数理解析研究所講究録
第 1704 巻 2010 年 195-202

Existence of multiple stable stationary patterns to some reaction-diffusion equation in heterogeneous environments

首都大学東京大学院理工学研究科　倉田和浩 (Kazuhiro Kurata)
Department of Mathematics and Information Sciences,
Tokyo Metropolitan University

1 Introduction and main results

In this paper we report some recent our mathematical research on the existence of multiple non-trivial stable stationary patterns for some rection-diffusion euquations with heterogeneous environments.

We emphasize that the existence of multiple stable stational patterns also depends not only on heterogenous environments but also on the nonlinearity. In fact, for the logistic model:

$$0 = d\Delta u + u(b(x) - u), \quad x \in \Omega, \quad \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega,$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, this phenomenon does not occur. It is well-known that a positive solution of this boundary value problem must be unique if exists. Actually, it exists if $(1/d) > \lambda_1(b) > 0$ when $\int_\Omega b\,dx < 0$, any $d > 0$ when $\int_\Omega b\,dx \geq 0$. Here, $\lambda_1(b)$ is the first positive eigenvalue of $-\Delta\phi = \lambda b(x)\phi$, $(x \in \Omega)$, $\partial\phi/\partial n = 0$, $(x \in \partial\Omega)$. Asymptotic behaviour of the unique solution $u = u_d$ is alos well-known as $d \to 0$.

In this paper we consider two reaction-diffusion model with weak Allee effect, which appears as a populationa growth model, and show existence of multiple stable stationary patterns under certain heterogeneous evnvironments. Throughout this paper, we say that a solution is stable, which is called weakly stable sometimes, if it is a local minimizers of the associated energy functional.

There are several works on the existence of stable staionary patterns for the rection-diffusion model with strong Allee effect and environment factor $a(x)$:

$$\frac{\partial u}{\partial t} = d\Delta u + u(u - a(x))(1 - u),$$

where $0 < a(x) < 1$ (see, e.g. Dancer-Yan([3]) and the references therein).

2 Problem 1(weak Allee Effect)

We consider the following stationary problem for a rection-diffusion model with weak Allee effect under Neumann boundary condition:

$$0 = d\Delta u + u(b(x)u - u^2), \quad x \in \Omega,$$

$$\frac{\partial u}{\partial n} = 0, \qquad x \in \partial\Omega,$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain. We consider the following situation: $\Omega = \Omega_+ \cup \Omega_-$ and $b(x) = 1$ on Ω_+, $b(x) = -1$ on Ω_-. This means that a habitat Ω consists of a good region Ω_+ and a bad region Ω_-.

Question: How this heterogeneous environment has an influence on patterns of stable positive solutions?

In [5], Ide, Kurata and Tanaka studied this problem and obtained the following results.

Theorem 1 *(a) If $\int_\Omega b\,dx < 0$, then for sufficiently small $d > 0$ there exists at least two type of positive solutions.*

(b) Assume $N = 1$ and Ω_+ has well-separated k-components. Then for sufficiiently small $d > 0$, there exists at least $2^k - 1$ types of stable positive solutions.

The precise condition on the meaning of the well-separated components, see [5]. In [8] we extended the part (b) of Theorem 1 (i.e. the existence of multiple stable patterns) in the higher dimensional case.

Theorem 2 *Suppose $1 \le N \le 3$, $\overline{\Omega_+} \subset \Omega$, Ω_+ has well-separated k-components. Then for sufficiently small $d > 0$, there exists at least $2^k - 1$ types of stable positive solutions. Actually, these stable solutions are local minimizers in $H^1(\Omega)$.*

Throughout this paper, stability means local minimizers in $H^1(\Omega)$. So it is sometomes called weak stability and it is not clear whether the solutions obtained above have a linearized stability.

3 Problem 2(Balanced Bistable Nonlinearity)

We consider another rection diffusion model with weak Allee effect:

$$-\epsilon^2\Delta u = f(x, u(x)), \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n} = 0, \quad \text{on} \quad \partial\Omega,$$

where $\epsilon > 0$, $\Omega \subset \mathbf{R}^N, N \ge 1$ is a bounded smooth domain. Here

$$f(x, u) = a(x)|u|^{p-1}u - |u|^{q-1}u, \quad 1 \le p < q,$$

and $a(x) > 0\,(x \in \Omega)$. An associated energy functional is as follows:

$$J_\epsilon(u) = \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x, u(x))\,dx,$$

$$F(x,u)=\int_0^u f(x,s)\,ds=\frac{1}{p+1}|u|^{p+1}a(x)-\frac{1}{q+1}|u|^{q+1}.$$

It is well-known that solutions of the boundary value problem above corresponds to critical points of $J_\epsilon(u)$ on $H^1(\Omega)$. This model has a so-called balanced bistable nonlinearity: for fixed $x\in\Omega$, $u=\pm a(x)^\gamma,\gamma=1/(q-p)$, are stable states in ODE sense and the potential $-F(x,u)$ takes its minimum same energy at two states $u=\pm a(x)^\gamma$. As a typical examples, we have as $p=1,q=3$,

$$-\epsilon^2\Delta u=u(x)a(x)-u(x)^3,\quad \text{in}\quad \Omega,$$

which is called Allen-Cahn equation. When $p=2,q=3$,

$$-\epsilon^2\Delta u=|u(x)|u(x)a(x)-u(x)^3,\quad \text{in}\quad \Omega.$$

Fo this model, it is rather easy to show that there exists a global minimizer $U_\epsilon(x)$ which is $U_\epsilon(x)>0$ $(x\in\Omega)$ and

$$\sup_{x\in\Omega}|U_\epsilon(x)-a(x)^\gamma|\to 0$$

as $\epsilon\to 0$. Moreover, global minimizers should not change sign.
Here we have a following question: Does there exist a stable solution (e.g. local minimizer) $u_\epsilon(x)$ such that

$$u_\epsilon(x)\sim a(x)^\gamma\quad\text{on}\quad\Omega^+,$$
$$u_\epsilon(x)\sim -a(x)^\gamma\quad\text{on}\quad\Omega^-$$

for some $\Omega^\pm\subset\Omega$ as $\epsilon\to 0$? Namely, does there exist a stable stationary pattern? In other word, does there exist a stable stationary sign-changing solution?

The following results are known:

- (H.Matano [9]) Let $p=1,q=3,N\geq 1$ and $a(x)\equiv 1$. Then, for any convex domain Ω, there are no non-constant local minimizer. However, for some non-convex domain Ω, there exists a sign-changing stable solution.

- (K. Nakashima[11]) Let $p=1,q=3,N=1$ and $a(x)$ has a non-degenerate minimum at $x=r_0\in\Omega=(0,1)$. Then there exists a stable solution $u(x)$ s.t. $u(x)\sim\sqrt{a(x)}$ on $(r_0,1)$ and $u(x)\sim-\sqrt{a(x)}$ on $(0,r_0)$.

In[6], Kurata and Matsuzawa studied this problem under the following situation: Let $p=1,q=3,N\geq 1$. Suppose there exit D_j $(j=1,2)$ s.t. $\overline{D_1}\cap\overline{D_2}=\emptyset,\overline{D_j}\subset\Omega$ $(j=1,2)$ and

$$a(x)=1\text{ on }D=D_1\cup D_2,\quad a(x)=O(\epsilon^2)\text{ on }\Omega\setminus\overline{D}.$$

Then we obtained the following result.

Theorem 3 *Under the assumptions above, there exists a local minimizer u_ϵ s.t. $u_\epsilon(x)\sim 1$ on D_1, $u(x)\sim -1$ on D_2 and $|u(x)|\leq C\epsilon$ on $\Omega\setminus\overline{D}$.*

In [8] (see also [13]), Kurata and Yanai studied this problem under the following situation: Let $p=2,q=3,N\geq 1$. Suppose there exit D_j $(j=1,2)$ s.t. $\overline{D_1}\cap\overline{D_2}=\emptyset,\overline{D_j}\subset\Omega$ $(j=1,2)$ and

$$a(x)=1\text{ on }D=D_1\cup D_2,\quad a(x)=O(\epsilon)\text{ on }\Omega\setminus\overline{D}.$$

Theorem 4 *Under the assumptions above, there exists a local minimizer u_ϵ s.t. $u_\epsilon(x)\sim 1$ on D_1, $u(x)\sim -1$ on D_2 and $|u(x)|\leq C\epsilon$ on $\Omega\setminus\overline{D}$.*

4 Basic Strategy for the Proofs of Theorem 2 and 4

In this section, we explain the basic strategy and the outline of the proof of Theorem 2 and the keypoints of the proof of Theorem 4. We omit the proof of Theorem 1 and 3. For the details, see [5], [8], [6], [8], [13].

Our basic strategy is to use varitaional methods and a sub-supersolution method. In the construction and estimates of a suitable sub-supersolution, we use boundary blow-up solutions to the following problem:

$$-\Delta V = g(V), \; V(x) > 0, \quad \text{in} \quad G,$$

$$V(x) = 0 \; (x \in \Gamma_1), \quad V(x) = +\infty \; (x \in \Gamma_2),$$

where G is a domain or an annular domain and $\partial G = \Gamma_1 \cup \Gamma_2$. Γ_1 may be an empty set. We choose the nonlinear function $g(t)$ suitably for each problem 1 and 2.

We use Dancer-Yan's compariosn lemma for minimizers.

Lemma 1 *Let $h_j(t), j = 1, 2$, be continuous functions s.t. $h_i(t) > 0$ $(t \leq 0)$; $h_i(t) \leq 0$ $(t \geq c)$. Let $H_j(t) = \int_0^t h_j(s)\,ds, j = 1, 2$ and*

$$J_{\lambda,j}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \lambda \int_\Omega H_j(v)\,dx.$$

Suppose $h_1(t) \geq h_2(t)$, $(0 \leq t \leq c)$, $\eta_j(x) \in H^1(\Omega) \cap C(\overline{\Omega}), j = 1, 2$ s.t. $c \geq \eta_1(x) \geq \eta_2(x) \geq 0$, $(x \in \partial\Omega)$, $\eta_1(x) \not\equiv \eta_2(x)$, $(x \in \partial\Omega)$. Let $u_j(x), j = 1, 2$ be minimizers to the minimizing problem:

$$\inf\{J_{\lambda,j}(v) \; : \; u(x) = \eta_j(x), \; (x \in \partial\Omega)\}.$$

Then we have $u_1(x) \geq u_2(x)$, $(x \in \Omega)$.

For the proof of Lemma 1, see [3] and [6].

4.1 Outline of the proof of Theorem 2

Let $f(x,t) = t_+^2(b(x) - t)$ and $F(x,t) = \int_0^t f(x,s)\,ds$. Let $1 \leq N \leq 3$. Define, for $u \in H^1(\Omega)$,

$$I(u) = I(u;\Omega) = \frac{d}{2}\int_\Omega |Du|^2\,dx - \int_\Omega F(x,u)\,dx,$$

The following propostion is easy to show.

Proposition 1 *Let $u = u_d$ be a global minimizer of $I(u;\Omega)$ on $H^1(\Omega)$. Then u_d tends to 1 as $d \to 0$ uniformly on any compact subset of Ω_+ and tends to 0 as $d \to 0$ uniformly on any compact subset of Ω_-.*

Now, we explain how to construct non-trivial local minimizers.

For simplicity, assume A is a one component of Ω_+, and show how to construct a local minimizer $\tilde{u}_d$ which behaves 1 on $\Omega_+ \setminus A$ and 0 on $\Omega_- \cup A$ as $d \to 0$. We also assume the following conditions(i.e. A is well-separated to other components of Ω_+):

(A): There exists $R_1 > r_0 > 0$ and $x_0 \in A$ such that $A \subset B_{r_0}(x_0)$, $B_{R_1(x_0)} \setminus B_{r_0}(x_0) \subset \Omega_-$.

Furthermore, R_1 should be large enough which will be specified later. Here $B_r(y) = \{x| \; |x-y| \le r\}$.

4.2 Construction of Sub-solution

Take $R = 2r_0 + \delta$ with sufficiently small $\delta \in (0, 1/8)$, e.g. $\delta = 1/16$ and fix R. Then R_1 should be large enough to satisfy $R_1 > 16r_0 + 1$ and $C_0/R_1^2 \le v(R)$, where $C_0 > 0$ is a certain universal constant and $v(R) > 0$ is a constant depending only on R, r_0. Note that this implies $R_1 > 8R$.

Let $\underline{v}_d$ be a minimizer of the problem:

$$\inf\Big\{I(u; \Omega \setminus B_R(x_0)) \mid u \in H^1(\Omega \setminus B_R(x_0)),\ u = 0 \text{ on } \partial B_R(x_0)\Big\}.$$

Now, define $\underline{u}_d(x) = \underline{v}_d(x)$ for $x \in \Omega \setminus B_R(x_0)$, and $= 0$ for $x \in B_R(x_0)$. Then it is easy to show

Lemma 2 *$\underline{u}_d$ is a subsolution of the problem.*

4.3 Construction of Super-solution

Define $\bar{b}(t) = 1$ for $t \in [0, r_0]$, and $= -1$ for $t \in [r_0, R]$. Then

$$\int_0^R \bar{b}(t)\, dt = -R + 2r_0 = -\delta < 0.$$

Now, it is known that the ODE problem

$$-v''(t) = v^2(t)\bar{b}(t),\ 0 < t < R, \quad v'(0) = v'(R) = 0$$

has a positive solution $v(t) > 0$ ($t \in [0, R]$) (this solution is a mountain pass type solution)([1]). It is easy to see $v'(t) < 0$ $(0 < t < R)$.

Lemma 3 *Let $v_d^*(x) = d\ v(|x - x_0|)$. Then we have*

$$-d\Delta v_d^* - f(x, v_d^*) \ge 0, \quad x \in B_R(x_0), \quad \frac{\partial v_d^*}{\partial n} = 0, \quad x \in \partial B_R(x_0).$$

Proof of Lemma 3: Note

$$b(x) \le \bar{b}(|x - x_0|), \quad x \in B_R(x_0).$$

Thus, using $(v_d^*)'(r) < 0$ with $r = |x - x_0|$, we have

$$-d\Delta v_d^* - f(x, v_d^*) \ge -d^2 v''(r) - d^2\bar{b}v^2 + (v_d^*)^3 \ge d^2(-v''(r) - \bar{b}v^2) = 0.$$

Let $\overline{v}_d$ be a minimizer of

$$\inf\Big\{I(u;\Omega\setminus B_R(x_0))\mid u\in H^1(\Omega\setminus B_R(x_0)),\ u=v_d^* \text{ on } \partial B_R(x_0)\Big\}.$$

Define $\overline{u}_d(x)=\overline{v}_d(x)$ for $x\in\Omega\setminus B_R(x_0)$, and $=v_d^*$ for $x\in B_R(x_0)$.

Note $\overline{u}_d\in C(\overline{\Omega})$ and a piecewise C^1 function. We want to show $\overline{u}_d$ is a supersolution and that $\overline{u}_d(x)\ge \underline{u}_d(x),\ x\in\Omega$. The following is a key lemma.

Lemma 4 *We have*

$$\frac{\partial\overline{v}_d}{\partial n}\ge 0,\ x\in\partial B_R(x_0),$$

where n is an inward unit normal on $\partial B_R(x_0)$.

Once we have this lemma, it is easy to see

Lemma 5 *$\overline{u}_d$ is a supersolution.*

Furthermore, the following lemma can be obtained by using Lemma 1.

Lemma 6 $\overline{u}_d(x)\ge\underline{u}_d(x),\ x\in\Omega.$

Now, we have the following by using the argument of Brezis-Nirenberg([2]), see also e.g. [12], [10] for Neumann boundary condition).

Theorem 5 *Suppose the assumption (A). Then there exists a solution $\tilde{u}_d$ such that*

$$\overline{u}_d(x)\ge\tilde{u}_d(x)\ge\underline{u}_d(x),\ x\in\Omega.$$

Actually, $\tilde{u}_d$ is a local minimizer of $I(u)$ on $H^1(\Omega)$.

Actually, $\tilde{u}_d$ tends to 1 uniformly on any compact subset of $\Omega_+\setminus A$ and tends to 0 uniformly on any compact subset of $\Omega_-\cup B_R(x_0)$.

Finally, we give the outline of the proof of Lemma 4.

4.4 Outline of the proof of Lemma 4:

We claim the following. **Claim 1:**

$$\overline{v}_d(x)\le dv(R)\equiv\alpha_d,\quad x\in B_{R_1/4}(x_0)\setminus B_R(x_0).$$

Note $\overline{v}_d=\alpha_d$ on $\partial B_R(x_0)$. So, if this claim is true, Lemma 4 follows easily.

To show Claim 1, we have two steps.

Step 1: We show

$$\overline{v}_d(x)\le dv(R),\quad x\in\partial B_{R_1/4}(x_0).$$

Proof of Step 1: Fix $x_1\in\partial B_{R_1/4}(x_0)$ and we want to show $\overline{v}_d(x_1)\le\alpha_d$. Note that $B_{R_1/8}(x_1)\subset B_{R_1}(x_0)\setminus B_R(x_0)\subset\Omega_-$, since $R_1>8R$.

Let w_d be a minimizer of

$$\inf\Big\{I(u;B_{R_1/8}(x_1))\mid u-1\in H_0^1(B_{R_1/8}(x_1))\Big\}.$$

Since $\overline{v}_d(x) \leq 1 = w_d(x)$, $x \in \partial B_{R_1/8}(x_1)$ we have $\overline{v}_d(x) \leq w_d(x)$, $x \in B_{R_1/8}(x_1)$ by Lemma 1.

Finally, we show **Claim 2:** There exists a positive constant C_0 independent of d, R_1 such that

$$(\sharp) \quad 0 \leq w_d(x_1) \leq \frac{C_0 d}{R_1^2}.$$

If ($\sharp$) is true,

$$\overline{v}_d(x_1) \leq C_0 \frac{d}{R_1^2} \leq \alpha_d = dv(R)$$

holds under the assumption

$$\frac{C_0}{R_1^2} \leq v(R).$$

Now, we can show the estimate in Claim 2 by comparing with the unique positive solution of

$$\Delta U = U^2, x \in B_1(0), \quad U(x) \to +\infty \ (|x| \to 1).$$

For this boundary blow-up problem, see e.g., [4]. Next, we can show the following.
Step 2: Then, since $\overline{v}_d = dv(R)$, we can show

$$\overline{v}_d(x) \leq dv(R), \quad x \in B_{R_1/4}(x_0) \setminus B_R(x_0)$$

by using Lemma 1.

4.5 Keypoints of the proof for Theorem 4.

Concerning the proof of Theorem 4, we just remark the following keypoints. First of all, we use a variational method and a sub-supersolution method as in the proof of Theorem 2. In the construction and estimates of a suitable sub-supersolution, we use boundary blow-up solutions to the following problem:

$$-\Delta V = V^2 - V^3, \ V(x) > 0, \quad \text{in} \quad G,$$

$$V(x) = 0 \ (x \in \Gamma_1), \quad V(x) = +\infty \ (x \in \Gamma_2),$$

where G is an annular domain and $\partial G = \Gamma_1 \cup \Gamma_2$. We note that existence of boundary blow-up solutions to the equation above also seems new. For the details, see [8] (see also [13]).

5 Summary and Future Problems

• **Summary**: We obtained non-trivial multiple stable patterns for reaction-diffusion equation with (weak) Allee effect and for a balanced bistable reaction-diffusion equations under certain heterogeneous environments. The method are based on the construction of suitable super-sub solutions by using variational methods. To estimate suitable sub-supersolution, we use certain boundary blow-up solutions.
• **Future Problems:**

1. We imposed certain restriction to the configuration of the heterogeneous environments. Can we obtain under more general heterogeneous configuration?
2. Can we obtain the same results for environments which change smoothly?
3. How about for a reaction-diffusin system?

References

[1] H.Berestycki, I. Capuzzo-Dolcetta, L.Nirenberg, Variational methods for indefinite superlinear homogeneous elliptic problems, NoDEA 2(1995), 553–572.

[2] H. Brezis and L. Nirenberg, H^1 versus C^1 local minimizers, C.R. Academic Science Paris Ser. I 317(1993), 465–472.

[3] N. Dancer and S. Yan, Construction of various types of solutions for an elliptic problem, Calc. Var. Partial Differential Equations **20**(2004),93-118.

[4] Y. Du, *Order Structure and Topological Methods in Nonlinear Partial Differential Equations, Vol.1*, World Scientific, 2006.

[5] T. Ide, K. Kurata, K. Tanaka, Multiple stable patterns for some reaction-diffusion equation in disrupted environments, Discrete and Conti. Dyna. Sys. 14(2006), 93–116.

[6] K. Kurata, H. Matsuzawa, Multiple stabel patterns in a balanced bistable equation with heterogeneous environments, to appear in Applicable Analysis.

[7] K.Kurata, S. Yanai, Existence of boundary blow-up solutions and its application to a pattern formation problem, in preparation.

[8] K. Kurata, S. Yan, Multiple stable patterns for some reaction-diffusion equation in disrupted environments: higher dimensional case, in preparation.

[9] H. Matano, Asymptotic behavior and stability of solutions of semilinear differential equations, Publ. Res.Inst.Math.Sci.**15(2)**(1979), 401–454.

[10] H. Matsuzawa, Stable transition layers in a balanced bistable equation with degeneracy, Nonlinear Analysis, **58**(2004), 45–67.

[11] K. Nakashima, Stable transition layers in a balanced bistable equation, Diff. Integral Eq.**13**(2000), 1025-1038.

[12] A.S. do Nascimento, Stable transition layers in a semilinear diffusion equation with spatial inhomogeneities in N- dimensional domains, J.Diff.Eq. **190**(2003), 16–38.

[13] S. Yanai, Existence of boundary blow-up solutions and its application to a pattern formation problem(in Jananese), Master thesis, 2009, Tokyo Metropolitan University.