数理解析研究所講究録
第 1702 巻 2010 年 71-87

Topics on the movement of Hot Spots for the Heat Equation with a Potential

東北大学大学院理学研究科 石毛和弘 (Kazuhiro Ishige)
Mathematical Institute, Tohoku University
大阪府立大学大学院工学研究科 壁谷喜継 (Yoshitsugu Kabeya)
Department of Mathematical Sciences, Osaka Prefecture University

1 Introduction

We consider the Cauchy problem of the heat equation with a potential,

$$(1.1) \qquad \begin{cases} \partial_t u = \Delta u - V(|x|)u & \text{in} \quad \mathbf{S} \equiv \mathbf{R}^N \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in} \quad \mathbf{R}^N, \end{cases}$$

where $N \geq 2$ and $\phi \in L^2(\mathbf{R}^N, \rho dx)$ with $\rho(x) = \exp(|x|^2/4)$. Throughout this paper we assume that the potential V satisfies the condition (V) for some $\omega \geq 0$ and $\theta > 0$:

$$(\mathbf{V}) \qquad \begin{cases} \text{(i)} & V = V(|x|) \in C^1(\mathbf{R}^N), \\ \text{(ii)} & V(r) \geq 0 \quad \text{on} \quad [0,\infty), \\ \text{(iii)} & \sup_{r\geq 1} r^{2+\theta} \left| V(r) - \dfrac{\omega}{r^2} \right| < \infty, \\ \text{(iv)} & \sup_{r\geq 1} \left| r^3 \left(\dfrac{d}{dr} V \right)(r) \right| < \infty. \end{cases}$$

We denote by $H(t)$ the set of the maximum points of the solution u of (1.1) over $\mathbf{R}^N \times \{t\}$, that is,

$$H(t) = \left\{ x \in \mathbf{R}^N \, : \, u(x,t) = \max_{y \in \mathbf{R}^N} u(y,t) \right\},$$

and call $H(t)$ the set of hot spots of the solution u at the time t.

The large time behavior of hot spots on non-compact domains has been studied since Chavel and Karp's interesting work [2] in 1990. In particular, they used the fundamental solution of the heat equation on $\mathbf{R}^N$, and proved that the hot spots on $\mathbf{R}^N$ tend to the center of mass of the initial data as $t \to \infty$ if the initial data is a nonnegative function having a compact support. Jimbo and Sakaguchi [13] treated the heat equation on the half space in $\mathbf{R}^N$, and studied the relation between the movement of hot spots and the boundary conditions. More precisely, they showed that $H(t)$ consists of a single point $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))$ for any large t and there holds $(2t)^{-1/2} x_N(t) \to 1$ $(x_N \to \infty)$ as $t \to \infty$ if the boundary condition is the Dirichlet or the Robin, whereas $x_N(t) = 0$ (on the boundary) for any large t if the boundary condition is the Neumann. Furthermore they also treated the movement of hot spots for the radial solutions in the exterior domain of a ball. Subsequently, the first author of this paper [6], [7] studied the movement of hot spots on the exterior domain of a ball without the radial symmetry of the initial data, by using rescale arguments and the radial symmetry of the domain effectively. Recently the authors of this paper [8] studied the decay rate of derivatives of the solution of the heat equations with a potential, and proved that the optimal decay rate of the derivatives of solutions was determined by the shape of the harmonic functions for $\Delta - V$.

From the result of [2], analyzing the heat kernel and showing their global bounds seem to be powerful to investigate the large time behavior of the hot spots. The global bounds of the heat kernel have been studied by Aronson [1], Davies [3], Fabes and Stroock [5], and many others (see also [15], [17], [18], and references therein). Among others, Zhang [17] studied the large time behavior of the heat kernel for the case which includes the potential V satisfying the condition (V). However, as stated in [13], the global bounds for the heat kernel do not seem useful to obtain particular large time behaviors of the hot spots for the equation (1.1).

In this paper, we make a survey on the large time behavior of the hot spots $H(t)$ of the solution u of (1.1) under the condition (V).

We introduce some notations in order to give the main results of this paper. Let $\Delta_{\mathbb{S}^{N-1}}$ be the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$ and $k = 0, 1, 2 \ldots$. Let ω_k be the k-th eigenvalues of

$$-\Delta_{\mathbb{S}^{N-1}} Q = \omega Q \quad \text{on} \quad \mathbb{S}^{N-1}, \qquad Q \in L^2(\mathbb{S}^{N-1}), \tag{1.2}$$

that is, $\omega_k = k(N+k-2)$. Furthermore let $\{Q_{k,i}\}_{i=1}^{l_k}$ and l_k be the complete orthonormal system and the dimension of the eigenspace corresponding to

ω_k, respectively. In particular, $l_0 = 0$, $l_1 = N$, and we may write

$$(1.3)\qquad Q_{0,1}\left(\frac{x}{|x|}\right) = \kappa_0, \qquad Q_{1,i}\left(\frac{x}{|x|}\right) = \kappa_1\frac{x_i}{|x|}, \quad i = 1, \dots, N,$$

where κ_0 and κ_1 are positive constants. Let $\alpha = \alpha(\omega)$ be a positive root of the equation $\alpha(\alpha + N - 2) = \omega$, that is,

$$(1.4)\qquad \alpha(\omega) = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\omega}}{2} > 0.$$

Then, under the condition (V), there exists a unique solution U_k of the ordinary differential equation

$$(O)\qquad U'' + \frac{N-1}{r}U' - V_k(r)U = 0 \quad \text{in} \quad (0,\infty)$$

with

$$(1.5)\qquad \limsup_{r\to 0} |U(r)| < \infty, \qquad \lim_{r\to\infty} r^{-\alpha(\omega+\omega_k)}U(r) = 1,$$

where $V_k(r) = V(r) + \omega_k r^{-2}$

We are ready to state the main results, which can be seen in [9, 10, 11]. We put

$$M \equiv \int_{\mathbf{R}^N} \phi(x)U_0(|x|)\,dx$$

and we denote the $L^2(\mathbf{R}^N, \rho dy)$ -norm by $\|\cdot\|$ and the usual $L^p(\mathbf{R}^N)$-norm by $\|\cdot\|_{L^p(\mathbf{R}^N)}$. We first give a result on the large time behaviors of the solution of (1.1) and its hot spots when $\omega > 0$.

Theorem 1.1 ([9]) *Consider the Cauchy problem* (1.1) *under the condition* (V) *with* $\omega > 0$ *and* $\theta > 0$. *Then, for any* $L > 0$,

$$(1.6)\qquad \lim_{t\to\infty} \sup_{x\in B(0,L)} \left| t^{\frac{N}{2}+\alpha(\omega)}u(x,t) - cMU_0(|x|)\right| = 0,$$

and

$$(1.7)\qquad \lim_{t\to\infty} (1+t)^{\frac{N+\alpha(\omega)}{2}} u\left((1+t)^{\frac{1}{2}}y, t\right) = cM|y|^{\alpha(\omega)}e^{-\frac{|y|^2}{4}}$$

in $L^2(\mathbf{R}^N, \rho dy)$ *and* $L^\infty(\mathbf{R}^N)$, *where* $c = 1/\int_{\mathbf{R}^N} |x|^{2\alpha(\omega)}e^{-|x|^2/4}dx$. *Furthermore, if* $M > 0$, *for any* $t > 0$, $H(t) \neq \emptyset$, *and*

$$(1.8)\qquad \lim_{t\to\infty} \sup_{x\in H(t)} \left| t^{-\frac{1}{2}}|x| - \sqrt{2\alpha(\omega)}\right| = 0.$$

Next we give a result on the direction for the hot spots to tend to the space infinity and the number of the hot spots.

Theorem 1.2 ([9, 11]) *Consider the Cauchy problem* (1.1) *under the condition* (*V*) *with* $\omega \geqq 0$ *and* $\theta > 0$. *Furthermore assume* $M > 0$ *and*

$$A_\phi \equiv \int_{\mathbf{R}^N} \phi(x) U_1(|x|) \frac{x}{|x|} dx \neq 0.$$

Then there exist a constant $T > 0$ *and a curve* $x = x(t) \in C^1([T, \infty) : \mathbf{R}^N)$ *such that*

$$H(t) = \{x(t)\}, \qquad t \geq T \tag{1.9}$$

and

$$\lim_{t \to \infty} \left| \frac{x(t)}{|x(t)|} - \frac{A_\phi}{|A_\phi|} \right| = 0. \tag{1.10}$$

In case of $\omega = 0$, the situation will change from that in $\omega > 0$.

Theorem 1.3 ([10]) *Let* $N \geq 3$. *Suppose that* u *be the solution of the Cauchy problem* (1.1) *under the condition* (*V*) *with* $\omega = 0$. *Assume* $M > 0$. *Then, for any* $t > 0$, $H(t) \neq \emptyset$ *and*

$$\lim_{t \to \infty} \sup_{x \in H(t)} \left| \frac{t U_0'(|x|)}{|x|} - \frac{1}{2} \right| = 0.$$

In particular, there hold

$$\liminf_{t \to \infty} t^{-1/N} \inf_{x \in H(t)} |x| > 0,$$

$$\limsup_{t \to \infty} t^{-1/\alpha} \sup_{x \in H(t)} |x| < \infty$$

for some constant $\alpha \in (2, N]$.

As a corollary, we have the following (see Corollary 1.1 in [10]).

Corollary 1.1 *Assume the same conditions as in Theorem 1.3.*

(i) *If* $\lim_{r \to \infty} r^\kappa V(r) = \bar{\omega} > 0$ *for some* $\kappa \in (2, N)$, *then*

$$\begin{aligned} |x| &= \left(\frac{2\bar{\omega} t}{N - \kappa} \right)^{1/\kappa} (1 + o(1)) \quad (2 < \kappa < N), \\ |x| &= \left(\frac{2\bar{\omega}\, t \log t}{N} \right)^{1/\kappa} (1 + o(1)) \quad (\kappa = N), \end{aligned}$$

for all $x \in H(t)$ *and all sufficiently large* t.

(ii) *If* $V \in L^1(\mathbf{R}^N)$, *then*

$$|x| = \left[\frac{2t}{|\partial B(0,1)|}\int_{\mathbf{R}^N} V(|x|)U_0(|x|)dx\right]^{1/N} + o(t^{1/N})$$

for all $x \in H(t)$ *and all sufficiently large* t.

Thus, if V has a compact support, however small V is, the hot spot moves to infinity, never stays around the center of mass.

Finally, we treat the case $N = 2$ and $\omega = 0$. Hereafter, we denote the ball centered at a with its radius r by $B(a, r)$.

Theorem 1.4 ([11]) *Let* u *be the solution of the Cauchy problem* (1.1) *under the condition* (V) *and assume* $(N, \omega) = (2, 0)$. *Then, for any* $L > 0$,

$$\lim_{t\to\infty} t(\log t)^2 u(x,t) = \pi^{-1} M U_0(|x|)$$

in $C(B(0, L))$

$$\lim_{t\to\infty} t(\log t) u\left((1+t)^{1/2}y, t\right) = (2\pi)^{-1} M e^{-|y|^2/4}$$

in $C(\mathbf{R}^2 \setminus B(0, L))$ *and* $L^2(\mathbf{R}^2, \rho dy)$ *hold.*

Theorem 1.5 ([11]) *Let* u *be the solution of the Cauchy problem* (1.1) *under the condition* (V) *and assume* $(N, \omega) = (2, 0)$ *and* $M > 0$. *Then, for any* $t > 0$, $H(t) \neq \emptyset$ *and*

$$\lim_{t\to\infty} \sup_{x\in H(t)} \left|\frac{(\log t)|x|^2}{2t} - 1\right| = 0.$$

Now, we briefly explain the ideas of proving Theorems 1.1 – 1.5. Let $\phi \in L^2(\mathbf{R}^N, \rho dx)$ and $u \equiv S(t)\phi$ be the solution of (1.1). By the same arguments as in [6] and [7], we can decompose the initial value ϕ in the Fourier series:

$$\phi = \sum_{k=0}^{\infty}\sum_{i=1}^{l_k} \phi_{k,i}(|x|) Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad L^2(\mathbf{R}^N, \rho dx). \tag{1.11}$$

Put

$$\Phi_{k,i}(x) = \phi_{k,i}(|x|) Q_{k,i}\left(\frac{x}{|x|}\right), \qquad u_{k,i}(x,t) = (S(t)\Phi_{k,i})(x).$$

Then, by the standard arguments for the parabolic equation (see (2.10) and [14]), we see

$$u(x,t)=\sum_{k=0}^{\infty} u_{k,i}(x,t) \quad \text{in} \quad L^\infty(\mathbf{R}^N) \quad \text{and} \quad C^2(\mathbf{R}^N) \tag{1.12}$$

for all $t > 0$. Furthermore, for any $k = 0, 1, 2, \ldots$ and $i = 1, \ldots, l_k$, there exists a solution $v_{k,i} = v_{k,i}(x,t) \equiv S_k(t)\phi_{k,i}$ of

$$(P_k) \quad \begin{cases} \partial_t v = \Delta v - V_k(|x|)v & \text{in} \quad (\mathbf{R}^N \setminus \{0\}) \times (0,\infty), \\ \limsup_{|x|\to 0} |x|^{-k}|v(x,t)| < \infty & \text{for} \quad t > 0, \\ v(x,0) = \phi_{k,i}(|x|) & \text{in} \quad \mathbf{R}^N \end{cases}$$

such that

$$u_{k,i}(x,t) = v_{k,i}(x,t) Q_{k,i}\left(\frac{x}{|x|}\right), \qquad (x,t) \in \mathbf{S}. \tag{1.13}$$

Furthermore we put

$$w_{k,i}(y,s) = (1+t)^{\frac{N}{2}} v_{k,i}(x,t), \quad y = (1+t)^{-\frac{1}{2}}x, \quad s = \log(1+t), \tag{1.14}$$

the forward self-similar transformation. Then $w_{k,i} = w_{k,i}(y,s)$ satisfies

$$(L_k) \quad \partial_s w = L_k w \quad \text{in} \quad (\mathbf{R}^N \setminus \{0\}) \times (0,\infty), \quad w(y,0) = \phi_{k,i}(|y|) \quad \text{in} \quad \mathbf{R}^N,$$

where

$$L_k w = L_k^* w - \left[e^s V(e^{-\frac{s}{2}}y) - \frac{\omega}{|y|^2}\right] w, \quad L_k^* w = \frac{1}{\rho}\operatorname{div}(\rho \nabla_y w) - \frac{\omega + \omega_k}{|y|^2} w + \frac{N}{2} w.$$

In order to study the behavior of the solution u, we investigate the behaviors of $w_{k,i}(s)$ in the space $L^2(\mathbf{R}^N, \rho dy)$, by using the eigenfunctions of L_k^*. The lack of regularity of $w_{k,i}$ near the origin is driven from the potential term of L_k, and it seems to be difficult to obtain the asymptotic behaviors of the derivatives of $w_{k,i}(s)$ as $s \to \infty$, by the behaviors of $w_{k,i}(s)$ in $L^2(\mathbf{R}^N, \rho dy)$, directly.

We follow the strategy in [6]–[8], and obtain the asymptotic behaviors of the derivatives of $w_{k,i}(s)$ as $s \to \infty$. However, it also seems difficult to apply the same arguments as in [6]–[8] directly, because of the singularities of the potential $(\omega + \omega_k)r^{-2}$ at $r = 0$ and of $e^s V(e^{-s/2}y)$ at $(y,s) = (0,\infty)$. We use several properties of U_k and radial functions constructed from inhomogeneous elliptic problems, and construct two super-solutions of (P_k) to overcome the difficulty driven from the singularity of $(\omega+\omega_k)r^{-2}$ at $r=0$. By using these supersolutions, we can obtain Theorems 1.1 and 1.2.

The organization of this paper is as follows: In Section 2 we give basic lemmas on several properties of stationary solutions and the super-solutions of (P_k). In Section 3 we consider the transformed problem (L_k), and study the behaviors of the solutions. Proofs of Lemmas and Propositions in this paper are not provided. They can be found in [9, 10, 11]. In Section 4 we give sketchy proofs of Theorems 1.1 – 1.5 by using several results given in the previous sections.

2 Basic Lemmas

In this section we enumerate several lemmas which assure the fundamental properties of solutions.

We first introduce several notations. For any sets A and B, let $f = f(\lambda,\nu)$ and $g = g(\lambda,\mu)$ be maps from $A\times B$ to $(0,\infty)$. Then we say $f(\lambda,\mu)\preceq g(\lambda,\mu)$ for all $\lambda\in A$ if, for any $\mu\in B$, there exists a positive constant C such that $f(\lambda,\mu)\le Cg(\lambda,\mu)$ for all $\lambda\in A$. Furthermore, we say $f(\lambda,\mu)\asymp g(\lambda,\mu)$ for all $\lambda\in A$ if $f(\lambda,\mu)\preceq g(\lambda,\mu)$ and $g(\lambda,\mu)\preceq f(\lambda,\mu)$ for all $\lambda\in A$.

The following lemma on the solution of (O) is fundamental.

Lemma 2.1 *Assume* (V) *for some* $\omega>0$ *and* $\theta>0$. *Let* $k=0,1,2,\ldots$ *and* $R=\sup\{r\ge 0 : V(\tau)\equiv 0 \quad in \quad [0,r]\}\in[0,\infty)$.
(i) *There exists a unique solution* $U_k(r)$ *of the ordinary differential equation of* (O) *satisfying* (1.5). *Furthermore* U_k *satisfies*

$$U_k(r)\ge 0,\quad U_k'(r)\ge 0,\qquad r\ge 0, \tag{2.1}$$

$$U_k(r)\asymp r^k,\qquad 0\le r\le 1, \tag{2.2}$$

$$U_k(r)\asymp r^{\alpha_k},\quad U_k'(r)\asymp r^{\alpha_k-1},\quad r\ge R+1. \tag{2.3}$$

(ii) *Let* f *be a continuous function on* $[0,\infty)$ *such that* $|f(r)|\le AU_k(r)$ *on* $[0,\infty)$ *for some constant* A. *Put*

$$F_k[f](r)=U_k(r)\int_0^r s^{1-N}[U_k(s)]^{-2}\left(\int_0^s \tau^{N-1}U_k(\tau)f(\tau)d\tau\right)ds.$$

Then

$$|F_k[f](r)|\le\frac{A}{2N}r^2U_k(r),\quad |F_k[f]'(r)|\le\frac{A}{2N}[2rU_k(r)+r^2U_k'(r)] \tag{2.4}$$

for all $r \geq 0$. *Furthermore, for any solution* $v = v(r)$ *of*

$$U'' + \frac{N-1}{r} U' - V_k(r) U = f \quad in \quad (0, \infty), \tag{2.5}$$

satisfying $\limsup_{r \to 0} |v(r)| < \infty$, *there exists a constant* c *such that*

$$v(r) = c U_k(r) + F_k[f](r), \qquad r \geq 0. \tag{2.6}$$

(iii) *In case of* $N = 2$, *statements in (i) and (ii) except* (2.3) *hold. The relation* (2.3) *is replaced by*

$$\lim_{r \to \infty} r U_0'(r) = 1.$$

Next we construct a super-solution of (P_k) by using the solution U_k. We improve the argument in Lemma 4.1 of [8] for the exterior problem, and obtain the following lemma.

Lemma 2.2 *Let* $k = 0, 1, 2, \ldots$. *Assume that* (V_ω) *and* E_ω *hold for some* $\omega \in [0, \omega_*)$. *Let* $\gamma \geq 0$ *and* U_k *be a unique radial solution to* (O). *Then there exist a radial function*

$$\tilde{W}(x,t) = C_1 (1+t)^{-A} \left[U_k(|x|) - A(1+t)^{-1} F_k[U_k](|x|) \right]$$

with $A = \alpha_k + \gamma/2$ *satisfies the following properties: for any* $0 < \epsilon \leq \epsilon_0 = (A+2)^{-1}$ *and* $T \geq 0$, *there exists a constant* C *such that*

$$\partial_t \tilde{W} \geq \Delta \tilde{W} - V_k(|x|) \tilde{W} \quad in \quad \mathbf{S}, \tag{2.7}$$

$$0 \leq \tilde{W}(x,t) \leq C (1+t)^{-\frac{\alpha_k}{2} - \gamma} U_k(|x|) \quad in \quad D_\epsilon(T), \tag{2.8}$$

$$\tilde{W}(x,t) \geq (1+t)^{-\gamma} \quad on \quad \Gamma_\epsilon(T), \tag{2.9}$$

where

$$\begin{aligned} D_\epsilon(T) &= \left\{ (x,t) \in \mathbf{R}^N \times (T, \infty) \, : \, |x| < \epsilon (1+t)^{1/2} \right\}, \\ \Gamma_\epsilon(T) &= \left\{ (x,t) \in \mathbf{R}^N \times (T, \infty) \, : \, |x| = \epsilon (1+t)^{1/2} \right\} \\ &\qquad \cup \left\{ (x,T) \in \mathbf{R}^N \times \{T\} \, : \, |x| < \epsilon (1+T)^{1/2} \right\}. \end{aligned}$$

In the case of $N = 2$, a variant lemma is obtained.

Lemma 2.3 *Assume the condition (V). Then for any $i, j \geq 0$ and any sufficiently small $\epsilon > 0$, there exist a function $W = W(x,t)$ in* **S** *and positive constants C_1 and C_2 such that*

$$\begin{aligned} &\partial_t W \geq \Delta W - V(|x|)W \quad \text{in} \quad \mathbf{S}, \\ &W(x,t) \leq C_2(1+t)^{-i}[\log(2+t)]^{-j-1}U_0(|x|) \quad \text{in} \quad D_\epsilon(T_\epsilon), \\ &W(x,t) \geq C_1(1+t)^{-i}[\log(2+t)]^{-j} \quad \text{on} \quad \Gamma_\epsilon(T_\epsilon), \end{aligned}$$

where T_ϵ is the constant satisfying $\epsilon(1+T_\epsilon)^{1/2} = 1$.

Under (V), we can see that the L^p-L^q estimate holds.

Lemma 2.4 *Let $\phi \in L^2(\mathbf{R}^N)$. Then, for any $1 \leq q \leq p \leq \infty$, there exists a constant C such that*

$$\|S(t)\phi\|_{L^p(\mathbf{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|\phi\|_{L^q(\mathbf{R}^N)}, \quad t > 0, \tag{2.10}$$

$$\|\partial_t S(t)\phi\|_{L^2(\mathbf{R}^N)} \leq Ct^{-1}\|\phi\|_{L^2(\mathbf{R}^N)}, \quad t > 0. \tag{2.11}$$

Next we prove the existence of the solution of (P_k), and give some properties of the solution.

Lemma 2.5 *Assume the condition (V) with some $\omega > 0$ and $\theta > 0$. Let ϕ be a radial function in $L^2(\mathbf{R}^N, \rho dx)$ and $k = 0, 1, \ldots$. Then there exists a classical and radial solution of*

$$\partial_t v = \Delta v - V_k(|x|)v \quad \text{in} \quad [\mathbf{R}^N \setminus \{0\}] \times (0,\infty) \tag{2.12}$$

with the following properties:
(i) *For any $T > 0$, $v \in L^\infty(0,T : L^2(\mathbf{R}^N)) \cap L^2(0,T : H^1(\mathbf{R}^N))$. Furthermore, for any $i = 1, \ldots, l_k$, the function*

$$v(|x|,t)Q_{k,i}\left(\frac{x}{|x|}\right)$$

is a solution of (1.1) *with $u(x,0) = \phi(|x|)Q_{k,i}(x/|x|)$.*
(ii) *Assume $\phi \geq 0$ on $\mathbf{R}^N$. Then, for any $k,\, l \in \mathbf{N}$ with $k \leq l$,*

$$0 \leq S_l(t)\phi \leq S_k(t)\phi, \qquad t \geq 0.$$

(iii) *v satisfies* (2.10) *and* (2.11) *with $S(t)\phi$ replaced by $v(t)$.*
(iv) *Assume that there exist positive constants C_1 and d such that*

$$\|v(t)\|_{L^2(\mathbf{R}^N)} \leq C_1 t^{-d}, \quad t > 0. \tag{2.13}$$

Then, for any $T > 0$ and any sufficiently small $\epsilon > 0$, there exists a constant C_2 such that

$$\begin{aligned} (2.14)\qquad |(\partial_t^j v)(x,t)| &\le C_2 t^{-d-\frac{N}{4}-\frac{\alpha_k}{2}-j} U_k(|x|), \\ (2.15)\qquad |(\partial_r v)(|x|,t)| &\le C_2 t^{-d-\frac{N}{4}-\frac{\alpha_k}{2}} \left[|U_k'(|x|)| + t^{-1}|x| U_k(|x|)\right], \end{aligned}$$

for all $(x,t) \in D_\epsilon(T)$, where $j = 0, 1$.
(v) *In case of $(N,\omega) = (2,0)$, assume that there exist constants $C_3 > 0$ and $j \ge 0$ such that*

$$(2.16)\qquad \|v(t)\|_2 \le C_3 (1+t)^{-\frac{1}{2}} [\log(2+t)]^{-j}$$

for all $t > 0$. Then for any sufficiently small $\epsilon > 0$, there exists a constant C_4 such that

$$(2.17)\qquad \left|\partial_t^i \partial_r^l f(x,t)\right| \le C_4 (1+t)^{-2-i} [\log(2+t)]^{-1-j} |x|^{2-l} U_0(|x|)$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$ and $l = 0, 1$.

3 Self-similar transformation

In this section, we enumerate several properties of solutions to (L_k). To this end, we first recall the following lemma on the eigenvalue problem for the operator L_k^*,

$$(E_k)\qquad \begin{cases} L_k^* \varphi = -\lambda \varphi & \text{in } \mathbf{R}^N, \\ \varphi \text{ is a radial function in } \mathbf{R}^N \text{ with respect to } 0, \\ \varphi \in H^1(\mathbf{R}^N, \rho dy). \end{cases}$$

Lemma 3.1 (*See Lemma 2.1 in* [16].) *Let $\omega \ge 0$ and $k = 0, 1, 2, \ldots$. Let $\{\lambda_{k,i}\}_{i=0}^\infty$ be the eigenvalues of (E_k) such that $\lambda_{k,0} < \lambda_{k,1} < \ldots$. Then*

$$(3.1)\qquad \lambda_{k,i} = \frac{\alpha_k}{2} + i$$

and all the eigenvalues are simple. Furthermore the eigenfunction φ_k corresponding to $\lambda_{k,0}$ is given by

$$(3.2)\qquad \varphi_k(y) = c_k |y|^{\alpha(\omega+\omega_k)} \exp\left(-\frac{|y|^2}{4}\right),$$

where c_k is a positive constant such that $\|\varphi_k\| = 1$.

By using Lemmas 2.4 and 3.1, we have the following proposition on the decay rate of the functions v_k and w_k.

Proposition 3.1 *Assume the same conditions as in Lemma* 2.5. *Let* $v = v(|x|,t)$ *be the function constructed in Lemma* 2.5 *and* $w = w(|y|,s)$ *be defined by* (1.14). *Then there exists a positive constant* C_1 *such that*

$$\|w(s)\| \le C_1 e^{-\frac{\alpha_k}{2}s}\|\phi\|, \quad s > 0. \tag{3.3}$$

Proposition 3.2 *Assume the same conditions as in Proposition* 3.1. *Put*

$$a_k = c_k \int_{\mathbf{R}^N} U_k(|x|)\phi(x)dx. \tag{3.4}$$

Then, for any $L > 0$,

$$\lim_{s\to\infty} \left\| e^{\frac{\alpha_k}{2}s} w(s) - a_k\varphi_k \right\|_{C^2(\{L^{-1}\le|y|\le L\})} = 0. \tag{3.5}$$

Furthermore, if $a_k = 0$, *then* $\|w(s)\| = O(e^{-\frac{\alpha_k+2}{2}s})$ *as* $s \to \infty$, *and for any* $L > 0$, *there exists a constant* C_2 *such that*

$$\left\| e^{\frac{\alpha_k}{2}s} w(s) \right\|_{C^2(\{L^{-1}\le|y|\le L\})} \le C_2 e^{-s}, \qquad s \ge 1. \tag{3.6}$$

Here, we recall that $v(|x|,t)$ can be expressed as

$$v(|x|,t) = c(t)U_k(|x|) + F_k[(\partial_t v)(t)](|x|)$$

with $c(t)$ being a smooth function due to Lemma 2.1.

Proposition 3.3 *Assume the same conditions as in Proposition* 3.2. *Then, there hold*

$$\begin{aligned} \lim_{t\to\infty} t^{\frac{N}{2}+\alpha_0} c(t) &= a_0 c_0, \quad \text{if } (N,\omega) \ne (2,0), \\ \lim_{t\to\infty} t(\log t)^2 c(t) &= 2a_0 c_0, \quad \text{if } (N,\omega) = (2,0). \end{aligned}$$

4 Proofs of Theorems 1.1– 1.5

Let u be the solution of (1.1). Let $u_{k,i}$, $v_{k,i}$, and $w_{k,i}$ be the functions defined in Section 1. For any $m = 0,1,2\ldots$, put

$$u_m(x,t) = \sum_{k=m}^{\infty}\sum_{i=1}^{l_k} u_{k,i}(x,t) = u(x,t) - \sum_{k=0}^{m-1}\sum_{i=1}^{l_k} u_{k,i}(x,t).$$

Then we have the following lemma.

Lemma 4.1 *Assume (V) for some $\omega > 0$ and $\theta > 0$. Then, for any $m = 0, 1, 2, \ldots$ and $l = 0, 1, 2$, there exists a positive constant C_1 such that*

$$|(\nabla_x^l u_m)(x,t)| \le C_1 t^{-\frac{N+\alpha_m}{2}} \|\phi\| \qquad in \quad \mathbf{S}. \tag{4.1}$$

Furthermore, for any $\epsilon > 0$, there exists a positive constant L such that

$$|u(x,t)| \le \epsilon t^{-\frac{N+\alpha_0}{2}} \tag{4.2}$$

for all $(x,t) \in \mathbf{S}$ with $|x| \ge L(1+t)^{1/2}$.

Proof of Theorem 1.1. By (1.3), (1.11), and the orthonormality of $\{Q_{k,i}\}$, we have

$$\kappa_0 \int_{\mathbf{R}^N} \phi_{0,1}(|y|)U_0(|y|)\rho dy = \int_{\mathbf{R}^N} \phi(|x|)U_0(|x|)dx = M. \tag{4.3}$$

By Propositions 3.2, 3.3, and (4.3), there exist positive constants ϵ and T such that

$$(1+t)^{\frac{N}{2}+\alpha_0} v_{0,1}(x,t) = \kappa_0^{-1} c_0^2 M(1+o(1))U_0(|x|) + O(t^{-1}|x|^2 U_0(|x|)) \tag{4.4}$$

$$(1+t)^{\frac{N}{2}+\alpha_k} v_{k,i}(x,t) = O(U_k(|x|)) \quad \text{if} \quad k \ge 1, i = 1, \ldots, l_k \tag{4.5}$$

for all $(x,t) \in D_\epsilon(T)$. Let $m = 2, 3, \ldots$ such that $\alpha_m > 2\alpha_0$. Since

$$\begin{aligned} u(x,t) - u_m(x,t) &= \kappa_0 v_{0,1}(x,t) \\ &+ \kappa_1 \sum_{i=1}^{N} v_{1,i}(x,t)\frac{x_i}{|x|} + \sum_{k=2}^{m-1}\sum_{i=1}^{l_k} v_{k,i}(x,t) Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad \mathbf{S}, \end{aligned} \tag{4.6}$$

by (4.1), (4.4), and (4.5), we have (1.6). Furthermore, by (1.5), (4.1) with $(k,m) = (0,1)$, and (4.4), we have

$$(1+t)^{\frac{N+\alpha_0}{2}} u\left((1+t)^{1/2}y, t\right) = O(|y|^{\alpha_0}) + O(t^{\frac{\alpha_1-\alpha_0}{2}}) \tag{4.7}$$

for all $y \in \mathbf{R}^N$ with $|y| \le \epsilon$ and $t \ge T$. On the other hand, by Proposition 3.2, Lemma 4.1, (4.3), and (4.6), for any $L > 0$, we see that $u_{0,1}$ is the dominant term and have

$$\begin{aligned} &\lim_{t\to\infty}(1+t)^{\frac{N+\alpha_0+l}{2}}(\nabla_x^l u)\left((1+t)^{1/2}y, t\right) \\ &= \lim_{t\to\infty}(1+t)^{\frac{N+\alpha_0+l}{2}}(\nabla_x^l u_{0,1})\left((1+t)^{1/2}y, t\right) = c_0 M(\nabla_y^l \varphi_0)(y) \end{aligned} \tag{4.8}$$

uniformly for all $y \in \mathbf{R}^N$ with $L^{-1} \le |y| \le L$, where $l = 0,1,2$. Therefore, by (4.2), (4.7), and (4.8) with $k = 0$, we have (1.7) in $L^\infty(\mathbf{R}^N)$. Furthermore, we have

$$\lim_{t\to\infty}(1+t)^{\frac{N+\alpha_0}{2}}\kappa v_{0,1}\left((1+t)^{1/2}y,t\right) = c_0 M\varphi_0(y)$$

in $L^2(\{|y| \ge \epsilon'\}, \rho dy)$ for any $\epsilon' > 0$. These with (4.7) imply (1.7) in $L^2(\mathbf{R}^N, \rho dy)$.

We next assume $M > 0$. By (1.1) and (O), we see that

$$\frac{d}{dt}\int_{\mathbf{R}^N} u(x,t)U_0(|x|)dx = 0, \qquad t > 0,$$

and for any $t_0 > 0$, we have

$$\int_{\mathbf{R}^N} u(x,t_0)U_0(|x|)dx = \int_{\mathbf{R}^N}\phi(x)U_0(|x|)dx = M > 0.$$

So there exists a point x_0 such that $u(x_0,t_0) > 0$. Then, by (4.2), there exists a positive constant $L > 0$ such that $|u(x,t_0)| < u(x_0,t_0)$ for all $|x| \ge L$. This implies that $H(t_0) \ne \emptyset$. Furthermore, since $\varphi_0 = c_0 r^{\alpha_0}e^{-r^2/4}$ takes its maximum only at $r = \sqrt{2\alpha_0}$, by (1.7), we have (1.8), and the proof of Theorem 1.1 is complete. □

Next, in order to prove Theorem 1.2, we give the following lemma on $\alpha(\omega+\omega_k)$.

Lemma 4.2 *Let $\omega \ge 0$ and $k = 0,1,2,\ldots$. Then*

$$\alpha(\omega+\omega_{k+1}) \le \alpha(\omega+\omega_k)+1, \tag{4.9}$$

$$2\alpha(\omega+\omega_{k+1}) \le \alpha(\omega+\omega_{k+2})+\alpha(\omega+\omega_k). \tag{4.10}$$

Proof of Theorem 1.2. We may assume, without loss of generality, that $A_\phi = |A_\phi|e_1 \ne 0$, where $e_1 = (1,0,\ldots,0) \in \mathbf{R}^N$. By (1.3) and (1.11),

$$\begin{aligned}\int_{\mathbf{R}^N}\phi(x)U_1(|x|)\frac{x_i}{|x|}dx &= \kappa_1\int_{\mathbf{R}^N}\phi_{1,i}(|x|)U_1(|x|)\frac{x_i^2}{|x|^2}dx \\ &= \kappa_1 N^{-1}\int_{\mathbf{R}^N}\phi_{1,i}(|x|)U_1(|x|)dx, \quad i = 1,\ldots,N,\end{aligned}$$

and we have

$$\int_{\mathbf{R}^N}\phi_{1,i}(|x|)U_1(|x|)dx = \kappa_1^{-1}N|A_\phi|\delta_{1i}. \tag{4.11}$$

Then, by (3.5) and (4.11), for any $L > 0$ and $l = 0, 1, 2$,

$$(4.12)\quad \lim_{t\to\infty}(1+t)^{\frac{N+\alpha_1+l}{2}}(\partial_r^l v_{1,1})\left((1+t)^{1/2}|y|, t\right) = c_1\kappa_1^{-1}N|A_\phi|(\partial_r^l\varphi_1)(y)$$

uniformly for all y with $L^{-1} \le |y| \le L$. So, by (1.13) and (4.12), for any $L > 0$,

$$(4.13)\quad \begin{aligned} &\lim_{t\to\infty}(1+t)^{\frac{N+\alpha_1+l}{2}}(\nabla_x^l u_{1,1})\left((1+t)^{1/2}y, t\right) \\ &= c_1N|A_\phi|\nabla_y^l\left(\varphi_1(y)\frac{y_1}{|y|}\right) = c_1^2N|A_\phi|\nabla_y^l\left(|y|^{\alpha_1-1}e^{-|y|^2/4}y_1\right)\end{aligned}$$

uniformly for all y with $L^{-1} \le |y| \le L$. Similarly, by (4.9), (3.5), (3.6), and (4.11), for any $(k,i) \notin \{(0,1),(1,1)\}$, $l = 0,1,2$, and $L > 0$, there exists a constant C_1 such that

$$\left|(\partial_r^l v_{k,i})\left((1+t)^{1/2}|y|, t\right)\right| \le C_1 t^{-\frac{N+\alpha_2+l}{2}}$$

for all y with $L^{-1} \le |y| \le L$ and all sufficiently large t. Then, there exists a constant C_2 such that

$$(4.14)\quad \left|(\nabla_x^l u_{k,i})\left((1+t)^{1/2}y, t\right)\right| \le C_2 t^{-\frac{N+\alpha_2+l}{2}}$$

for all y with $L^{-1} \le |y| \le L$ and all sufficiently large t. Put $u^* = u_1 - u_{1,1}$. Then, by (4.2) and (4.14), for any $L > 0$, there exists a constant C_3 such that

$$(4.15)\quad \left|(\nabla_x^l u^*)\left((1+t)^{1/2}y, t\right)\right| \le C_3 t^{-\frac{N+\alpha_2+l}{2}}$$

for all y with $L^{-1} \le |y| \le L$ and all sufficiently large t.

Put

$$(4.16)\quad \zeta(x) = \max_{2\le i\le N}\frac{|x_i|}{|x|}.$$

Then,we have

$$\left|Q_{2,i}\left(\frac{x}{|x|}\right) - Q_{2,i}\left(\frac{|x_1|e_1}{|x|}\right)\right| \preceq \frac{|x - |x|e_1|}{|x|} \le \frac{|x| - x_1}{|x|} + (N-1)\zeta(x)$$

for all $x \in \mathbf{R}^N \setminus \{0\}$. Then, by Theorem 1.1, (4.12), and (4.15), there exists a positive constant c_* such that

$$
\begin{aligned}
(4.17) \qquad & 0 \le u(x,t) - u(|x|e_1, t) \\
&= \kappa_1 v_{1,1}(x,t)\frac{x_1 - |x|}{|x|} + u^*(x,t) - u^*(|x|e_1,t) \\
&= c_* t^{-\frac{N+\alpha_1}{2}}(1+o(1))\frac{x_1-|x|}{|x|} + O(t^{-\frac{N+\alpha_2}{2}})\zeta(x)
\end{aligned}
$$

for all $x \in H(t)$ and sufficiently large t. Thus, any point on $H(t)$ sits in $\mathbf{R}^N_+ \equiv \{x = (x_1, x') \in \mathbf{R}^N : x_1 > 0\}$ for all sufficiently large t and we see that

$$\lim_{t\to\infty} \sup_{x \in H(t)} \zeta(x) = 0.$$

Then we have

$$(4.18) \qquad 1 - \frac{x_1}{|x|} = \left(1 + \frac{x_1}{|x|}\right)^{-1}\left(1 - \frac{x_1^2}{|x|^2}\right) \asymp \zeta(x)^2$$

for all $x \in H(t)$ and all sufficiently large t. Furthermore, by (4.17) and (4.18), there exists a positive constant C_4 such that

$$0 \le -\zeta(x)^2 + C_4 t^{-\frac{\alpha_2-\alpha_1}{2}}\zeta(x)$$

for all $x \in H(t)$ and all sufficiently large t. Therefore, putting

$$C(t) = \left\{x \in \mathbf{R}^N_+ : \inf_{x\in H(t)} |x| \le |x| \le \sup_{x \in H(t)} |x|,\ \zeta(x) \le C_4 t^{-\frac{\alpha_2-\alpha_1}{2}}\right\},$$

we have

$$(4.19) \qquad H(t) \subset C(t)$$

for all sufficiently large t.

Let

$$(4.20) \qquad H_0(t) = \left\{x \in \mathbf{R}^N : u_{0,1}(x,t) = \max_{z \in \mathbf{R}^N} u_{0,1}(z,t)\right\}.$$

Since $(r^{\alpha_0}e^{-r^2/4})'' < 0$ near $r = \sqrt{2\alpha_0}$, we can estimate the Hessian $(\partial_a\partial_\beta u)$ to show the uniqueness of the hot spot. □

Proofs of Theorems 1.3– 1.5. These will be done by using almost the same arguments as above with noting the differences of the asymptotic behaviors stated in Sections 2 and 3. Details can be seen in [10, 11] and we omit the detail. □

Concluding remarks. Throughout this paper, we consider the case when $V \geq 0$. There aries one question: what happens to hot spots when $V \leq 0$? To have a positive solution, V should be restricted so that the Hardy type inequality holds. That is, V sould be a function such that

$$\int_{\mathbf{R}^N} |\nabla u|^2 \, dx \geq \int_{\mathbf{R}^N} V(|x|) u^2 \, dx$$

holds for any $u \in \mathscr{D} := \{u \mid \int_{\mathbf{R}^N} |\nabla u|^2 \, dx < \infty\}$. For such V, we can disucss as in this paper. However, in this case, any points on the hot spot will converge to the origin if $\omega > 0$ unlike the cases in this paper. Details will appear in [12]. For the cases when V changes its sign, more careful analysis will be needed to investigate the behavior of hot spots. For the cases other than commented in this paper, we are required to make other kinds of tools to analyze this type of problems.

References

[1] D. G. Aronson, Non-negative solutions of linear parabolic equations, Ann. Scuola Norm. Sup. Pisa 22 (1968), 607–694.

[2] I. Chavel and L. Karp, Movement of hot spots in Riemannian manifolds, J. Analys Math., 55 (1990), 271–286.

[3] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.

[4] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions of the heat equation, Nonlinear Anal. T. M. A., 11 (1987), 1103–1133.

[5] E. B. Fabes and D. W. Stroock, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash, Arch. Rational Mech. Anal. 96 (1986), 327–338.

[6] K. Ishige, Movement of hot spots on the exterior domain of a ball under the Neumann boundary condition, J. Differential Equations 212 (2005), 394–431.

[7] K. Ishige, Movement of hot spots on the exterior domain of a ball under the Dirichlet boundary condition, preprint.

[8] K. Ishige and Y. Kabeya, Decay rates of the derivatives of the solutions of the heat equations in the exterior domain of a ball, in J. Math. Soc. Japan.

[9] K. Ishige and Y. Kabeya, Large time behaviors of hot spots for the heat equation with a potential, J. Differential Equations **244** (2008), 2934–2962; Corrigendum in J. Differential Equations **245** (2008), 2352–2354.

[10] K. Ishige and Y. Kabeya, Hot spots for the heat equation with a rapidly decaying negative potential, Adv. Differential Equations **14** (2009), 643–662.

[11] K. Ishige and Y. Kabeya, Hot spots for the two dimensional heat equation with a rapidly decaying negative potential, to appear in Discrete. Contin. Dyn. Syst. Ser. S.

[12] K. Ishige and Y. Kabeya, Large time behaviors of hot spots for the heat equation with a potential II: positive potential case, in preparation.

[13] S. Jimbo and S. Sakaguchi, Movement of hot spots over unbounded domains in $\mathbf{R}^N$, J. Math. Anal. Appl. 182 (1994), 810–835.

[14] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., Providence, 1968.

[15] J. R. Norris, Long-time behaviour of heat flow: global estimates and exact asymptotics, Arch. Rational Mech. Anal. 140 (1997), 161–195.

[16] N. Mizoguchi, H. Ninomiya, and E. Yanagida, Critical exponent for the bipolar blowup in a semilinear parabolic equation, J. Math. Anal. Appl. 218 (1998), 495–518.

[17] Qi. S. Zhang, The quantizing effect of potentials on the critical number of reaction-diffusion equations, J. Differential Equations 170 (2001), 188–214.

[18] Qi. S. Zhang, A sharp comparison result concerning Schrödinger heat kernels, Bull. London Math. Soc. 35 (2003), 461–472.