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RECTIFIABLE, UNRECTIFIABLE AND FRACTAL OSCILLATIONS OF SOLUTIONS OF LINEAR AND HALF-LINEAR DIFFERENTIAL EQUATIONS OF SECOND-ORDER

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Abstract. In this expository paper, we pay attention to a new kind of oscillations of solutions of the second-order differential equations on the finite interval. It is the so-called rectifiable, unrectifiable and fractal oscillations of real functions and solutions of differential equations introduced in Pašić [8], [9] and Wong [15], and continued to study in [4], [6], [10], [11], [12], and [16].

1. Motivation for the oscillations near $x = 0$

We consider famous Euler linear differential equation,

$$y'' + \lambda x^{-2} y = 0,\ x \in (0,\infty),\ \lambda > 0. \tag{1}$$

DEFINITION 1. A function $y(x)$ oscillates near $x = 0$ if there is a decreasing sequence $a_n \in (0,1]$ such that $a_n \searrow 0$ and $y(a_n) = 0$. A function $y(x)$ oscillates near $x = \infty$ if there is an increasing sequence $a_n \in [T,\infty)$, for some $T > 0$, such that $a_n \to \infty$.

The following basic facts on the equation (1) are very well known:

- if $\lambda > 1/4$, then all solutions $y(x)$ of (1) are given by the formula

$$y(x) = c_1\sqrt{x}\cos(\rho \ln x) + c_2\sqrt{x}\sin(\rho \ln x),$$

where $\rho = \sqrt{\lambda - 1/4}$;

- $y(x)$ are oscillating near both $x = 0$ and $x = \infty$, see Figures 1 and 2 below:

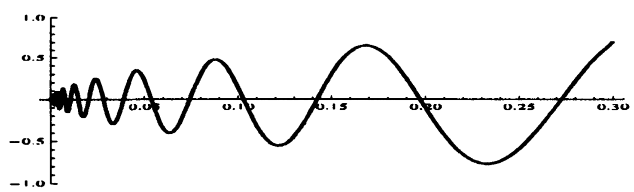

Figure 1: oscillations near $x = 0$

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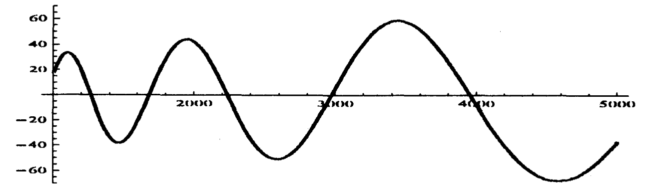


Figure 2: oscillations near $x = \infty$

2. Definition of the rectifiable and unrectifiable oscillations on $[0,1]$

Let $G(y) \subseteq \mathbb{R}^2$ denote the graph of a function $y : [0,1] \to \mathbb{R}$, defined by

$$G(y) = \{(x,y) \in \mathbb{R}^2 : x \in [0,1], y = y(x)\}.$$

Its length is defined by:

$$\operatorname{length} G(y) = \sup \sum_{i=1}^{m} ||(t_i, y(t_i)) - (t_{i-1}, y(t_{i-1}))||_2,$$

where $0 = t_0 < t_1 < \cdots < t_m = 1$ is a partition of the unit interval. Ofcourse, in the case when $y \in C^1((0,1])$, then the length of $G(y)$ can be calculated by the formula,

$$\operatorname{length} G(y) = \lim_{\delta \to 0} \int_{\delta}^{1} \sqrt{1 + y'^2(x)}\, dx.$$

DEFINITION 2. A function $y(x)$ is *rectifiable oscillatory* on $[0,1]$ if $y(x)$ oscillates near $x = 0$ and $\operatorname{length} G(y) < \infty$. A function $y(x)$ is *unrectifiable oscillatory* on $[0,1]$ if $y(x)$ oscillates near $x = 0$ and $\operatorname{length} G(y) = \infty$.

EXAMPLE 1. All solutions of the Euler equation

$$y'' + \lambda x^{-2} y = 0,\ x \in (0,1],\ \lambda > 1/4,$$

are rectifiable oscillatory on $[0,1]$, where $y(x)$ are explicitly given by:

$$y(x) = c_1 \sqrt{x} \cos(\rho \ln x) + c_2 \sqrt{x} \sin(\rho \ln x),\ \rho = \sqrt{\lambda - 1/4}.$$

EXAMPLE 2. All solutions $y(x)$ of the linear equation,

$$y'' + \lambda x^{-4} y = 0,\ x \in (0,1],\ \lambda > 0,$$

are unrectifiable oscillatory on $[0,1]$, where $y(x)$ are explicitly given by:

$$y(x) = c_1 x \cos(\sqrt{\lambda}/x) + c_2 x \sin(\sqrt{\lambda}/x).$$

3. Rectifiable and unrectifiable oscillations of linear differential equations

According to Examples 1 and 2, it is naturaly to pose the following questions: what is about the rectifiable and unrectifiable oscillations of the linear second-order differential equation of Euler type:

$$y'' + \lambda x^{-\sigma} y = 0,\ x \in (0,1], \tag{2}$$

where $\lambda > 0$ and $\sigma \geqslant 2$? Does it depend on the values of σ? In particular for $\sigma = 2$ and $\sigma = 4$, the answer is given in Examples 1 and 2. However, a complete answer to this question is given in the following result.

THEOREM 1. *We have:*
(i) *if* $2 \leqslant \sigma < 4$, *then all solution of Eq.* (2) *are rectifiable oscillatory on* $[0,1]$;
(ii) *if* $\sigma \geqslant 4$, *then all solution of Eq.* (2) *are unrectifiable oscillatory on* $[0,1]$.

The proof of Theorem 1 was published in [8] and [15]. Precisely, Theorem 1 in [8] was considered where the following properties of solutions $y(x)$ of Eq. (2) are presumed:

$$|y(x)| \leqslant cx^{\sigma/4} \text{ and } |y'(x)| \leqslant cx^{-\sigma/4} \text{ near } x = 0.$$

In [15], the previous statement is verified for all solutions of the equation Eq. (2), see also Lemma 4 below.

The proof of Theorem 1 is based on the following two lemmas.

LEMMA 1. (see [8]) *Let* $y \in C([0,1])$ *oscillate near* $x = 0$. *Let* $s_n \in (0,1]$ *be a decreasing sequence,* $s_n \searrow 0$ *and* $y'(s_n) = 0$. *Then we have:*

$$\text{length}\, G(y) < \infty \quad \textit{if and only if} \quad \sum_{n=1}^{\infty} |y(s_n)| < \infty.$$

LEMMA 2. *Let* $y(x)$ *be a solution of Eq.* (2). *Let* $s_n \in (0,1]$ *be a decreasing sequence,* $s_n \searrow 0$ *and* $y'(s_n) = 0$. *Then there are two positive constants* c_1 *and* c_2 *such that:*
(i) (see [15])

$$c_1 s_n^{\sigma/4} \leqslant |y(s_n)| \leqslant c_2 s_n^{\sigma/4} \ \ (\textit{i. e.}\ |y(s_n)| \sim s_n^{\sigma/4}),$$

(ii) (see [8])

$$c_1 n^{-2/(\sigma-2)} \leqslant s_n \leqslant c_2 n^{-2/(\sigma-2)} \ \ (\textit{i. e.}\ s_n \sim n^{-2/(\sigma-2)}).$$

Involving the precise asymptotic behaviour of s_n and $|y(s_n)|$ from Lemma 2 into Lemma 1, we get the proof of Theorem 1.

The result of Theorem 1 could be generalized to the general linear differential equations:

$$y'' + f(x)y = 0,\ x \in (0,1], \tag{3}$$

where $f \in C^2((0,1])$ satisfies:

$$f(x) > 0 \text{ and } f'(x) < 0 \text{ on } (0,1],\ f(0+) = \infty, \tag{4}$$

$$\sqrt{f} \notin L^1(0,1) \text{ and } f^{-1/4}(f^{-1/4})'' \in L^1(0,1). \tag{5}$$

THEOREM 2. *Let $f(x)$ satisfy the conditions (4) and (5). Then all solutions of Eq. (3) oscillate near $x = 0$. Moreover,*

$$\text{length}\, G(y) < \infty \ \textit{if and only if} \ \int_{0+}^{1} \sqrt[4]{f(x)}dx < \infty.$$

The proof of Theorem 2 was published in [4]. It is based on the following three lemmas.

LEMMA 3. (see the book [1]) *Let $y \in C([0,1])$. Then we have:*

$$\text{length}\, G(y) < \infty \ \textit{if and only if} \ \int_{0+}^{1} |y'(x)|dx < \infty.$$

LEMMA 4. (see [15]) *Let $f(x)$ satisfy (4) and (5). There is a positive constant c such that for all solutions of Eq. (3) we have:*

$$|y(x)| \leqslant \frac{c}{\sqrt[4]{f(x)}} \ \textit{and} \ |y'(x)| \leqslant c\sqrt[4]{f(x)} \ \textit{near}\ x = 0.$$

LEMMA 5. *Let $y(x)$ be a solution of Eq. (3). Let $s_n \in (0,1]$ be a decreasing sequence, $s_n \searrow 0$ and $y'(s_n) = 0$. There are $c_1 > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\frac{c_1}{\sqrt[4]{f(s_n)}} \leqslant |y(s_n)|,\ \forall n \geqslant n_0 \ \textit{and} \ \int_{s_n}^{1} \sqrt{f(x)}dx \sim n \ \textit{as}\ n \to \infty.$$

It is not difficult to check that the function $f(x) = x^{-\sigma}$, $x \in (0,1]$ and $\sigma > 2$, satisfies the conditions (4) and (5). Moreover,

$$\textstyle\int_{0+}^{1} \sqrt[4]{f(x)}dx < \infty \ \text{if and only if} \ \sigma < 4.$$

Hence, Theorem 1 is an easy consequence of Theorem 2.

4. Some consequences of Theorem 2

According to Theorem 2, we are able to establish the rectifiable and unrectifiable oscillations for some classes of linear differential equations which are different than the Euler type Eq. (2).

COROLLARY 1. *Let* $f(x)$ *satisfy the conditions* (4) *and* (5). *Let* $f(x) \sim x^{-\sigma}$ *near* $x = 0$. *Then we have:*
(i) *if* $2 \leqslant \sigma < 4$, *then all solution of* $y'' + f(x)y = 0$, $x \in (0,1]$, *are rectifiable oscillatory on the interval* $[0,1]$;
(ii) *if* $\sigma \geqslant 4$, *then all solution of* $y'' + f(x)y = 0$, $x \in (0,1]$, *are unrectifiable oscillatory on the interval* $[0,1]$.

COROLLARY 2. *We consider the following chirp's equation*

$$y'' + x^{-2}[\beta^2 x^{-2\beta} + (1-\beta^2)/4]y = 0, \; x \in (0,1]. \tag{6}$$

Then we have:
(i) *if* $0 < \beta < 1$, *then all solution of Eq.* (6) *are rectifiable oscillatory on* $[0,1]$;
(ii) *if* $\beta \geqslant 1$, *then all solution of Eq.* (6) *are unrectifiable oscillatory on* $[0,1]$.

COROLLARY 3. (see Wong [15]) *We consider the following linear equation*

$$y'' + \lambda x^{-4} e^{\frac{2}{x}} y = 0, \; x \in (0,1], \; \lambda > 0. \tag{7}$$

Then all solution of Eq. (7) *are unrectifiable oscillatory on* $[0,1]$.

5. On Hartman-Wintner conditions

Let us recall that the Hartman-Wintner conditions (5) plays the esential role in the proof of Theorem 2. Therefore, it is worthile to find the answer to the following open question: does it possible to construct the coefficient $f(x)$ satisfying (4) and the first Hartman-Wintner condition from (5) but does not satisfy the second one from (5)? That is to say, we would like to find $f(x)$ with the following properties:

$$f(x) > 0 \text{ and } f'(x) < 0 \text{ on } (0,1], \; f(0+) = \infty, \tag{8}$$

$$\sqrt{f} \notin L^1(0,1) \text{ and } f^{-1/4}(f^{-1/4})'' \notin L^1(0,1). \tag{9}$$

In the solving of this problem, we find that the following lemma could be of some interest.

LEMMA 6. *Let* $f(x)$ *satisfy* (8) *and the second Hartman-Wintner condition from* (5). *Then* $f(x)$ *must satisfy the first Hartman-Wintner condition from* (9) *and the following two:*

$$\lim_{x \to 0} f^{-\frac{3}{2}}(x) f'(x) = 0 \;\; \textit{and} \;\; [f^{-\frac{3}{2}} f']' \in L^1(0,1).$$

In order to prove this lemma, we suggest reader to follow a method presented in the proof of [11, Lemma 2].

6. Coexistence of the rectifiable and unrectifiable oscillations

According to Theorem 2, we observe the following consequence.

COROLLARY 4. *Let $f(x)$ satisfy the condition* (4) *and* (5). *Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of $y'' + f(x)y = 0$, $x \in (0,1]$. Then $y_1(x)$ and $y_2(x)$ are both rectifiable oscillatory on $[0,1]$ at the same time.*

Hence it is reasonable to pose the following question: it is possible to construct the coefficient $f(x)$ satisfying (8) and (9) such that $y_1(x)$ is rectifiable and at the same time $y_2(x)$ is unrectifiable oscillatory on the interval $[0,1]$?
The answer is yes and it could be found in the last section of [4].

7. Rectifiable and unrectifiable oscillations of solutions of the half-linear differential equations

In this section, we consider half-linear differential equation,

$$(|y'|^{p-2}y')' + f(x)|y|^{p-2}y = 0,\ x \in (0,1], \tag{10}$$

where $f(x)$ besides (4) satisfies the following Hartman-Wintner type conditions generalizing the related ones in (5) from $p=2$ to $p>1$:

$$f^{\frac{1}{p}} \notin L^1(0,1) \text{ and } f^{-\frac{1}{pq}}\left[f^{-\frac{1}{p^2}}\right]'' \in L^1(0,1). \tag{11}$$

In particular for $p=2$, obviously Eq. (10) becames the linear equation Eq. (3) considered in previous sections. The following result is a natural generalization of Theorem 2 from linear to the half-linear equations.

THEOREM 3. *Let $f(x) > 0$ and $f'(x) < 0$ on $(0,1]$, $f(0+) = \infty$ and satisfy* (11). *Then all solutions of Eq.* (10) *oscillate near $x = 0$. Moreover,*

$$\operatorname{length} G(y) < \infty \text{ if and only if } \int_{0+}^{1} f^{\frac{1}{p^2}}(x)dx < \infty.$$

The proof of Theorem 3 has been published in [11]. It is based on the follwing two steps.

First step. Every solution $y(x)$ of Eq. (10) could be written in the form:

$$y(x) = (p-1)^{\frac{1}{pq}} f^{-\frac{1}{pq}}(x) V^{\frac{1}{p}}(x) w(\varphi(x)),$$

$$|y'|^{p-2}y' = -(p-1)^{-\frac{1}{pq}} f^{\frac{1}{pq}}(x) V^{\frac{1}{q}}(x) |w'(\varphi(x))|^{p-2} w'(\varphi(x)),$$

where the function $w = w(t)$, $t > 0$, is the so-called generalized sine function,

$$(|w'(x)|^{p-2}w'(x))' + (p-1)|w(x)|^{p-2}w(x) = 0,\ w(0) = 0,\ w'(0) = 1,$$

$$|w'(t)|^p + |w(t)|^p \equiv 1 \text{ for all } t > 0.$$

Second step. It is important to show that the functions $V(x)$ and $\varphi(x)$ satisfy the equations:

$$\varphi'(x) = \frac{-1}{(p-1)^{\frac{1}{p}}} f^{\frac{1}{p}}(x) + \frac{1}{p}\frac{f'(x)}{f(x)} |w'(\varphi(x))|^{p-2} w'(\varphi(x)) w(\varphi(x)),$$

$$V'(x) = \left[(p-1)^{\frac{1}{p}} f^{-\frac{1}{p}}(x)\right]' |y'|^p + \left[(p-1)^{-\frac{1}{q}} f^{\frac{1}{q}}(x)\right]' |y|^p,$$

and the following asymptotic conditions:

$$\varphi'(x) < 0 \text{ for all } x \in (0,1] \quad \text{and} \quad \lim_{x \to 0+} \varphi(x) = \infty,$$

$$0 < \lim_{x \to 0+} V(x) < +\infty.$$

Now, according to the previous two steps and by using the same geometric lemmas as in the of Theorem 2, one can derive the proof of Theorem 3.

8. Further generalization: two-point oscillations

In this section, we present the oscillations of solutions of the Dirichlet problem on the unit interval which was introduced in [12].

DEFINITION 3. A function $y(x)$ is *two-point oscillatory* on $[0,1]$ if $y(x)$ oscillates at the same time near $x = 0$ and $x = 1$. That is, if there is a decreasing sequence $a_n \in (0,1]$ and increasing sequence $b_n \in [0,1)$ such that: $a_n \searrow 0$, $b_n \nearrow 1$, and $y(a_n) = y(b_n) = 0$, see figure below:

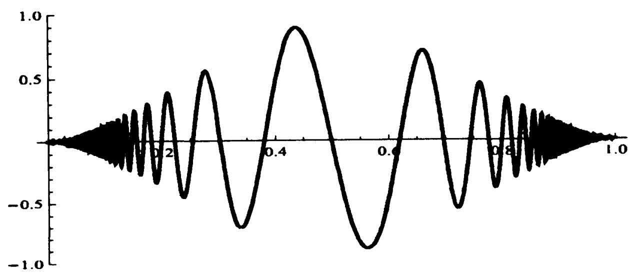


Figure 3: two-point oscillations with higher density near $x = 0$ and $x = 1$

The main motivation to study this kind of oscillations we obtain from the oscillations of the so-called Riemann-Weber version of Euler linear differential equation,

$$y'' + x^{-2}\left(\frac{1}{4} + \frac{\lambda}{|\ln x|^2}\right)y = 0,\ x \in (0,1),\ \lambda > 0. \tag{12}$$

About this equation it is known:

- if $\lambda > 1/4$, then all solutions $y(x)$ of (12) are given by the formula:

$$y(x) = \sqrt{x \ln \frac{1}{x}} \Big[c_1 \cos\big(\rho \ln\ln \frac{1}{x}\big) + c_2 \sin\big(\rho \ln\ln \frac{1}{x}\big)\Big],$$

where $\rho = \sqrt{\lambda - 1/4}$;

- $y(x)$ are oscillating near $x = 0$ and $x = 1$ at the same time.

9. The existence of two-point oscillations

We start with two linearly independent functions $y_1(x)$ and $y_2(x)$ in the form:

$$y_1(x) = |q'(x)|^{-\frac{1}{2}} \cos q(x) \quad \text{and} \quad y_2(x) = |q'(x)|^{-\frac{1}{2}} \sin q(x).$$

It is not difficult to see that the equation which corresponds to the fundamental set of solutions $y(x) = c_1 y_1(x) + c_2 y_2(x)$, it is:

$$y'' + \big[\frac{1}{2} S(q')(x) + (q')^2(x)\big] y = 0, \; x \in (0,1), \tag{13}$$

where $S(q')(x)$ denotes as usual the Schwarzian derivative of $q(x)$ defined by

$$S(q')(x) = \frac{q'''(x)}{q'(x)} - \frac{3}{2}\big[\frac{q''(x)}{q'(x)}\big]^2, \; x \in (0,1).$$

THEOREM 4. *Let $q(x)$ satisfy the following condition:*

$$q \in C^3(0,1), \tag{14}$$

$$|q(0+)| = |q(1-)| = +\infty \;\; \textit{and} \;\; |q'(0+)| = |q'(1-)| = +\infty, \tag{15}$$

$$q'(x) < 0 \; \textit{for all} \; x \in (0,1) \;\; \textit{and} \;\; S(q') \in C(0,1). \tag{16}$$

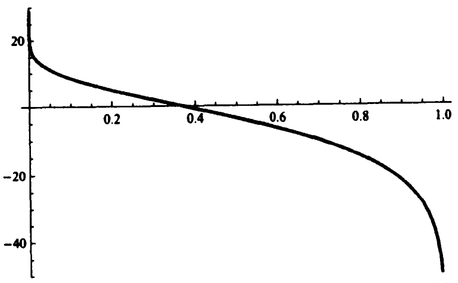
Then all solutions of Eq. (13) *are two-point oscillatory on* $(0,1)$. *Moreover, for any function* $f \in C(0,1)$ *such that*

$$f(x) \geqslant \big[\frac{1}{2} S(q')(x) + (q')^2(x)\big], \; x \in (0,1),$$

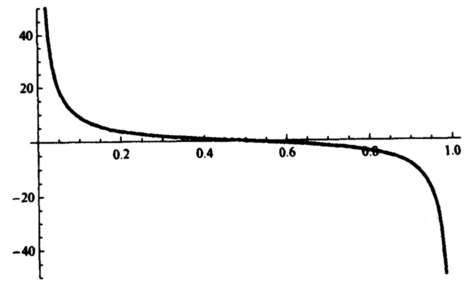
then all solutions of the equation $y'' + f(x)y = 0$, $x \in (0,1)$, *are two-point oscillatory on* $[0,1]$.

The proof of this theorem has been published in [12].

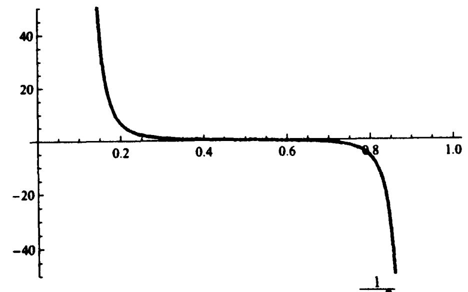
Some classes of the frequences $q(x)$ which satisfy (14), (15), and (16) are given in the following pictures:



a logarithmic class: $q(x) = \rho \ln\ln \frac{1}{x}$, $\rho = \sqrt{\lambda - \frac{1}{4}}$



a polynomial class: $q(x) = \frac{1-2x}{(x-x^2)^\beta}$, $\beta > 0$



an exponential class: $q(x) = a(1-2x)e^{\frac{1}{x-x^2}}$, $a > 0$

10. Some consequences of Theorem 4

COROLLARY 5. *Let* $\rho = \sqrt{\lambda - \frac{1}{4}}$ *and* $\lambda > \frac{1}{4}$. *Then all solutions of Riemann-Weber equation* (13) *are two-point oscillatory on* $[0,1]$.

Proof. The function $q(x) = \rho \ln\ln \frac{1}{x}$ satisfies the conditions (14), (15), and (16). Moreover,

$$\tfrac{1}{2}S(q')(x) + (q')^2(x) = \tfrac{1}{x^2}\left(\tfrac{1}{4} + \tfrac{\lambda}{|\ln x|^2}\right),$$

and thus:

$$y'' + \frac{1}{x^2}\left(\frac{1}{4} + \frac{\lambda}{|\ln x|^2}\right)y = y'' + \left[\frac{1}{2}S(q')(x) + (q')^2(x)\right]y = 0,\ x \in (0,1).$$

Hence by Theorem 4, all solutions of Riemann-Weber equation (13) are two-point oscillatory on $[0,1]$. Q. E. D.

COROLLARY 6. *Let* $c(x)$ *be smooth and positive on* $[0,1]$ *and let* $\sigma > 2$. *Then all solutions of the equation:*

$$y'' + \frac{c(x)}{(x-x^2)^\sigma}y = 0,\ x \in (0,1), \tag{17}$$

are two-point oscillatory on $[0,1]$.

Proof. At the first, the function $q(x) = \frac{1-2x}{(x-x^2)^\beta}$, $\beta > 0$, satisfies the conditions (14), (15), and (16). Since $c(x) > 0$ and $\sigma > 2$, there is an $\beta > 0$ and $m > 0$ such that $2\beta+2<\sigma$ and

$$f(x) := \frac{c(x)}{(x-x^2)^\sigma} \geqslant \frac{m}{(x-x^2)^{2\beta+2}} \geqslant \left[\frac{1}{2}S(q')(x)+(q')^2(x)\right],$$

where $q(x) = \frac{1-2x}{(x-x^2)^\beta}$. Hence by Theorem 4, all solutions of Eq. (17) are two-point oscillatory on $[0,1]$. Q. E. D.

COROLLARY 7. *Let $c(x)$ be a continuous function on $[0,1]$ such that $c(x) \geqslant 1$ for all $x \in (0,1)$. Then all solutions of the equation:*

$$y''(x)+c(x)e^{\frac{4}{x-x^2}}y(x)=0,\ x\in(0,1), \tag{18}$$

are two-point oscillatory on $[0,1]$.

Proof. The function $q(x)=(1-2x)e^{\frac{1}{x-x^2}}$ satisfies the conditions (14), (15), and (16). Now, we have:

$$f(x) := c(x)e^{\frac{4}{x-x^2}} \geqslant \frac{e^{\frac{2}{x-x^2}}}{(x-x^2)^4} \geqslant \left[\frac{1}{2}S(q')(x)+(q')^2(x)\right],$$

where $q(x) = (1-2x)e^{\frac{1}{x-x^2}}$. Hence by Theorem 4, all solutions of Eq. (18) are two-point oscillatory on $[0,1]$. Q. E. D.

11. Two-point rectifiable and unrectifiable oscillations

DEFINITION 4. A function $y(x)$ is *two-point rectifiable oscillatory* on $[0,1]$ if $y(x)$ is two-point oscillatory on $[0,1]$ and length $G(y)<\infty$. A function $y(x)$ is *two-point unrectifiable oscillatory* on $[0,1]$ if $y(x)$ is two-point oscillatory on $[0,1]$ and length $G(y)=\infty$.

THEOREM 5. *Let $q(x)$ satisfy the previous conditions (14), (15), and (16). There holds true:*

(i) *if $\left(|q'|^{-\frac{3}{2}}|q''|+|q'|^{\frac{1}{2}}\right) \in L^1(0,1)$, then all solutions of Eq. (13) are two-point rectifiable oscillatory on $(0,1)$;*

(ii) *if $|q'(x)|^{-1}$ is increasing near $x=0$ and decreasing near $x=1$ and the series,*

$$\sum_k |q'(q^{-1}(k\pi))|^{-\frac{1}{2}} \ \text{or}\ \sum_k |q'(q^{-1}(-k\pi))|^{-\frac{1}{2}}$$

is divergent, then all solutions of Eq. (13) are two-point unrectifiable oscillatory on $(0,1)$.

The main consequence of Theorem 5 is the following.

COROLLARY 8. *Let* $c(x)$ *be smooth and positive on* $[0,1]$. *We have:*

(i) *if* $\sigma \in (2,4)$, *then equation* (17) *is two-point rectifiable oscillatory on* $[0,1]$.

(ii) *if* $\sigma \geqslant 4$, *then equation* (17) *is two-point unrectifiable oscillatory on* $[0,1]$.

The proofs of Theorem 5 and Corollary 8 have been published in [12].

12. Motivation to introduce and study the so-called fractal oscillations

In the application (acoustic, telecomunication, signal processing etc.), a signal is called *chirp* if its frequence is growing up or down in the time:

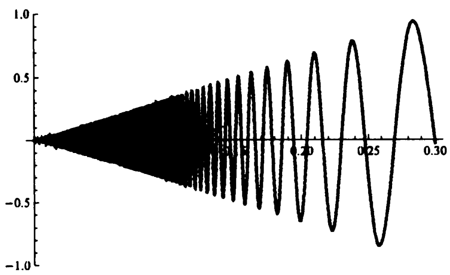


Figure 4: the (α,β)-chirp: $y(x) = x^{\alpha}\cos(x^{-\beta})$ or $y(x) = x^{\alpha}\sin(x^{-\beta})$

On the rectifiable and unrectifiable oscillations of the (α,β)-chirp one can say the following.

THEOREM 6. (see the book [14]) *Let* $y(x)$ *be the* (α,β)*-chirp, that is,* $y(x) = x^{\alpha}\cos(x^{-\beta})$ *or* $y(x) = x^{\alpha}\sin(x^{-\beta})$. *Then we have:*

$$\text{length} G(y) = \infty \iff \beta \geqslant \alpha.$$

How to estimate the density of an area filled by a chirp near $x = 0$, see figure above? In order to give the answer to this question, we need to recall some notions from the fractal geometry of plane curves like the ε-neighbourhood, Minkowski-Bouligand dimension (box dimension) and the s-dimensional Minkowski content of the graph $G(y)$ denoted respectively by $G_{\varepsilon}(y)$, $\dim_M G(y)$ and $M^s(G(y))$, and defined respectively by:

$$G_{\varepsilon}(y) = \{(t_1,t_2) \in \mathbb{R}^2 : d((t_1,t_2),G(y)) \leqslant \varepsilon\},$$
$$\dim_M G(y) = \lim_{\varepsilon\to 0}\left(2 - \frac{\log|G_{\varepsilon}(y)|}{\log\varepsilon}\right),$$
$$M^s(G(y)) = \lim_{\varepsilon\to 0}(2\varepsilon)^{s-2}|G_{\varepsilon}(y)|,\ s \in [1,2].$$

Let us remark that in the general case, in previous definitions it is required the term 'lim' to replace by 'lim sup'.

It is elementary to obtain the following properties:

(i) $|G_\varepsilon(y)| \to 0$ as $\varepsilon \to 0$ and the density of an area filled by $G(y)$ is equivalent to the asymptotics of $|G_\varepsilon(y)|$ as $\varepsilon \to 0$;

(ii) $\dim_M G(y) = s,\ 0 < M^s(G(y)) < \infty \iff |G_\varepsilon(y)| \sim \varepsilon^{2-s}$ as $\varepsilon \to 0$.

Now, the density of an area filled by a chirp near $x = 0$, it could be described by the following result.

THEOREM 7. (see the book [14]) *Let $y(x)$ be the (α,β)-chirp, that is, $y(x) = x^\alpha \cos(x^{-\beta})$ or $y(x) = x^\alpha \sin(x^{-\beta})$. Then we have:*

$$\dim_M G(y) = 2 - \frac{1+\alpha}{1+\beta} \quad \textit{and} \quad |G_\varepsilon(y)| \sim \varepsilon^{\frac{1+\alpha}{1+\beta}} \ \textit{as}\ \varepsilon \to 0.$$

Let us remark that the box dimension satisfies the following axiomatic properties. That is, if $P(\mathbb{R}^2)$ denotes the partitive set of $\mathbb{R}^2$, then we have:

(i) $\dim_M : P(\mathbb{R}^2) \to [1,2]$;

(ii) if $E \subseteq F$, then $\dim_M(E) \leqslant \dim_M(F)$ (*monotonicity*);

(iii) $\dim_M(E \cup F) = \max\{\dim_M(E), \dim_M(F)\}$ (*finite stability*);

(iv) if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a bi-Lipschitz transformation, then for all $E \subseteq \mathbb{R}^2$, $\dim_M(f(E)) = \dim_M(E)$ (*bi-Lipschitz invariance*).

13. Fractal oscillations of real functions

Now, we recall the definition of the so-called fractal oscillations of real functions introduced in [9].

DEFINITION 5. Let $s \in [1,2]$. A function $y(x)$ is the *s-dimensional fractal oscillatory* on $[0,1]$ if $y(x)$ oscillates near $x = 0$ and $\dim_M G(y) = s$ and $0 < M^s(G(y)) < \infty$. A function $y(x)$ is called to be *fractal oscillatory* on $[0,1]$ if there is an $s \in [1,2]$ such that $y(x)$ is the *s-dimensional fractal oscillatory* on $[0,1]$.

THEOREM 8. (a generalization of Theorem 7) *Let $y(x)$ be the (α,β)-chirp, that is, $y(x) = x^\alpha \cos(x^{-\beta})$ or $y(x) = x^\alpha \sin(x^{-\beta})$. Then we have:*
(i) *if $\alpha > \beta > 0$, then $y(x)$ is the 1-dimensional fractal oscillatory on $[0,1]$;*
(ii) *if $\alpha = \beta > 0$, then $y(x)$ is not fractal oscillatory on $[0,1]$, since $\dim_M G(y) = 1$ and $M^1(G(y)) = \infty$;*
(iii) *if $0 < \alpha < \beta$, then $y(x)$ is the s-dimensional fractal oscillatory on $[0,1]$, where the dimensional number s satisfies $s = 2 - \frac{1+\alpha}{1+\beta}$.*

14. Fractal oscillations of linear differential equations

Such a kind of results presented in Theorem 8, it could be also verified to the case of all solutions of linear differential equations of second-order.

DEFINITION 6. Let $s \in [1,2]$ be an arbitrarily given real number. A linear equation (P): $y'' + f(x)y = 0$ is said to be the s-dimensional fractal oscillatory on $[0,1]$, if all its solutions $y(x)$ are the *s-dimensional fractal oscillatory* on $[0,1]$.

We know that:

- Euler equation

$$y'' + \lambda x^{-2} y = 0,\ x \in (0,1],\ \lambda > 1/4,$$

is the *1-dimensional fractal oscillatory* on $[0,1]$;

- The $(2,3)$-chirp equation,

$$y'' + (9x^{-8} - 2x^{-2})y = 0,\ x \in (0,1],$$

is the s-dimensional fractal oscillatory on $[0,1]$, where $s = 5/4$.

What is about the fractal oscillations of the linear second-order differential equation of Euler type:

$$y'' + \lambda x^{-\sigma} y = 0,\ x \in (0,1], \tag{19}$$

where $\lambda > 0$ and $\sigma \geqslant 2$? Does it depend on the values of σ? The answer is given in the following result.

THEOREM 9. *We have:*

(i) *if* $2 \leqslant \sigma < 4$, *then Eq.* (19) *is the 1-dimensional fractal oscillatory on* $[0,1]$*;*

(ii) *if* $\sigma = 4$, *then Eq.* (19) *is not fractal oscillatory on* $[0,1]$, *since* $\dim_M G(y) = 1$ *and* $M^1(G(y)) = \infty$*;*

(iii) *if* $\sigma > 4$, *then Eq.* (19) *is the s-dimensional fractal oscillatory on* $[0,1]$, *where the dimensional number s satisfies* $s = \frac{3}{2} - \frac{2}{\sigma}$.

REMARK 1. For $\sigma = 4$ we have a kind of the s-dimensional Minkowski degeneration. That is, the equation, $y'' + \lambda x^{-4} y = 0$, $x \in (0,1]$, $\lambda > 0$, is not fractal oscillatory on $[0,1]$, since $\dim_M G(y) = 1$ and $M^1(G(y)) = \infty$ for all solutions $y(x)$. That is, $G(y)$ is not M^1 measurable.

The proof of Theorem 9 has been published in [9].

Next, we consider the linear second-order differential equation:

$$y'' + f(x)y = 0,\ x \in (0,1], \tag{20}$$

where $f \in C^2((0,1])$ satisfies:

$$f(x) > 0 \text{ and } f'(x) < 0 \text{ on } (0,1],\ f(0+) = \infty, \tag{21}$$

$$\sqrt{f} \notin L^1(0,1) \text{ and } f^{-1/4}(f^{-1/4})'' \in L^1(0,1). \tag{22}$$

THEOREM 10. *Let $f(x)$ satisfy the conditions* (21) *and* (22), *and let* $f(x) \sim x^{-\sigma}$ *near* $x = 0$. *Then we have:*
(i) *if* $2 \leqslant \sigma < 4$, *then Eq.* (20) *is the* 1*-dimensional fractal oscillatory on* $[0,1]$;
(ii) *if* $\sigma = 4$, *then Eq.* (20) *is not fractal oscillatory on the interval* $[0,1]$, *since* $\dim_M G(y) = 1$ *and* $M^1(G(y)) = \infty$;
(iii) *if* $\sigma > 4$, *then Eq.* (20) *is the* s*-dimensional fractal oscillatory on* $[0,1]$, *where the dimensional number* s *satisfies* $s = \frac{3}{2} - \frac{2}{\sigma}$.

The proof of Theorem 10 has been published in [4].

As a consequence we observe a generalization of Theorem 8.

COROLLARY 9. *We consider the* (α,β)*-chirp equation:*

$$y'' + \left(\frac{\beta^2}{x^{2\beta+2}} + \frac{1-\beta^2}{4x^2}\right)y = 0,\ x \in (0,1], \tag{23}$$

where $\alpha = (\beta+1)/2$. *Then we have:*
(i) *if* $0 < \beta < 1$, *then Eq.* (23) *is the* 1*-dimensional fractal oscillatory on* $[0,1]$;
(ii) *if* $\beta = 1$, *then Eq.* (23) *is not fractal oscillatory on* $[0,1]$, *since* $\dim_M G(y) = 1$ *and* $M^1(G(y)) = \infty$;
(iii) *if* $\beta > 1$, *then Eq.* (23) *is the* s*-dimensional fractal oscillatory on* $[0,1]$, *where the dimensional number* s *satisfies* $s = \frac{3}{2} - \frac{1}{\beta+1}$.

15. What is the essences in the fractal oscillations?

The following statements are equivalent:

(i) $y(x)$ is fractal oscillatory on $[0,1]$;

(ii) there is an $s \in [1,2]$ such that:

$$\dim_M G(y) = s \text{ and } 0 < M^s(G(y)) < \infty;$$

(iii) $|G_\varepsilon(y)| \sim \varepsilon^{2-s}$ when $\varepsilon \to 0$, that is, there are $c_1, c_2 > 0$ such that:

$$c_1\varepsilon^{2-s} \leqslant |G_\varepsilon(y)| \leqslant c_2\varepsilon^{2-s} \text{ for small } \varepsilon > 0.$$

Hence, in order to prove that an oscillatory function $y(x)$ is the fractal oscillatory on $[0,1]$, we need to estimate $|G_\varepsilon(y)|$ from below and above by the term ε^{2-s}, for some $s \in [1,2]$. Therefore one can observe that the essential tools in the fractal oscillations play the following two lemmas.

LEMMA 7. (a lower bound of $|G_\varepsilon(y)|$) *Let* $y \in C([0,1])$ *and let* $a_n \in (0,1]$ *be a decreasing sequence of consecutive zeros of* $y(x)$ *such that* $a_n \searrow 0$. *Then there is a function* $k : (0,\varepsilon_0) \to \mathbb{N}$ *for some* $\varepsilon_0 > 0$, $k = k(\varepsilon)$, *such that:*

$$|a_n - a_{n+1}| \leqslant \varepsilon \text{ for all } n \geqslant k(\varepsilon) \text{ and } \varepsilon \in (0,\varepsilon_0), \tag{24}$$

$$k(\varepsilon) \text{ is increasing and } k(\varepsilon) \to \infty \text{ as } \varepsilon \to 0. \tag{25}$$

Moreover, for any function $k(\varepsilon)$ *satisfying* (24) *and* (25), *we have*

$$\sum_{n=k(\varepsilon)}^{\infty} \max_{x\in[a_{n+1},a_n]} |y(x)|(a_n - a_{n+1}) \leqslant |G_\varepsilon(y)| \text{ for small } \varepsilon > 0. \tag{26}$$

The proof of Lemma 7 was published in [4, Appendix] and in a corresponding integral form in [7, Section 2].

REMARK 2. Obviously, the term $\max_{x\in[a_{n+1},a_n]} |y(x)|$ in (26) could be replaced by weaker one: $|y(s_n)|$, where $s_n \in [a_{n+1}, a_n]$.

LEMMA 8. (an upper bound of $|G_\varepsilon(y)|$) *Let* $y \in C([0,1]) \cap C^2((0,1])$, $y(0) = 0$ *and let* $a_n \in (0,1]$ *be a decreasing sequence of inflexion-points of* $y(x)$ *such that* $a_n \searrow 0$. *Then there is a function* $m : (0, \varepsilon_0) \to \mathbb{N}$ *for some* $\varepsilon_0 \in (0,1)$, $m = m(\varepsilon)$, *such that:*

$$|a_n - a_{n+1}| \geqslant 4\varepsilon \text{ for all } n = 1,2,\ldots,m(\varepsilon) \text{ and } \varepsilon \in (0,\varepsilon_0), \tag{27}$$

$$m(\varepsilon) \text{ is increasing and } m(\varepsilon) \to \infty \text{ as } \varepsilon \to 0. \tag{28}$$

Next, there is a positive constant $M > 0$ *such that for any function* $m(\varepsilon)$ *satisfying* (27) *and* (28), *we have*

$$|G_\varepsilon(y)| \leqslant M\big[\varepsilon + a_{m(\varepsilon)} \max_{x\in[0,a_{m(\varepsilon)}]} |y(x)|\big]$$
$$+ M\Big[\varepsilon \sum_{n=2}^{m(\varepsilon)} \max_{x\in[a_{n+1},a_n]} |y(x)| + \varepsilon^2 m(\varepsilon)\Big] \text{ for small } \varepsilon > 0.$$

The proof of Lemma 7 was published in [7, Section 2] in the special case when the boundary curves of $G_\varepsilon(y)$ are regular.

16. Open problem A: fractal oscillations of self-adjont linear differential equations

In this section, we consider the self-adjont equation:

$$(p(x)y')' + q(x)y = 0,\ x \in (0,1], \tag{29}$$

where $y \in C([0,1]) \cap C^2((0,1])$ and the coefficients $p(x)$ and $q(x)$ satisfy:

$$p \in C^1([0,1]),\ p(x) > 0 \text{ and } p'(x) \geqslant 0 \text{ on } (0,1], \tag{30}$$

$$q \in C^2((0,1]),\ q(x) > 0 \text{ and } q'(x) < 0 \text{ on } (0,1],\ q(0+) = \infty, \tag{31}$$

and satisify the following Hartman-Wintner type conditions:

$$\sqrt{\frac{q}{p}} \notin L^1(0,1) \text{ and } \frac{1}{\sqrt[4]{pq}}\Big(p\big(\frac{1}{\sqrt[4]{pq}}\big)'\Big)' \in L^1(0,1). \tag{32}$$

We can propose the following conjecture.

CONJECTURE 1. *Let the functions* $p(x)$ *and* $q(x)$ *satisfy the conditions* (30), (31) *and* (32), *and let*

$$p(x) \sim x^{\mu} \text{ and } q(x) \sim x^{-\sigma} \text{ near } x = 0,$$

where $\mu \geqslant 0$, $\sigma > 0$, $\sigma + \mu > 2$, *and* $\sigma - \mu > -4$. *Then the equation*

$$(p(x)y')' + q(x)y = 0, \; x \in (0,1],$$

is the s *- dimensional fractal oscillatory on* $[0,1]$, *where the dimensional number* s *satisfies:* $s = \frac{3}{2} + \frac{\mu-2}{\sigma+\mu}$.

It is clear that Conjecture 1 in particular for $p(x) \equiv 1$, $q(x) = f(x)$, $\mu = 0$, and $\sigma = \sigma$, generalizes both Theorem 9 and Theorem 10 on the fractal oscillations of the equation $y'' + f(x)y = 0$, $x \in (0,1]$.

When Conjecture 1 is verified, then we have the following consequence.

COROLLARY 10. (a conditional consequence of Conjecture 1) *Let* $f(x) > 0$ *and* $f'(x) < 0$ *on* $(0,1]$, $f(0+) = \infty$ *and satisfy the Hartman-Wintner type conditions:*

$$\sqrt{f} \notin L^1(0,1) \quad and \quad f^{-1/4}(f^{-1/4})'' \in L^1(0,1).$$

If $\mu \geqslant 0$ *and* $f(x) \sim x^{-\sigma}$, $\beta \geqslant 1$, *then the linear equation:*

$$y'' + \mu x^{-1} y' + f(x)y = 0, \quad x \in (0,1],$$

is the s*-dimensional fractal oscillatory on* $[0,1]$, *where the dimensional number* $s = \frac{3}{2} + \frac{\mu-2}{\sigma}$.

It is clear that previous conditional consequence of Conjecture 1 in particular for $\mu = 0$, generalizes both Theorem 9 and Theorem 10 on the fractal oscillations of the equation $y'' + f(x)y = 0$, $x \in (0,1]$. Also, it motivates a study presented in the following section.

Some results on the Conjecture 1 will appear in a forthcoming paper [10].

17. Open problem B: fractal oscillations of linear differential equations with damping term

What kind of asymptotic properties near $x = 0$ are proposed to the coefficients $f(x)$ and $g(x)$ such that the linear equation:

$$y'' + g(x)y' + f(x)y = 0, \; x \in (0,1],$$

is the s-dimensional fractal oscillatory on $[0,1]$ for some $s \in (1,2]$?

A motivation to solve this problem we find in the book [14], which could be formulated in this way: if $0 < \alpha < \beta$, then the (α, β)-chirp equation:

$$y'' + \frac{\beta - 2\alpha + 1}{x} y' + \Big(\frac{\beta^2}{x^{2\beta+2}} - \frac{\alpha(\beta - \alpha)}{x^2} \Big) y = 0, \; x \in (0,1].$$

is the s-dimensional fractal oscillatory on $[0,1]$, where $s = 2 - \frac{\alpha+1}{\beta+1}$.

18. Open problem C: rectifiable and unrectifiable oscillations of radially symmetric solutions of some pde's

Let $N \geqslant 2$ and let $B = \{x \in \mathbb{R}^N : |x| < 1\}$ be a unit ball centered at the origin with its boundary ∂B.

DEFINITION 7. A function $u : B \setminus \{0\} \to \mathbb{R}$ is said to be *radially symmetric* if there is a function $y = y(r)$, $y : (0,1] \to \mathbb{R}$ such that $u(x) = y(|x|)$, $x \in B \setminus \{0\}$.
A radially symmetric function $u : B \setminus \{0\} \to \mathbb{R}$ is said to be *oscillatory near* ∂B if corresponding function $y = y(r)$ oscillates near $r = 1$.
A radially symmetric function $u : B \setminus \{0\} \to \mathbb{R}$ is said to be *s-dimensional fractal oscillatory near* ∂B if corresponding function $y = y(r)$ oscillates near $r = 1$ and $\dim_M G(y) = s$ and $0 < M^s(G(y)) < \infty$, for some $s \in [0,1]$.

EXAMPLE 3. We consider the radially symmetric solutions of the Dirichlet problem:

$$\begin{cases} -\Delta u = \frac{\lambda}{|x|^2 \ln^2 |x|} u & \text{in } B \setminus \{0\} \subseteq \mathbb{R}^2,\ \lambda > 1/4, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{33}$$

All radially symmetric solutions $u(x)$ of (33) are the 1-dimensional fractal oscillatory near ∂B. It is because:

$$u(x) = \sqrt{\ln \frac{1}{|x|}} \left[c_1 \cos\left(\rho \ln\ln \frac{1}{|x|}\right) + c_2 \sin\left(\rho \ln\ln \frac{1}{|x|}\right)\right],$$

where $x \in B \setminus \{0\} \subseteq \mathbb{R}^2, \lambda > 1/4$, $\rho = \sqrt{\lambda - 1/4}$, and $c_1, c_2 \in \mathbb{R}$.

EXAMPLE 4. We consider the radially symmetric solutions of the Dirichlet problem:

$$\begin{cases} -\Delta u = \frac{\lambda}{|x|^2 \ln^4 |x|} u & \text{in } B \setminus \{0\} \subseteq \mathbb{R}^2,\ \lambda > 0, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{34}$$

All radially symmetric solutions $u(x)$ of (34) are not fractal oscillatory near ∂B. It is because:

$$u(x) = \ln|x| \left[c_1 \cos\left(\frac{\sqrt{\lambda}}{\ln|x|}\right) + c_2 \sin\left(\frac{\sqrt{\lambda}}{\ln|x|}\right)\right],$$

where $x \in B \setminus \{0\} \subseteq \mathbb{R}^2$, $\lambda > 0$ and $c_1, c_2 \in \mathbb{R}$.

Now, we consider the one-parameter Dirichlet problem:

$$\begin{cases} -\Delta u = \frac{\lambda}{|x|^2 (-\ln|x|)^\sigma} u & \text{in } B \setminus \{0\} \subseteq \mathbb{R}^2,\ \sigma \geqslant 2, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{35}$$

where $u \in C^2(B \setminus \{0\}) \cap C(\overline{B} \setminus \{0\})$ and $\lambda > 0$.

CONJECTURE 2. (the case when $N=2$) *We have:*
(i) *if* $2 \leqslant \sigma < 4$, *then all radially symmetric solutions* $u(x)$ *of Eq.* (35) *are the* 1*-dimensional fractal oscillatory near* ∂B;
(ii) *if* $\sigma = 4$, *then radially symmetric solutions* $u(x)$ *of Eq.* (35) *are not fractal oscillatory near* ∂B, *since* $\dim_M G(y) = 1$ *and* $M^1(G(y)) = \infty$;
(iii) *if* $\sigma > 4$, *then radially symmetric solutions* $u(x)$ *of Eq.* (35) *are the* s*-dimensional fractal oscillatory near* ∂B, *where the dimensional number* s *satisfies* $s = \frac{3}{2} - \frac{2}{\sigma}$.

Also, we consider the one-parameter Dirichlet problem:

$$\begin{cases} -\Delta u = \frac{\lambda(2-N)^2}{|x|^{2N-2}(|x|^{2-N}-1)^{\sigma}} u & \text{in } B \setminus \{0\} \subseteq \mathbb{R}^N, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{36}$$

where $u \in C^2(B \setminus \{0\}) \cap C(\overline{B} \setminus \{0\})$, $N \geqslant 3$, $\sigma \geqslant 2$ and $\lambda > 0$.

CONJECTURE 3. (the case when $N \geqslant 3$) *We have:*
(i) *if* $2 \leqslant \sigma < 4$, *then all radially symmetric solutions* $u(x)$ *of Eq.* (36) *are the* 1*-dimensional fractal oscillatory near* ∂B;
(ii) *if* $\sigma = 4$, *then radially symmetric solutions* $u(x)$ *of Eq.* (36) *are not fractal oscillatory near* ∂B, *since* $\dim_M G(y) = 1$ *and* $M^1(G(y)) = \infty$;
(iii) *if* $\sigma > 4$, *then radially symmetric solutions* $u(x)$ *of Eq.* (36) *are the* s*-dimensional fractal oscillatory near* ∂B, *where the dimensional number* s *satisfies* $s = \frac{3}{2} - \frac{2}{\sigma}$.

For more details about two previous conjecture we propose the forthcoming paper [6].

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