is reduced to (1.1) with $u(a) = 0$, where $a = R^{2-N}$ and

$$p(t) = \frac{1}{(N-2)^2} t^{-(2N-2)/(N-2)} V(t^{-1/(N-2)}).$$

By the direct calculation, we find that

$$\int_0^R r^{N-1} v_r(r)^2 dr = (N-2) \int_a^\infty |u'(t)|^2 dt.$$

By Theorems 1, 2 and 3, we obtain the following:

Theorem 4. *Assume that*

$$\lim_{r \to 0} r^{2-N} \int_0^r s^{N-1} V(s) ds = \alpha, \quad 0 < \alpha < \infty. \tag{1.11}$$

If

$$r^{2-N} \int_0^r s^{N-1} V(s) ds \leq \alpha \quad \textit{for } 0 < r < R, \tag{1.12}$$

then (1.10) *has no positive solution* $v \in H_0^1(B_R)$.

Remark. For the existence of the principal eigenvalue to the problem (1.10), the main assumption in [13] reads $S_V^0 = S_V < \lambda_1(\Omega)$. Thus Theorem 4 implies that $S_V = \lambda_1(\Omega)$ when (1.11) and (1.12) hold.

Theorem 5. *Assume that* (1.11) *holds, and that*

$$r^{2-N} \int_0^r s^{N-1} V(s) ds \quad \textit{is increasing for sufficiently small } r > 0.$$

(i) *If*

$$\limsup_{r \to 0} \left(r^{2-N} \int_0^r s^{N-1} V(s) ds - \alpha \right)^{1/2} |\log r| > (N-2)^{3/2} k\pi \alpha^{1/2}$$

with some $k \in \mathbf{N}$, *then there exist eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_k \in (0, \overline{\lambda})$.

(ii) *If*

$$\limsup_{r \to 0} \left(r^{2-N} \int_0^r s^{N-1} V(s) ds - \alpha \right)^{1/2} |\log r| = \infty$$

then there exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ *such that* $\lambda_n \to \overline{\lambda}$ *as* $n \to \infty$.

REFERENCES

[1] W. Allegretto, Principal eigenvalues for indefinite elliptic problems in $\mathbf{R}^N$, Proc. Amer. Math. Soc. **116** (1992) 701–706.

[2] K. J. Brown, C. Cosner, and J. Fleckinger, Principal eigenvalues for problems with indefinite with function on $\mathbf{R}^N$, Proc. Am. Math. Soc. **109** (1990) 147–155.

[3] D. Daners, Principal eigenvalues for some periodic-parabolic operators on $\mathbf{R}^N$ and related topics, J. Differential Equations **121** (1995) 293–313.

[4] P. Hartman, "Ordinary Differential Equations", John Wiley, New York (1964).

[5] E. Hille, Non-oscillation theorems, Trans. Amer. Math. Soc. **64** (1948) 234–252.

[6] Z. Jin, Principal eigenvalues with indefinite weight functions, Trans. Am. Math. Soc. **349** (1997) 1945–1959.

[7] T. Kusano and M. Naito, A singular eigenvalue problem for the Sturm-Liouville equation, Differential Equations **34** (1998) 302-311.

[8] T. Kusano and M. Naito, On the number of zeros of nonoscillatory solutions to half-linear ordinary differential equations involving a parameter, Trans. Amer. Math. Soc. **354** (2002) 4751–4767.

[9] M. Murata, Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbf{R}^N$, Duke Math. J. **53** (1986) 869–943.

[10] Z. Nehari, Oscillation criteria of solutions of second-order linear differential equations, Trans. Amer. Math. Soc. **85** (1957) 428-445.

[11] Y. Pinchover, Criticality and ground states for second-order elliptic equations, J. Differential Equations **80** (1989) 237–250.

[12] G. Rozenblum, M. Solomyak, On principal eigenvalues for indefinite problems in Euclidean space, Math. Nachr. **192** (1998) 205–223.

[13] D. Smets, A concentration-compactness lemma with applications to singular eigenvalue problems, J. Functional Analysis **167** (1999) 463–480.

[14] C. A. Swanson, "Comparison and oscillation theory of linear differential equations", Academic Press, New York, London, 1968.

[15] A. Tertikas, Critical phenomena in linear elliptic problems, J. Funct. Anal. **154** (1998) 42–66.