

Nonlinear diffusion and geometry of domain *

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1 Introduction

This is based on the author's recent work with R. Magnanini [MS3]. Let Ω be a C^2 domain in \mathbb{R}^N with $N \geq 2$, and let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ satisfy

$$\phi \in C^2(\mathbb{R}), \quad \phi(0) = 0, \quad \text{and} \quad 0 < \delta_1 \leq \phi'(s) \leq \delta_2 \quad \text{for } s \in \mathbb{R}, \quad (1.1)$$

where δ_1, δ_2 are positive constants. Consider the unique bounded solution $u = u(x, t)$ of either the initial-boundary value problem:

$$\partial_t u = \Delta \phi(u) \quad \text{in } \Omega \times (0, +\infty), \quad (1.2)$$

$$u = 1 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (1.3)$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \quad (1.4)$$

or the initial value problem:

$$\partial_t u = \Delta \phi(u) \quad \text{in } \mathbb{R}^N \times (0, +\infty) \quad \text{and} \quad u = \chi_{\Omega^c} \quad \text{on } \mathbb{R}^N \times \{0\}, \quad (1.5)$$

where χ_{Ω^c} denotes the characteristic function of the set $\Omega^c = \mathbb{R}^N \setminus \Omega$. By the maximum principle, we know that

$$0 < u < 1 \quad \text{either in } \Omega \times (0, +\infty) \quad \text{or in } \mathbb{R}^N \times (0, +\infty). \quad (1.6)$$

*This research was partially supported by a Grant-in-Aid for Scientific Research (B) (#20340031) of Japan Society for the Promotion of Science.

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Here, we have

$$\int_0^1 \frac{\phi'(\xi)}{\xi} d\xi = +\infty, \quad (1.7)$$

which means that the equation $\partial_t u = \Delta\phi(u)$ has the property of *infinite* speed of propagation of disturbances from rest. Let $\Phi = \Phi(s)$ be the function defined by

$$\Phi(s) = \int_1^s \frac{\phi'(\xi)}{\xi} d\xi \quad \text{for } s > 0. \quad (1.8)$$

Note that if $\phi(s) \equiv s$, then $\Phi(s) = \log s$. This is the case corresponding to the heat equation. Define the distance function $d = d(x)$ by

$$d(x) = \text{dist}(x, \partial\Omega) \quad \text{for } x \in \Omega. \quad (1.9)$$

Then, in [MS2] we have a generalization of a result of Varadhan [Va] to the nonlinear diffusion equation.

Theorem 1.1 ([MS2, Theorem 1.1 and Theorem 4.1]) *Let u be the solution of either problem (1.2)-(1.4) or problem (1.5). Then*

$$\lim_{t \rightarrow 0^+} -4t\Phi(u(x, t)) = d(x)^2 \quad (1.10)$$

uniformly on every compact set in Ω .

This theorem gives us an interaction between nonlinear diffusion and geometry of domain, since the distance function $d(x)$ is deeply related to the geometry of Ω .

Remark. In [MS2], only the case where $\partial\Omega$ is bounded is treated. *Here, let us show that Theorem 1.1 holds also when $\partial\Omega$ is unbounded.*

Take any point $x_0 \in \Omega$. For each $\varepsilon > 0$, there exist a point $z \in \mathbb{R}^N \setminus \bar{\Omega}$ and $\delta > 0$ such that $|x_0 - z| < d(x_0) + \varepsilon$ and $B_\delta(z) \subset \mathbb{R}^N \setminus \bar{\Omega}$, where $B_\delta(z)$ denotes the open ball in \mathbb{R}^N with radius δ and centered at z .

Consider problem (1.2)-(1.4) first. Let $u^\pm = u^\pm(x, t)$ be bounded solutions of the following initial-boundary value problems:

$$\partial_t u^+ = \Delta\phi(u^+) \quad \text{in } B_{d(x_0)}(x_0) \times (0, +\infty), \quad (1.11)$$

$$u^+ = 1 \quad \text{on } \partial B_{d(x_0)}(x_0) \times (0, +\infty), \quad (1.12)$$

$$u^+ = 0 \quad \text{on } B_{d(x_0)}(x_0) \times \{0\}, \quad (1.13)$$

and

$$\partial_t u^- = \Delta \phi(u^-) \quad \text{in } (\mathbb{R}^N \setminus \overline{B_\delta(z)}) \times (0, +\infty), \quad (1.14)$$

$$u^- = 1 \quad \text{on } \partial B_\delta(z) \times (0, +\infty), \quad (1.15)$$

$$u^- = 0 \quad \text{on } (\mathbb{R}^N \setminus \overline{B_\delta(z)}) \times \{0\}, \quad (1.16)$$

respectively. Then it follows from the comparison principle that

$$u^-(x_0, t) \leq u(x_0, t) \leq u^+(x_0, t) \quad \text{for every } t > 0, \quad (1.17)$$

which gives

$$-4t\Phi(u^-(x_0, t)) \geq -4t\Phi(u(x_0, t)) \geq -4t\Phi(u^+(x_0, t)) \quad \text{for every } t > 0.$$

By [MS2, Theorem 1.1], letting $t \rightarrow 0^+$ yields that

$$(d(x_0) + \varepsilon)^2 \geq \limsup_{t \rightarrow 0^+} (-4t\Phi(u(x_0, t))) \geq \liminf_{t \rightarrow 0^+} (-4t\Phi(u(x_0, t))) \geq d(x_0)^2.$$

This implies (1.10). By a scaling argument, for each $0 < \rho_0 \leq \rho_1 < +\infty$, in every subset E of $\{x \in \Omega : \rho_0 \leq d(x) \leq \rho_1\}$ where $\delta > 0$ can be chosen independently of each point $x \in E$, the convergence in (1.10) is uniform.

It remains to consider problem (1.5). Let $u^\pm = u^\pm(x, t)$ be bounded solutions of the following initial value problems:

$$\partial_t u^+ = \Delta \phi(u^+) \quad \text{in } \mathbb{R}^N \times (0, +\infty) \quad \text{and} \quad u^+ = \chi_{B_{d(x_0)}(x_0)^c} \quad \text{on } \mathbb{R}^N \times \{0\}, \quad (1.18)$$

and

$$\partial_t u^- = \Delta \phi(u^-) \quad \text{in } \mathbb{R}^N \times (0, +\infty) \quad \text{and} \quad u^- = \chi_{\overline{B_\delta(z)}} \quad \text{on } \mathbb{R}^N \times \{0\}, \quad (1.19)$$

respectively. Then by the comparison principle we get (1.17). Thus, with the aid of [MS2, Theorem 4.1], this gives us the conclusion (1.10) similarly.

Let us state our main theorem which also gives us another interaction between nonlinear diffusion and geometry of domain.

Theorem 1.2 ([MS3]) *Let u be the solution of either problem (1.2)-(1.4) or problem (1.5). Let $x_0 \in \Omega$. Assume that $B_R(x_0) \subset \Omega$ and $\overline{B_R(x_0)} \cap \partial\Omega = \{y_0\}$ for some*

$y_0 \in \partial\Omega$. Then we have

$$\lim_{t \rightarrow 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x, t) dx = c(\phi, N) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}}, \quad (1.20)$$

where $\kappa_1(y_0), \dots, \kappa_{N-1}(y_0)$ denote the principal curvatures of $\partial\Omega$ at $y_0 \in \partial\Omega$ with respect to the interior normal direction to $\partial\Omega$, and $c(\phi, N)$ is a positive constant depending only on ϕ and N (Of course, $c(\phi, N)$ depends on the problems (1.2)-(1.4) and (1.5)). When $\kappa_j(y_0) = \frac{1}{R}$ for some $j \in \{1, \dots, N-1\}$, the formula (1.20) holds by setting the right-hand side to be $+\infty$.

Remark. Notice that we have

$$\kappa_j(y_0) \leq \frac{1}{R} \quad \text{for every } j \in \{1, \dots, N-1\}.$$

When $\partial\Omega$ is bounded and ϕ satisfies either $\int_0^1 \frac{\phi'(\xi)}{\xi} d\xi < +\infty$ or $\phi(s) \equiv s$, Theorem 1.2 was proved in [MS1] for problem (1.2)-(1.4). The method of the proof of the present article will enable us to show the same results also when $\partial\Omega$ is unbounded. In [MS1], the supersolutions and subsolutions to problem (1.2)-(1.4) were constructed in $\Omega_\rho \times (0, \tau]$ for sufficiently small $\rho > 0$ and $\tau > 0$, where

$$\Omega_\rho = \{x \in \Omega : d(x) < \rho\}. \quad (1.21)$$

In those processes, the property of *finite* speed of propagation of disturbances from rest coming from $\int_0^1 \frac{\phi'(\xi)}{\xi} d\xi < +\infty$ plays a useful role, since on $\Gamma_\rho \times (0, \tau]$ the solution u equals zero, where we put

$$\Gamma_\rho = \{x \in \Omega : d(x) = \rho\}. \quad (1.22)$$

Therefore, the estimates on $\Gamma_\rho \times (0, \tau]$ were easy. In the case where $\phi(s) \equiv s$, both the linearity of the heat equation and the result of Varadhan [Va] were used in constructing the supersolutions and subsolutions. Here, by using Theorem 1.1, we construct the supersolutions and subsolutions.

2 Outline of the proof of Theorem 1.2

In this section we give an outline of the proof of Theorem 1.2. For the details, see [MS3]. We distinguish two cases:

(I) $\partial\Omega$ is bounded; (II) $\partial\Omega$ is unbounded.

Let us show that case (I) implies case (II). We can find two C^2 domains, say Ω_1, Ω_2 , having bounded boundaries such that Ω_1 and $\mathbb{R}^N \setminus \overline{\Omega_2}$ are bounded, $B_R(x_0) \subset \Omega_1 \subset \Omega \subset \Omega_2$, and there exists $\beta > 0$ satisfying

$$B_\beta(y_0) \cap \partial\Omega \subset \partial\Omega_1 \cap \partial\Omega_2 \quad \text{and} \quad \overline{B_R(x_0)} \cap (\mathbb{R}^N \setminus \Omega_i) = \{y_0\} \quad \text{for } i = 1, 2. \quad (2.1)$$

Let $u_i = u_i(x, t)$ ($i = 1, 2$) be the two bounded solutions of either problem (1.2)-(1.4) or problem (1.5) where Ω is replaced by Ω_1, Ω_2 , respectively. Since $\Omega_1 \subset \Omega \subset \Omega_2$, it follows from the comparison principle that

$$u_2 \leq u \quad \text{in } \Omega \times (0, +\infty) \quad \text{and} \quad u \leq u_1 \quad \text{in } \Omega_1 \times (0, +\infty).$$

Therefore, it follows that for every $t > 0$

$$t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u_2(x, t) \, dx \leq t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x, t) \, dx \leq t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u_1(x, t) \, dx,$$

which shows that case (I) implies case (II).

Thus it suffices to consider case (I). We distinguish two cases:

(IBVP) problem (1.2)-(1.4) ; (IVP) problem (1.5).

Let us consider case (IBVP) first. We quote a result from Atkinson and Peletier [AtP]. It was shown in [AtP] that, for every $c > 0$, there exists a unique C^2 solution $f_c = f_c(\xi)$ of the problem:

$$(\phi'(f_c)f_c)' + \frac{1}{2}\xi f_c' = 0 \quad \text{in } [0, +\infty), \quad (2.2)$$

$$f_c(0) = c, \quad f_c(\xi) \rightarrow 0 \quad \text{as } \xi \rightarrow +\infty, \quad (2.3)$$

$$f_c' < 0 \quad \text{in } [0, +\infty). \quad (2.4)$$

By writing $v_c = v_c(\xi) = \phi(f_c(\xi))$ for $\xi \in [0, +\infty)$, we have:

$$-v'_c(0) = \frac{1}{2} \int_0^\infty f_c(s) ds \text{ for } c > 0; \quad (2.5)$$

$$0 < f_{c_1} < f_{c_2} \text{ on } [0, +\infty) \text{ if } 0 < c_1 < c_2 < +\infty; \quad (2.6)$$

$$0 > v'_{c_1}(0) > v'_{c_2}(0) \text{ if } 0 < c_1 < c_2 < +\infty. \quad (2.7)$$

Furthermore, [AtP, Lemma 4, p. 383] tells us that, for every compact interval I contained in $(0, +\infty)$,

$$\frac{-4\Phi(f_c(\xi))}{\xi^2} \rightarrow 1 \text{ as } \xi \rightarrow +\infty \text{ uniformly for } c \in I. \quad (2.8)$$

Note that, if we put $w(s, t) = f_c(t^{-\frac{1}{2}}s)$ for $s > 0$ and $t > 0$, then w satisfies the one-dimensional problem:

$$\partial_t w = \partial_s^2 \phi(w) \text{ in } (0, +\infty)^2, w = c \text{ on } \{0\} \times (0, +\infty), \text{ and } w = 0 \text{ on } (0, +\infty) \times \{0\}.$$

Let $0 < \varepsilon < \frac{1}{4}$. We can find a sufficiently small $0 < \eta_\varepsilon \ll \varepsilon$ and two C^2 functions $f_\pm = f_\pm(\xi)$ for $\xi \geq 0$ satisfying:

$$f_\pm(\xi) = f_{1\pm\varepsilon} \left(\sqrt{1 \mp 2\eta_\varepsilon} \xi \right) \text{ if } \xi \geq \eta_\varepsilon; \quad (2.9)$$

$$f'_\pm < 0 \text{ in } [0, +\infty); \quad (2.10)$$

$$f_- < f_1 < f_+ \text{ in } [0, +\infty); \quad (2.11)$$

$$(\phi'(f_\pm)f'_\pm)' + \frac{1}{2}\xi f'_\pm = h_\pm(\xi)f'_\pm \text{ in } [0, +\infty), \quad (2.12)$$

where $h_\pm = h_\pm(\xi)$ is defined by

$$h_\pm(\xi) = \begin{cases} \pm\eta_\varepsilon\xi & \text{if } \xi \geq \eta_\varepsilon, \\ \pm\eta_\varepsilon^2 & \text{if } \xi \leq \eta_\varepsilon. \end{cases} \quad (2.13)$$

Here, in order to use the function h_\pm also for case (IVP) later, we defined $h_\pm(\xi)$ for all $\xi \in \mathbb{R}$. Then we notice that

$$h_+ = -h_- \geq \eta_\varepsilon^2 \text{ on } \mathbb{R} \text{ and } f_\pm \rightarrow f_1 \text{ as } \varepsilon \rightarrow 0 \text{ uniformly on } [0, +\infty). \quad (2.14)$$

Set $\Psi = \Phi^{-1}$. Then it follow from (2.8) that there exists $\xi_\varepsilon > 1$ such that

$$\Psi \left(-\frac{\xi^2}{4} \left(1 - \frac{\eta_\varepsilon}{2} \right) \right) > f_c(\xi) > \Psi \left(-\frac{\xi^2}{4} \left(1 + \frac{\eta_\varepsilon}{2} \right) \right) \text{ for } \xi \geq \xi_\varepsilon \text{ and } c \in I_\varepsilon, \quad (2.15)$$

where we set $I_\varepsilon = [1 - 2\varepsilon, 1 + 2\varepsilon]$.

Since $\partial\Omega$ is of class C^2 and bounded, there exists $\rho_0 > 0$ such that the distance function d belongs to $C^2(\overline{\Omega_{\rho_0}})$. Set $\rho_1 = \max\{2R, \rho_0\}$. Theorem 1.1 yields that

$$-4t\Phi(u(x, t)) \rightarrow d(x)^2 \quad \text{as } t \rightarrow 0^+ \quad \text{uniformly on } \overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}. \quad (2.16)$$

Then there exists $\tau_{1,\varepsilon} > 0$ such that for every $t \in (0, \tau_{1,\varepsilon}]$ and every $x \in \overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}$

$$|-4t\Phi(u(x, t)) - d(x)^2| < \frac{1}{2}\eta_\varepsilon\rho_0^2 \leq \frac{1}{2}\eta_\varepsilon d(x)^2.$$

Thus it follows that for every $t \in (0, \tau_{1,\varepsilon}]$ and every $x \in \overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}$

$$\Psi\left(-\frac{(1 - \frac{1}{2}\eta_\varepsilon) d(x)^2}{4t}\right) > u(x, t) > \Psi\left(-\frac{(1 + \frac{1}{2}\eta_\varepsilon) d(x)^2}{4t}\right). \quad (2.17)$$

From (2.15), we have

$$f_+(\xi) = f_{1+\varepsilon}(\sqrt{1 - 2\eta_\varepsilon} \xi) > \Psi\left(-\frac{\xi^2}{4}\left(1 - \frac{\eta_\varepsilon}{2}\right)\right) \quad \text{if } \xi \geq \frac{\xi_\varepsilon}{\sqrt{1 - 2\eta_\varepsilon}}; \quad (2.18)$$

$$f_-(\xi) = f_{1-\varepsilon}(\sqrt{1 + 2\eta_\varepsilon} \xi) < \Psi\left(-\frac{\xi^2}{4}\left(1 + \frac{\eta_\varepsilon}{2}\right)\right) \quad \text{if } \xi \geq \frac{\xi_\varepsilon}{\sqrt{1 + 2\eta_\varepsilon}}. \quad (2.19)$$

Define the two functions $w_\pm = w_\pm(x, t)$ by

$$w_\pm(x, t) = f_\pm\left(t^{-\frac{1}{2}}d(x)\right) \quad \text{for } (x, t) \in \Omega \times (0, +\infty). \quad (2.20)$$

Hence it follows from (2.17), (2.18) and (2.19) that there exists $\tau_{2,\varepsilon} \in (0, \tau_{1,\varepsilon}]$ satisfying

$$w_- < u < w_+ \quad \text{in } (\overline{\Omega_{\rho_1}} \setminus \Omega_{\rho_0}) \times (0, \tau_{2,\varepsilon}]. \quad (2.21)$$

Since $d \in C^2(\overline{\Omega_{\rho_0}})$ and $|\nabla d| = 1$ in $\overline{\Omega_{\rho_0}}$, we have

$$\partial_t w_\pm - \Delta\phi(w_\pm) = -f'_\pm t^{-1} \left\{ h_\pm + \sqrt{t} \phi'(f_\pm) \Delta d \right\} \quad \text{in } \overline{\Omega_{\rho_0}} \times (0, +\infty). \quad (2.22)$$

Therefore, it follows from the former formula of (2.14) that there exists $\tau_{3,\varepsilon} \in (0, \tau_{2,\varepsilon}]$ satisfying

$$\partial_t w_- - \Delta\phi(w_-) < 0 < \partial_t w_+ - \Delta\phi(w_+) \quad \text{in } \Omega_{\rho_0} \times (0, \tau_{3,\varepsilon}]. \quad (2.23)$$

Observe that

$$w_- = u = w_+ = 0 \text{ in } \Omega_{\rho_0} \times \{0\}, \quad (2.24)$$

$$w_- = f_-(0) < 1 = f_1(0) = u < f_+(0) = w_+ \text{ on } \partial\Omega \times (0, \tau_{3,\varepsilon}], \quad (2.25)$$

$$w_- < u < w_+ \text{ on } \Gamma_{\rho_0} \times (0, \tau_{3,\varepsilon}]. \quad (2.26)$$

Note that (2.26) comes from (2.21). Thus it follows from the comparison principle and (2.21) that

$$w_- \leq u \leq w_+ \text{ in } \overline{\Omega_{\rho_1}} \times (0, \tau_{3,\varepsilon}]. \quad (2.27)$$

Here we quote a geometric lemma from [MS1] adjusted to our situation:

Lemma 2.1 ([MS1, Lemma 2.1, p. 376]) *Suppose that $\kappa_j(y_0) < \frac{1}{R}$ for every $j = 1, \dots, N-1$. Then we have:*

$$\lim_{s \rightarrow 0^+} s^{-\frac{N-1}{2}} \mathcal{H}^{N-1}(\Gamma_s \cap B_R(x_0)) = 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}}, \quad (2.28)$$

where \mathcal{H}^{N-1} is the standard $(N-1)$ -dimensional Hausdorff measure, and ω_{N-1} is the volume of the unit ball in \mathbb{R}^{N-1} .

First of all, let us consider the case where $\kappa_j(y_0) < \frac{1}{R}$ for every $j = 1, \dots, N-1$.

It follows from (2.27) that

$$\int_{B_R(x_0)} w_- \, dx \leq \int_{B_R(x_0)} u \, dx \leq \int_{B_R(x_0)} w_+ \, dx \text{ for every } t \in (0, \tau_{3,\varepsilon}]. \quad (2.29)$$

Since with the aid of the co-area formula we have

$$\int_{B_R(x_0)} w_{\pm} \, dx = t^{\frac{N+1}{4}} \int_0^{2Rt^{-\frac{1}{2}}} f_{\pm}(\xi) \xi^{\frac{N-1}{2}} \left(t^{\frac{1}{2}} \xi \right)^{-\frac{N-1}{2}} \mathcal{H}^{N-1} \left(\Gamma_{t^{\frac{1}{2}} \xi} \cap B_R(x_0) \right) d\xi,$$

by using Lebesgue's dominated convergence theorem and Lemma 2.1 we get

$$\lim_{t \rightarrow 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} w_{\pm} \, dx = 2^{\frac{N-1}{2}} \omega_{N-1} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}} \int_0^{\infty} f_{\pm}(\xi) \xi^{\frac{N-1}{2}} d\xi.$$

Therefore, since $\varepsilon > 0$ is arbitrarily small, the latter formula of (2.14) yields (1.20), where we set

$$c(\phi, N) = 2^{\frac{N-1}{2}} \omega_{N-1} \int_0^{\infty} f_1(\xi) \xi^{\frac{N-1}{2}} d\xi.$$

It remains to consider the case where $\kappa_j(y_0) = \frac{1}{R}$ for some $j \in \{1, \dots, N-1\}$. Choose a sequence of balls $\{B_{R_k}(x_k)\}_{k=1}^{\infty}$ satisfying:

$R_k < R$, $y_0 \in \partial B_{R_k}(x_k)$ and $B_{R_k}(x_k) \subset B_R(x_0)$ for every $k \geq 1$, and $\lim_{k \rightarrow \infty} R_k = R$.

Since $\kappa_j(y_0) \leq \frac{1}{R} < \frac{1}{R_k}$ for every $j = 1, \dots, N-1$ and every $k \geq 1$, we can apply the previous case to each $B_{R_k}(x_k)$ to see that for every $k \geq 1$

$$\begin{aligned} \liminf_{t \rightarrow 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x, t) dx &\geq \liminf_{t \rightarrow 0^+} t^{-\frac{N+1}{4}} \int_{B_{R_k}(x_k)} u(x, t) dx \\ &= c(\phi, N) \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R_k} - \kappa_j(y_0) \right) \right\}^{-\frac{1}{2}}. \end{aligned}$$

Hence, letting $k \rightarrow \infty$ yields that

$$\liminf_{t \rightarrow 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} u(x, t) dx = +\infty,$$

which completes the proof for problem (1.2)-(1.4).

Let us consider case (IVP) and let $u = u(x, t)$ be the solution of problem (1.5). We replace problem (2.2)-(2.4) by the following problem for every $c > 0$:

$$(\phi'(f_c) f_c')' + \frac{1}{2} \xi f_c' = 0 \text{ in } \mathbb{R}, \quad (2.30)$$

$$f_c(\xi) \rightarrow c \text{ as } \xi \rightarrow -\infty, \quad f_c(\xi) \rightarrow 0 \text{ as } \xi \rightarrow +\infty, \quad (2.31)$$

$$f_c' < 0 \text{ in } \mathbb{R}. \quad (2.32)$$

By writing $v_c = v_c(\xi) = \phi(f_c(\xi))$ for $\xi \in \mathbb{R}$, we have:

$$-v_c'(0) = \frac{1}{2} \int_0^{\infty} f_c(s) ds \text{ for } c > 0; \quad (2.33)$$

$$0 < f_{c_1} < f_{c_2} \text{ on } \mathbb{R} \text{ if } 0 < c_1 < c_2 < +\infty; \quad (2.34)$$

$$0 > v_{c_1}'(0) > v_{c_2}'(0) \text{ if } 0 < c_1 < c_2 < +\infty. \quad (2.35)$$

Then [AtP, Lemma 4, p. 383] tells us that (2.8) also holds for the solution f_c of this problem. Note that if we put $w(s, t) = f_c(t^{-\frac{1}{2}}s)$ for $s \in \mathbb{R}$ and $t > 0$, then w satisfies the one-dimensional initial value problem:

$$\partial_t w = \partial_s^2 \phi(w) \text{ in } \mathbb{R} \times (0, +\infty) \text{ and } w = c\chi_{(-\infty, 0]} \text{ on } \mathbb{R} \times \{0\}.$$

Let $0 < \varepsilon < \frac{1}{4}$. We can find a sufficiently small $0 < \eta_\varepsilon \ll \varepsilon$ and two C^2 functions $f_\pm = f_\pm(\xi)$ for $\xi \in \mathbb{R}$ satisfying:

$$f_\pm(\xi) = f_{1\pm\varepsilon} \left(\sqrt{1 \mp 2\eta_\varepsilon} \xi \right) \quad \text{if } \xi \geq \eta_\varepsilon; \quad (2.36)$$

$$f'_\pm < 0 \quad \text{in } \mathbb{R}; \quad (2.37)$$

$$f_-(-\infty) < 1 = f_1(-\infty) < f_+(-\infty) \quad \text{and} \quad f_- < f_1 < f_+ \quad \text{in } \mathbb{R}; \quad (2.38)$$

$$\left(\phi'(f_\pm) f'_\pm \right)' + \frac{1}{2} \xi f'_\pm = h_\pm(\xi) f'_\pm \quad \text{in } \mathbb{R}. \quad (2.39)$$

Then we also have (2.14).

Moreover, it follows from (2.8) that there exists $\xi_\varepsilon > 1$ satisfying (2.15). Proceeding similarly yields (2.16), (2.17), (2.18) and (2.19). Let us consider the signed distance function $d^* = d^*(x)$ of $x \in \mathbb{R}^N$ to the boundary $\partial\Omega$ defined by

$$d^*(x) = \begin{cases} \text{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\text{dist}(x, \partial\Omega) & \text{if } x \notin \Omega. \end{cases} \quad (2.40)$$

Since $\partial\Omega$ is of class C^2 and bounded, there exists a number $\rho_0 > 0$ such that $d^*(x)$ is C^2 -smooth on a compact neighborhood \mathcal{N} of the boundary $\partial\Omega$ given by

$$\mathcal{N} = \{x \in \mathbb{R}^N : -\rho_0 \leq d^*(x) \leq \rho_0\}. \quad (2.41)$$

For simplicity we have used the same $\rho_0 > 0$ as in (2.16). Define $w_\pm = w_\pm(x, t)$ by

$$w_\pm(x, t) = f_\pm \left(t^{-\frac{1}{2}} d^*(x) \right) \quad \text{for } (x, t) \in \mathbb{R}^N \times (0, +\infty). \quad (2.42)$$

Then we also have (2.21). Since $d^* \in C^2(\mathcal{N})$ and $|\nabla d^*| = 1$ in \mathcal{N} , we have

$$\partial_t w_\pm - \Delta \phi(w_\pm) = -f'_\pm t^{-1} \left\{ h_\pm + \sqrt{t} \phi'(f_\pm) \Delta d^* \right\} \quad \text{in } \mathcal{N} \times (0, +\infty). \quad (2.43)$$

Therefore, it follows from the former formula of (2.14) that there exists $\tau_{3,\varepsilon} \in (0, \tau_{2,\varepsilon}]$ satisfying:

$$\partial_t w_- - \Delta \phi(w_-) < 0 < \partial_t w_+ - \Delta \phi(w_+) \quad \text{in } \mathcal{N} \times (0, \tau_{3,\varepsilon}], \quad (2.44)$$

$$w_- \leq u \leq w_+ \quad \text{in } \mathcal{N} \times \{0\}, \quad (2.45)$$

$$w_- < u < w_+ \quad \text{on } \partial\mathcal{N} \times (0, \tau_{3,\varepsilon}]. \quad (2.46)$$

Note that in (2.46) the inequality on $\Gamma_{\rho_0} \times (0, \tau_{3,\varepsilon}]$ comes from (2.21) and the inequality on $(\partial\mathcal{N} \setminus \Gamma_{\rho_0}) \times (0, \tau_{3,\varepsilon}]$ comes from the former formula of (2.38). Thus it follows from the comparison principle and (2.21) that

$$w_- \leq u \leq w_+ \quad \text{in } \overline{\mathcal{N} \cup \Omega_{\rho_1}} \times (0, \tau_{3,\varepsilon}]. \quad (2.47)$$

Then, with the aid of (2.47) the rest of the proof runs similarly.

Acknowledgement.

The author would like to thank Professor Hitoshi Ishii for the idea introducing the sequence of balls $\{B_{R_k}(x_k)\}_{k=1}^\infty$.

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