数理解析研究所講究録
第 1702 巻 2010 年 98-107

Uniform stability and attractivity for linear systems with periodic coefficients

杉江実郎 (Jitsuro Sugie)

Department of Mathematics and Computer Science, Shimane University

1 Introduction

In this paper, we consider the linear system

$$\mathbf{x}' = A(t)\mathbf{x} = \begin{pmatrix} -r(t) & p(t) \\ -p(t) & -q(t) \end{pmatrix}\mathbf{x}, \tag{1}$$

where the prime denotes d/dt; the coefficients $p(t)$, $q(t)$ and $r(t)$ are continuous for $t \geq 0$, and $p(t)$ is a periodic function with period $\omega > 0$. System (1) has the zero solution $\mathbf{x}(t) \equiv \mathbf{0} \in \mathbb{R}^2$. We say that the zero solution of (1) is *attractive* if every solution $\mathbf{x}(t)$ of (1) tends to $\mathbf{0}$ as time t increases.

If $q(t)$ and $r(t)$ are also periodic functions with period ω, Floquet's theorem is available. Let $\Phi(t)$ be the fundamental matrix of (1) with $\Phi(0) = E$, the 2×2 identity matrix. Then $\Phi(\omega)$ is called the *monodromy matrix* of (1). Let μ_1 and μ_2 be the eigenvalues of the monodromy matrix $\Phi(\omega)$. The eigenvalues μ_1 and μ_2 are often called the Floquet multipliers of (1). By Abel's formula,

$$\det\Phi(\omega) = \det\Phi(0)\exp\left(-\int_0^\omega (q(s)+r(s))ds\right) = \exp\left(-\int_0^\omega (q(s)+r(s))ds\right).$$

Thus, the Floquet multipliers μ_1 and μ_2 are the roots of the equation

$$\mu^2 - \mathrm{tr}\Phi(\omega)\mu + \exp\left(-\int_0^\omega (q(s)+r(s))ds\right) = 0.$$

It is well-known that the zero solution of (1) is attractive if and only if the Floquet multipliers μ_1 and μ_2 have magnitudes strictly less than 1. Hence, in the case where $p(t)$, $q(t)$ and $r(t)$ are periodic, necessary and sufficient conditions for the zero solution of (1) to be attractive are that

$$|\mathrm{tr}\Phi(\omega)| < 1 + \exp\left(-\int_0^\omega (q(s)+r(s))ds\right)$$

and

$$\exp\left(-\int_0^\omega (q(s)+r(s))ds\right) < 1.$$

For example, we can find Floquet's theorem in the books [2, 3, 5, 8, 16].

Although the above conditions are necessary and sufficient for the zero solution of (1) to be attractive, it is difficult to estimate the absolute value of the trace of $\Phi(\omega)$, because it is impossible to find a fundamental matrix of (1) in general. Of course, Floquet's theorem is useless when $q(t)$ or $r(t)$ is not periodic. Then, without knowledge of a fundamental matrix of (1), can we decide whether the zero solution is attractive? What kind of condition on $A(t)$ will guarantee the attractivity of the zero solution of (1)?

2 The main theorem

To give an answer to the above question, we prepare some notations. Let

$$R(t) = \int_0^t r(s)ds \quad \text{and} \quad \psi(t) = 2(q(t) - r(t))$$

for $t \geq 0$ and denote a positive part and a negative part of $\psi(t)$ by

$$\psi_+(t) = \max\{0, \psi(t)\} \quad \text{and} \quad \psi_-(t) = \max\{0, -\psi(t)\},$$

respectively. Note that $\psi(t) = \psi_+(t) - \psi_-(t)$ and $|\psi(t)| = \psi_+(t) + \psi_-(t)$.

We introduce an important concept here. A nonnegative function $\phi(t)$ is said to be *weakly integrally positive* if

$$\int_I \phi(t)dt = \infty$$

for every set $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$ such that $\tau_n + \delta < \sigma_n < \tau_{n+1} < \sigma_n + \Delta$ for some $\delta > 0$ and $\Delta > 0$. For example, $1/(1+t)$ and $\sin^2 t/(1+t)$ are weakly integrally positive functions (see [6, 7, 13–15]).

Our main result is stated as follows:

Theorem 1. *Suppose that $q(t)$ and $R(t)$ are bounded for $t \geq 0$. Suppose also that*

(i) *$\psi_+(t)$ is weakly integrally positive;*

(ii) $\displaystyle\int_0^\infty \psi_-(t)dt < \infty.$

Then the zero solution of (1) *is attractive.*

To prove Theorem 1, we need some lemmas. We present the lemmas without the proofs.

Lemma 2. *Suppose that assumption* (ii) *in Theorem* 1 *holds. Let $v(t)$ be nonnegative and continuously differentiable on $[t_0, \infty)$ for some $t_0 > 0$. If*

$$v'(t) \leq \psi_-(t)v(t) \quad \textit{for } t \geq t_0, \tag{2}$$

then $v'(t)$ is absolutely integrable, and therefore $v(t)$ has a nonnegative limiting value.

Lemma 3. *Suppose that $R(t)$ is bounded for $t \geq 0$. If assumption* (ii) *in Theorem* 1 *holds, then all solutions of* (1) *are uniformly stable and uniformly bounded.*

Recall that $p(t)$ is a periodic function with period $\omega > 0$. Let

$$\overline{p} = \max_{t\in[0,\omega]} p(t) \quad \text{and} \quad \underline{p} = \min_{t\in[0,\omega]} p(t).$$

Taking $\overline{p} \geq \underline{p}$ into account, we see that if $\overline{p}+\underline{p} \geq 0$, then $\overline{p} > 0$; if $\overline{p}+\underline{p} < 0$, then $\underline{p} < 0$. Since $p(t)$ is continuous for $t \geq 0$, we see that $p(t)$ has the following property (we omit the proof).

Lemma 4. *Let m be any integer. If $\underline{p}+\overline{p} \geq 0$, then there exist numbers a and b with $0 \leq a < b \leq \omega$ such that*

$$p(t) \geq \frac{1}{2}\overline{p} > 0 \quad \textit{for} \quad m\omega + a \leq t \leq m\omega + b.$$

If $\underline{p}+\overline{p} < 0$, then there exist numbers a and b with $0 \leq a < b \leq \omega$ such that

$$p(t) \leq \frac{1}{2}\underline{p} < 0 \quad \text{for} \quad m\omega + a \leq t \leq m\omega + b.$$

3 Proof of the main theorem

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $\mathbf{x}(t; t_0, \mathbf{x}_0)$ be a solution of (1) passing through $(t_0, \mathbf{x}_0) \in [0,\infty)\times\mathbb{R}^2$. It follows from Lemma 3 that for any $\alpha > 0$, there exists a $\beta(\alpha) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \alpha$ imply that

$$\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \beta \quad \text{for} \quad t \geq t_0. \tag{3}$$

For the sake of brevity, we write $(x(t), y(t)) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ and

$$v(t) = V(t, x(t), y(t)).$$

Then, we have

$$v(t) = \frac{1}{2}e^{2R(t)}(x^2(t) + y^2(t)) \tag{4}$$

and

$$v'(t) = -(q(t) - r(t))e^{2R(t)}y^2(t) \leq \psi_-(t)v(t) \tag{5}$$

for $t \geq t_0$. Hence, from Lemma 2, we see that $v(t)$ has a limiting value $v_0 \geq 0$. If $v_0 = 0$, then by (4) the solution $(x(t), y(t))$ tends to $\mathbf{0}$ as $t \to \infty$. This completes the proof. Thus, we need consider only the case in which $v_0 > 0$. We will show that this case does not occur.

Because of (3), we see that $|y(t)|$ is bounded for $t \geq t_0$. Hence, $|y(t)|$ has an inferior limit and a superior limit. First, we will show that the inferior limit of $|y(t)|$ is zero, and we will then show that the superior limit of $|y(t)|$ is also zero.

Suppose that $\liminf_{t\to\infty}|y(t)| > 0$. Then, there exist a $\gamma > 0$ and a $T_1 \geq t_0$ such that $|y(t)| > \gamma$ for $t \geq T_1$. It follows from (5) and Lemma 2 that

$$\begin{aligned}\infty > \int_{t_0}^{\infty}|v'(s)|ds &= \frac{1}{2}\int_{t_0}^{\infty}|\psi(s)|e^{2R(s)}y^2(s)ds \\ &\geq \frac{1}{2}\gamma^2\int_{T_1}^{\infty}\psi_+(s)e^{2R(s)}ds \geq \frac{1}{2}\gamma^2 e^{-2L}\int_{T_1}^{\infty}\psi_+(s)ds,\end{aligned}$$

where L is the number given in the proof of Lemma 3. This contradicts assumption (i). Thus, we see that $\liminf_{t\to\infty}|y(t)| = 0$.

Suppose that $\limsup_{t\to\infty}|y(t)| > 0$. Let $\nu = \limsup_{t\to\infty}|y(t)|$. Since $q(t)$ is bounded, we can find a $\overline{q} > 0$ such that

$$|q(t)| \leq \overline{q} \quad \text{for } t \geq 0. \tag{6}$$

Since $v(t)$ tends to a positive value v_0 as $t \to \infty$, there exists a $T_2 \geq t_0$ such that

$$0 < \frac{1}{2}v_0 < v(t) < \frac{3}{2}v_0 \quad \text{for } t \geq T_2. \tag{7}$$

Let ε be so small that

$$0 < \varepsilon < \min\left\{\frac{1}{2}\nu, \sqrt{\frac{\overline{p}^2 e^{-2L}v_0}{4(\overline{q}+2/(b-a))^2+\overline{p}^2}}, \sqrt{\frac{\underline{p}^2 e^{-2L}v_0}{4(\overline{q}+2/(b-a))^2+\underline{p}^2}}\right\}, \tag{8}$$

where a and b are the numbers given in Lemma 4. Then, since $\liminf_{t\to\infty}|y(t)| = 0$, we can select two intervals $[\tau_n, \sigma_n]$ and $[t_n, s_n]$ with $[t_n, s_n] \subset [\tau_n, \sigma_n]$, $T_2 < \tau_n$ and $\tau_n \to \infty$ as $n \to \infty$ such that $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$, $|y(t_n)| = \nu/2$, $|y(s_n)| = 3\nu/4$ and

$$|y(t)| \geq \varepsilon \quad \text{for } \tau_n < t < \sigma_n, \tag{9}$$

$$|y(t)| \leq \varepsilon \quad \text{for } \sigma_n < t < \tau_{n+1}, \tag{10}$$

$$\frac{1}{2}\nu < |y(t)| < \frac{3}{4}\nu \quad \text{for } t_n < t < s_n. \tag{11}$$

By (4), (7) and (10), we have

$$|x(t)| = \sqrt{2e^{-2R(t)}v(t) - y^2(t)} \geq \sqrt{e^{-2L}v_0 - \varepsilon^2} \tag{12}$$

for $\sigma_n \leq t \leq \tau_{n+1}$.

Claim. The sequences $\{\tau_n\}$ and $\{\sigma_n\}$ satisfy $\tau_{n+1} - \sigma_n \leq 2\omega$ for any integer n.

Suppose that there exists an $n_0 \in \mathbb{N}$ such that $\tau_{n_0+1} - \sigma_{n_0} > 2\omega$. We can choose an $m \in \mathbb{N}$ such that $(m-1)\omega < \sigma_{n_0} \leq m\omega$. Hence, we have

$$\tau_{n_0+1} > \sigma_{n_0} + 2\omega > (m-1)\omega + 2\omega = (m+1)\omega,$$

and therefore $[m\omega, (m+1)\omega] \subset [\sigma_{n_0}, \tau_{n_0+1}]$. There are two cases to consider: (a) $\overline{p} + \underline{p} \geq 0$ and (b) $\overline{p} + \underline{p} < 0$. In case (a), by Lemma 4, $p(t) \geq \overline{p}/2 > 0$ for $t \in [a + m\omega, b + m\omega] \subset$

$[m\omega, (m+1)\omega]$. Hence, using the second equation in system (1) with (6), (10) and (12), we have

$$|y'(t)| \geq |p(t)||x(t)| - |q(t)||y(t)| \geq \frac{1}{2}\bar{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon \tag{13}$$

for $a + m\omega < t < b + m\omega$. It follows from (8) that

$$\frac{1}{2}\bar{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon > \frac{2}{b-a}\varepsilon. \tag{14}$$

From (10) and (13), we can estimate that

$$\begin{aligned} 2\varepsilon &\geq |y(b+m\omega)| + |y(a+m\omega)| \geq \left| \int_{a+m\omega}^{b+m\omega} y'(s)ds \right| \\ &= \int_{a+m\omega}^{b+m\omega} |y'(s)|ds \geq (b-a)\left(\frac{1}{2}\bar{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon\right). \end{aligned}$$

This contradicts (14). In case (b), by Lemma 4, $p(t) \leq \underline{p}/2 < 0$ for $t \in [a + m\omega, b + m\omega] \subset [m\omega, (m+1)\omega]$. Hence, combining this with (6), (10) and (12), we obtain

$$|y'(t)| \geq |p(t)||x(t)| - |q(t)||y(t)| \geq -\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon \tag{15}$$

for $a + m\omega < t < b + m\omega$. It follows from (8) that

$$-\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon > \frac{2}{b-a}\varepsilon. \tag{16}$$

From (10) and (15), we can estimate that

$$\begin{aligned} 2\varepsilon &\geq |y(b+m\omega)| + |y(a+m\omega)| \geq \left| \int_{a+m\omega}^{b+m\omega} y'(s)ds \right| \\ &= \int_{a+m\omega}^{b+m\omega} |y'(s)|ds \geq (b-a)\left(-\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon\right). \end{aligned}$$

This contradicts (16). Thus, the claim is proved.

Let $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$. Then, by means of Lemma 2 with (5) and (9), we get

$$\begin{aligned} \infty > \int_{t_0}^{\infty} |v'(s)|ds &= \frac{1}{2}\int_{t_0}^{\infty} |\psi(s)|e^{2R(s)}y^2(s)ds \\ &\geq \frac{1}{2}e^{-2L}\int_{t_0}^{\infty} \psi_+(s)y^2(s)ds \geq \frac{1}{2}\varepsilon^2 e^{-2L}\int_I \psi_+(s)ds. \end{aligned}$$

Hence, it follows from assumption (i) and the claim that $\liminf_{n\to\infty}(\sigma_n - \tau_n) = 0$. Since $[t_n, s_n] \subset [\tau_n, \sigma_n]$, it follows that

$$\liminf_{n\to\infty}(s_n - t_n) = 0. \tag{17}$$

By (4), (7) and (11), we have

$$|x(t)| = \sqrt{2e^{-2R(t)}v(t) - y^2(t)} \le \sqrt{3e^{2L}v_0 - \frac{\nu^2}{4}}$$

for $t_n \le t \le s_n$. Let $K = \max\{|\overline{p}|, |\underline{p}|\}$. Then, from (6) and (11), we see that

$$|y'(t)| \le |p(t)||x(t)| + |q(t)||y(t)| < K\sqrt{3e^{2L}v_0 - \frac{\nu^2}{4}} + \frac{3}{4}\overline{q}\nu$$

for $t_n \le t \le s_n$. Letting $N = K\sqrt{3e^{2L}v_0 - \nu^2/4} + 3\overline{q}\nu/4$ and integrating this inequality from t_n to s_n, we obtain

$$\begin{aligned}\frac{1}{4}\nu &= |y(s_n)| - |y(t_n)| \le |y(s_n) - y(t_n)| \\ &= \left|\int_{t_n}^{s_n} y'(s)ds\right| \le \int_{t_n}^{s_n} |y'(s)|ds \le N(s_n - t_n).\end{aligned}$$

This contradicts (17). We therefore conclude that $\limsup_{t\to\infty} |y(t)| = \nu = 0$.

In summary, $y(t)$ tends to zero as $t \to \infty$. Hence, there exists a $T_3 \ge T_2$ such that

$$|y(t)| < \varepsilon \quad \text{for } t \ge T_3. \tag{18}$$

Let l be an integer satisfying $l\omega > T_3$. Using (18) instead of (10) and following the same process as in the proof of the claim, we see that if $\overline{p} + \underline{p} \ge 0$, then

$$\begin{aligned}2\varepsilon &\ge |y(b + l\omega)| + |y(a + l\omega)| \ge \left|\int_{a+l\omega}^{b+l\omega} y'(s)ds\right| \\ &= \int_{a+l\omega}^{b+l\omega} |y'(s)|ds \ge (b - a)\left(\frac{1}{2}\overline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \overline{q}\varepsilon\right) > 2\varepsilon,\end{aligned}$$

which is a contradiction; if $\overline{p} + \underline{p} < 0$, then

$$\begin{aligned}2\varepsilon &\ge |y(b + l\omega)| + |y(a + l\omega)| \ge \left|\int_{a+l\omega}^{b+l\omega} y'(s)ds\right| \\ &= \int_{a+l\omega}^{b+l\omega} |y'(s)|ds \ge (b - a)\left(-\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \overline{q}\varepsilon\right) > 2\varepsilon,\end{aligned}$$

which is again a contradiction. Thus, the case of $v_0 > 0$ cannot happen.

The proof of Theorem 1 is thus complete. □

4 Examples

We illustrate our main result with simple examples in which $p(t)$, $q(t)$ and $r(t)$ are periodic. It is well-known that if the zero solution of a linear periodic system is attractive, then it is uniformly asymptotically stable (for example, see [5, 18]).

Example 1. Let $\lambda > 0$. Consider system (1) with

$$p(t) = \cos t, \quad q(t) = \frac{\lambda}{2 - \sin t} \quad \text{and} \quad r(t) = 0. \tag{19}$$

Then the zero solution is attractive.

Since $\lambda/3 \leq q(t) \leq \lambda$ and $R(t) \equiv 0$, it is clear that $q(t)$ and $R(t)$ are bounded for $t \geq 0$. Also, assumptions (i) and (ii) are satisfied. In fact, we have

$$\psi(t) = 2(q(t) - r(t)) = \frac{2\lambda}{2 - \sin t},$$

and therefore

$$\psi_+(t) = \frac{2\lambda}{2 - \sin t} \quad \text{and} \quad \psi_-(t) = 0$$

for $t \geq 0$. Hence, $\psi_+(t)$ is weakly integrally positive and

$$\int_0^\infty \psi_-(t)dt = 0.$$

Thus, by means of Theorem 1, we conclude that the zero solution is attractive.

Figure 1(a) shows a positive orbit of (1) with (19) and $\lambda = 0.1$. The starting point $\mathbf{x}_0$ is $(-1, 0)$ and the initial time t_0 is 0. The positive orbit moves around the origin $\mathbf{0}$ in a clockwise and a counter-clockwise direction alternately, because $p(t)$ changes its sign. The positive orbit approaches the origin $\mathbf{0}$ as it goes up and down.

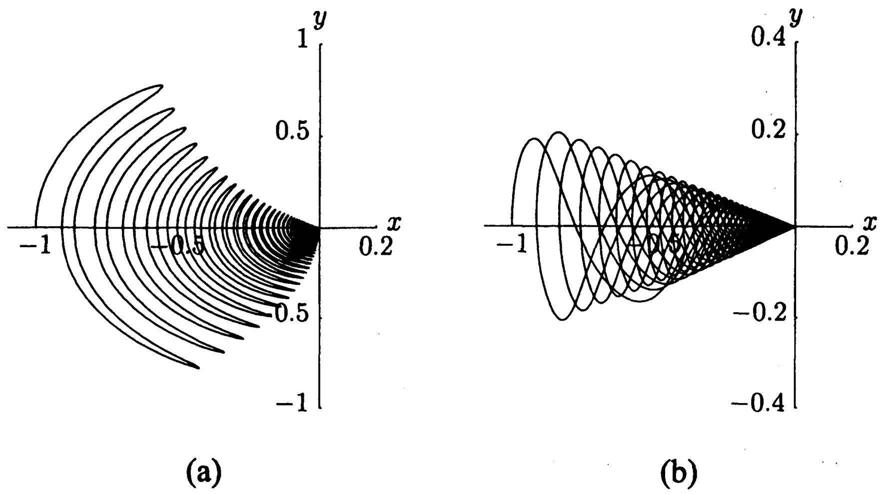

Figure 1: (a) A positive orbit of (1) with (20); (b) a positive orbit of (1) with (21)

Example 2. Let $\lambda \geq 1$. Consider system (1) with

$$p(t) = \cos \lambda t, \quad q(t) = \cos^2 t + \sin t \quad \text{and} \quad r(t) = \sin t. \tag{20}$$

Then the zero solution is attractive.

It is easy to check that $q(t)$ and $R(t)$ are bounded for $t \geq 0$ and that assumptions (i) and (ii) are satisfied. We omit the details.

In Figure 1(b), we show a positive orbit of (1) with (20) and $\lambda = 4$. The positive orbit starts from the point $(-1, 0)$ at the initial time 0. The positive orbit goes to the right and then goes to the left, and it repeats such a movement regularly. Although the positive orbit displays intricate behavior, it approaches the origin 0 ultimately.

In Examples 1 and 2, all coefficients of (1) are periodic functions with period 2π. However, we cannot find the monodromy matrix $\Phi(2\pi)$. It is particularly hard to estimate the absolute value of the trace of $\Phi(2\pi)$. For this reason, we cannot apply Floquet's theorem to Examples 1 and 2 directly. Theorem 1 has the advantage of being applicable to cases where the monodromy matrix of (1) cannot be found and cases where $q(t)$ or $r(t)$ is not periodic.

Fortunately, in Examples 1 and 2 the Floquet multipliers μ_1 and μ_2 can be calculated by a numerical scheme. As shown in Tables 1 and 2, $|\mu_1| < 1$ and $|\mu_2| < 1$. Hence, we see that the zero solution of (1) is attractive.

λ	μ_1	μ_2
1	0.3351718550789	0.0793024028529
0.1	0.8888872982404	0.7827240687567
0.01	0.9882826823640	0.9758079535053
0.001	0.9988220356864	0.9975540561378

Table 1: Floquet multipliers of (1) with (20)

λ	μ_1	μ_2
1	0.5569470757759	0.0775907086028
10	0.9845517600942	0.0438919719768
100	0.9998429464892	0.0432207062297
1000	0.9999986933319	0.0432139974342

Table 2: Floquet multipliers of (1) with (21)

Remark. The zero solution of system (1) with (19) is attractive if and only if $\lambda > 0$. In fact, if $\lambda \leq 0$, then

$$\exp\left(-\int_0^\omega (q(s) + r(s))ds\right) = \exp\left(-\int_0^\omega \frac{\lambda}{2 - \sin s} ds\right) \geq 1.$$

Hence, as mentioned in Section 1, the zero solution is not attractive in this case.

References

[1] A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory*, 2nd ed., Springer-Verlag, Berlin–Heidelberg–New York, 2005.

[2] F. Brauer and J. Nohel, *The Qualitative Theory of Ordinary Differential Equations*, W. A. Benjamin, New York and Amsterdam, 1969; (revised) Dover, New York, 1989.

[3] J. Cronin, *Differential Equations: Introduction and Qualitative Theory*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics 180, Marcel Dekker, New York–Basel–Hong Kong, 1994.

[4] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York–London, 1966.

[5] J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York–London–Sydney, 1969; (revised) Krieger, Malabar, 1980.

[6] L. Hatvani, *On the asymptotic stability by nondecrescent Ljapunov function*, Nonlinear Anal., **8** (1984), 67–77.

[7] L. Hatvani, *On the asymptotic stability for a two-dimensional linear nonautonomous differential system*, Nonlinear Anal., **25** (1995), 991–1002.

[8] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An introduction to Dynamical Systems*, 3rd ed., Oxford Texts in Applied and Engineering Mathematics 2, Oxford University Press, Oxford, 1999.

[9] J. P. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, Mathematics in Science and Engineering 4, Academic Press, New York–London, 1961.

[10] D. R. Merkin, *Introduction to the Theory of Stability*, Texts in Applied Mathematics 24, Springer-Verlag, New York–Berlin–Heidelberg, 1997.

[11] A. N. Michel, L. Hou and D. Liu, *Stability Dynamical Systems: Continuous, Discontinuous, and Discrete Systems*, Birkhäuser, Boston–Basel–Berlin, 2008.

[12] N. Rouche, P. Habets and M. Laloy, *Stability Theory by Liapunov's Direct Method*, Applied Mathematical Sciences 22, Springer-Verlag, New York–Heidelberg–Berlin, 1977.

[13] J. Sugie, *Convergence of solutions of time-varying linear systems with integrable forcing term*, Bull. Austral. Math. Soc., **78** (2008), 445–462.

[14] J. Sugie, *Influence of anti-diagonals on the asymptotic stability for linear differential systems*, Monatsh. Math., **157** (2009), 163–176.

[15] J. Sugie and Y. Ogami, *Asymptotic stability for three-dimensional linear differential systems with time-varying coefficients*, to appear in Quart. Appl. Math.

[16] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Springer-Verlag, New York–Berlin–Heidelberg, 1990.

[17] T. Yoshizawa, *Note on the boundedness and the ultimate boundedness of solutions of* $x' = F(t, x)$, Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math., **29** (1955), 275–291.

[18] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, Math. Soc. Japan, Tokyo, 1966.

[19] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Applied Mathematical Sciences 14, Springer-Verlag, New York–Heidelberg–Berlin, 1975.