数理解析研究所講究録
第 1705 巻 2010 年 134-146

CUBIC AND TERNARY ALGEBRAS, TERNARY SYMMETRIES, AND THE LORENTZ GROUP

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Abstract

We show how the Lorentz symmetry represented by the $SL(2, C)$ group can be introduced without any notion of metric, just as the symmetry of Z_3-graded cubic algebra ands its constitutive relations. Parallelly, ternary algebra of cubic matrices (three-valenced covariant tensors) is introduced, and its representation is found in terms of the Pauli matrices. The relationship of such algebraic constructions with quark states is also considered.

1 Introduction: the importance of number 3

There are many fundamental facts in physics involving some sort of triality, or the presence of the magic number "3". They come from different phenomenological realms and apparently do not have much in common except for the pertinence of the number "3".

Here are a few important phenomena around us in which the number **3** is present in a very fundamental manner:

- We live in three space dimensions;

- There are three fundamental inversions in physics: P, C and T, respectively, the space reflection, the charge conjugation and the time reversal. They apply to the elementary particle physics, and although there are some exceptions to each of them, the combination of all three, the PCT-symmetry, is exact and knows no exceptions.

- There are three fundamental gauge fields in nature, the electromagnetic field, the weak force field, and the strong (color) field. We deliberately do not count in the gravitational interaction, which we believe to be of radically different nature.

- The carriers of elementary charges also go by packs of three: there are three types of quarks, and three types of leptons;

- There is an additional symmetry that applies to quarks, the color $SU(3)$ group: three different colors are needed to combine three quarks into a hadron;

- The neutrino oscillations mix up three types of particles, each of different lepton family

It is tempting to relate somehow at least a part of these phenomena, imagining that there is some common reason for three space dimensions, three quarks in a baryon, three colors and three quark and three lepton families (sometimes called generations). But if at least some of these "ternary" features are related, another pertinent question can be asked, which aspect can be derived from the other one, in other words, can one imagine that one of these facts leads inevitably to another as its direct consequence. And the most exciting perspective would be to relate the three sace dimensions of the Minkowskian space-time with three quarks of three different colors necessary to form a baryon, and maybe also the three quark and lepton generations.

Einstein's dream was to be able to derive the properties of matter, and perhaps its very existence, from the properties of fields defined on the space-time, and if possible, from the geometry and topology of the space-time itself. As a follower of Maxwell and Faraday, he believed in the primary role of fields and tried to derive the equations of motion as characteristic behavior of singularities of the fields, or the singularities of the space-time curvature.

This dream was pursued in other forms by physicists who tried to derive fundamental properties of elementary particles from geometrical and topological aspects of supposedly underlying geometry, often more than four-dimensional. The efforts made in order to deduce discrete particle charges and symmetries from the topology and geometry of seven-dimensional spheres, Calabi-Yau manifolds or orbifolds belong to this category.

Yet an alternative point of view can be considered, supposing that the existence of matter is primary with respect to that of the space-time. One of the hints that it might be so indeed is the strange discrepancy between proton's lifetime, estimated by now as being above the threshold of 10^{33} years, and the age of our Universe (and according to the current standard theory, the space-time itself as we know it), which is of the order of 10^{15} years. In this light, the idea to derive the geometric properties of space-time, and perhaps its very existence, from fundamental symmetries and interactions proper to matter's most fundamental building blocks seems quite natural.

Historically, the laws of Quantum Mechanics were formulated without any mention of space and time, as presented in the first articles written by M. Born, P. Jordan and W. Heisenberg [1] in their version of matrix mechanics; also in the fundamental manual by P.A.M. Dirac [2], or in J. von Neumann's [3] formulation of quantum theory in terms of the C^* algebras. The non-commutative geometry [5] gives another example of interpreting the space-time relationships in purely algebraic terms.

2 The Z_3-graded algebra of quark states

At present, the most successful theoretical descriptions of fundamental interactions are based on the quark model, despite the fact that isolated quarks cannot be observed, although the deep elastic scattering of high-energy photons reveals the presence of point-like objects inside the nucleons. The only states accessible to direct observation are either three-quark or three-anti-quark combinations (fermions) or the quark-anti-quark states (bosons). Whenever one has to do with a tri-linear combination of fields (or operators), one must investigate the behavior of such states under permutations.

If the the space-time is to be derived from the interactions of fundamental constituents of matter, then it seems reasonable to choose the most stable particles known to us, which are baryons, and the strongest ineractions available, which are the interactions between quarks. The principal difficulty resides in the fact that we should define these "quarks" (or their states) without any mention of space-time.

The minimal requirements for the definition of quarks at the initial stage of model building are the following:

i) The mathematical entities representing quarks should form a linear space over complex numbers, so that we could produce their linear combinations with complex coefficients.

ii) They should also form an associative algebra, so that we could consider their multilinear combinations;

iii) There should exist two isomorphic algebras of this type corresponding to quarks and anti-quarks, and the conjugation transformation that maps one of these algebras onto another, $\mathcal{A} \rightarrow \bar{\mathcal{A}}$.

iv) The three quark (or three anti-quark) and the quark-anti-quark combinations should be distinguished in a certain way, for example, they should form a subalgebra in the algebra spanned by the generators.

With this in mind we can start to explore the algebraic properties of quarks that would lead to more general symmetries, that of space and time, appearing as a consequence of covariance requirements imposed on the discrete relations between the generators of the hypothetical quark algebra.

Let us introduce N generators spanning a linear space over complex numbers, satisfying the following relations which are a cubic generalization of anti-commutation in the ususal (binary) case (see e.g. [6], [7]):

$$\theta^A \theta^B \theta^C = j\, \theta^B \theta^C \theta^A = j^2\, \theta^C \theta^A \theta^B, \tag{1}$$

with $j = e^{i\pi/3}$, the primitive cubic root of 1. We have $\bar{j} = j^2$ and $1 + j + j^2 = 0$. We shall also introduce a similar set of *conjugate* generators, $\bar{\theta}^{\dot{A}}$, $\dot{A}, \dot{B}, ... = 1, 2, ..., N$, satisfying similar condition with j^2 replacing j:

$$\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2\, \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} = j\, \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}, \tag{2}$$

Let us denote this algebra by $\mathcal{A}$. We shall endow this algebra with a natural Z_3 grading, considering the generators θ^A as grade 1 elements, and their conjugates $\bar{\theta}^{\dot{A}}$ of grade 2. The grades add up modulo 3, and the products $\theta^A\theta^B$ span a linear subspace of grade 2, while the cubic products $\theta^A\theta^B\theta^C$ are of grade 0. Similarly, all quadratic expressions in conjugate generators, $\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}$ are of grade $2+2 = 4_{mod\,3} = 1$, whereas their cubic products are again of grade 0, like the cubic products od θ^A's.

Combined with the associativity property, these cubic relations impose finite dimension on the algebra generated by the Z_3 graded generators. As a matter of fact, cubic expressions are the highest order that does not vanish identically. The proof is immediate:

$$\theta^A\theta^B\theta^C\theta^D = j\,\theta^B\theta^C\theta^A\theta^D = j^2\,\theta^B\theta^A\theta^D\theta^C =$$
$$= j^3\,\theta^A\theta^D\theta^B\theta^C = j^4\,\theta^A\theta^B\theta^C\theta^D,$$

and because $j^4 = j \neq 1$, the only solution is

$$\theta^A\theta^B\theta^C\theta^D = 0. \tag{3}$$

Therefore the total dimension of the algebra defined via the cubic relations (1) is equal to $N + N^2 + (N^3 - N)/3$: the N generators of grade 1, the N^2 independent products of two generators, and $(N^3 - N)/3$ independent cubic expressions, because the cube of any generator must be zero by virtue of (1), and the remaining $N^3 - N$ ternary products are divided by 3, also by virtue of the constitutive relations (1).

The conjugate generators $\bar{\theta}^{\dot{B}}$ span an algebra $\bar{\mathcal{A}}$ isomorphic with $\mathcal{A}$. If we want the products between the generators θ^A and the conjugate ones $\bar{\theta}^{\dot{B}}$ to be included into the greater algebra spanned by both types of generators, we should consider all possible products, which will be included in the linear subspaces with a definite grade. of the resulting algebra $\mathcal{A}\otimes\bar{\mathcal{A}}$. In order to decide which expressions are linearly dependent, and what is the overall dimension of the enlarged algebra generated by θ^A's and their conjugate variables $\bar{\theta}^{\dot{D}}$'s, we must impose on them some binary commutation relations.

The fact that the conjugate generators are of grade 2 may suggest that they behave like products of two ordinary generators $\theta^A\theta^B$. Such a choice was often made (see, e.g., [6], [10] and [7]). However, this does not enable one to make a distinction between the conjugate generators and the products of two ordinary ones, and it would be better to be able to make the difference. Due to the binary nature of "mixed" products, another choice is possible, namely, to impose the following relations:

$$\theta^A\bar{\theta}^{\dot{B}} = -j\,\bar{\theta}^{\dot{B}}\theta^A, \qquad \bar{\theta}^{\dot{B}}\theta^A = -j^2\,\theta^A\bar{\theta}^{\dot{B}}, \tag{4}$$

In what follows, we shall deal with the simplest realization of such algebras, spanned by two generators. Consider the case when $A, B, .. = 1, 2$. The algebra A contains numbers, two generators of grade 1, θ^1 and θ^2, their four independent products (of grade 2), and two independent cubic expressions, $\theta^1\theta^2\theta^1$ and $\theta^2\theta^1\theta^2$. Similar expressions can be produced with conjugate generators $\bar{\theta}^{\dot{C}}$; finally, mixed expressions appear, like four independent grade 0 terms $\theta^1\bar{\theta}^{\dot{1}}$, $\theta^1\bar{\theta}^{\dot{2}}$, $\theta^2\bar{\theta}^{\dot{1}}$ and $\theta^2\bar{\theta}^{\dot{2}}$.

3 The $SL(2, \mathrm{C})$ as invariance group of Z_3-graded cubic relations

Let us consider multilinear forms defined on the algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$. Because only cubic relations are imposed on products in $\mathcal{A}$ and in $\bar{\mathcal{A}}$, and the binary relations on the products of ordinary and conjugate elements, we shall fix our attention on tri-linear and bi-linear forms, conceived as mappings of $\mathcal{A} \otimes \bar{\mathcal{A}}$ into linear spaces over complex numbers.

Consider a tri-linear form ρ^{α}_{ABC}. We shall call this form Z_3-invariant if we can write:

$$\rho^{\alpha}_{ABC}\, \theta^A \theta^B \theta^C = \frac{1}{3}\left[\rho^{\alpha}_{ABC}\, \theta^A \theta^B \theta^C + \rho^{\alpha}_{BCA}\, \theta^B \theta^C \theta^A + \rho^{\alpha}_{CAB}\, \theta^C \theta^A \theta^B\right] =$$

$$= \frac{1}{3}\left[\rho^{\alpha}_{ABC}\, \theta^A \theta^B \theta^C + \rho^{\alpha}_{BCA}\, (j^2\, \theta^A \theta^B \theta^C) + \rho^{\alpha}_{CAB}\, j\, (\theta^A \theta^B \theta^C)\right],$$

by virtue of the commutation relations (1).

From this it follows that we should have

$$\rho^{\alpha}_{ABC}\ \theta^A \theta^B \theta^C = \frac{1}{3}\left[\rho^{\alpha}_{ABC} + j^2\, \rho^{\alpha}_{BCA} + j\, \rho^{\alpha}_{CAB}\right] \theta^A \theta^B \theta^C, \tag{5}$$

from which we get the following properties of the ρ-cubic matrices:

$$\rho^{\alpha}_{ABC} = j^2\, \rho^{\alpha}_{BCA} = j\, \rho^{\alpha}_{CAB}. \tag{6}$$

Even in this minimal and discrete case, there are covariant and contravariant indices: the lower and the upper indices display the inverse transformation property. If a given cyclic permutation is represented by a multiplication by j for the upper indices, the same permutation performed on the lower indices is represented by multiplication by the inverse, i.e. j^2, so that they compensate each other.

Similar reasoning leads to the definition of the conjugate forms $\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{B}\dot{A}}$ satisfying the relations similar to (6) with j replaced be its conjugate, j^2:

$$\bar{\rho}^{\dot{\alpha}}_{\dot{A}\dot{B}\dot{C}} = j\, \bar{\rho}^{\dot{\alpha}}_{\dot{B}\dot{C}\dot{A}} = j^2\, \bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{A}\dot{B}} \tag{7}$$

In the simplest case of two generators, the j-skew-invariant forms have only two independent components:

$$\rho^1_{121} = j\, \rho^1_{211} = j^2\, \rho^1_{112}, \quad \rho^2_{212} = j\, \rho^2_{122} = j^2\, \rho^2_{221},$$

and we can set

$$\rho^1_{121} = 1,\ \rho^1_{211} = j^2,\ \rho^1_{112} = j,$$
$$\rho^2_{212} = 1,\ \rho^2_{122} = j^2,\ \rho^2_{221} = j.$$

A tensor with three covariant indices can be interpreted as a "cubic matrix". One can introduce a ternary multiplication law for cubic matrices defined below:

$$(a * b * c)_{ikl} := \sum_{pqr} a_{piq} b_{qkr} c_{rlp} \tag{8}$$

in which any *cyclic* permutation of the matrices in the product is equivalent to the same permutation on the indices:

$$(a * b * c)_{ikl} = (b * c * a)_{kli} = (c * a * b*)_{lik} \tag{9}$$

The ternary mutliplication law involving summation on certain indices should be in fact interpreted as contraction of covariant and contravariant indices, which requires the introduction of an analog of a metric tensor. In the case introduced above, with

$$(a * b * c)_{ikl} := \sum_{pqr} a_{piq} b_{qkr} c_{rlp} \tag{10}$$

the "metric" is juct the Kronecker delta:

$$(a * b * c)_{ikl} := a_{nij} b_{pkr} c_{slm}\, \delta^{jp} \delta^{rs} \delta^{mn}. \tag{11}$$

If we want to keep a particular symmetry under such ternary composition, i.e. make it define a closed ternary algebra, we should introduce a new composition law that follows the particular symmetry of the given type of cubic matrices. For example, let us define:

$$\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\} := \rho^{(\alpha)} * \rho^{(\beta)} * \rho^{(\gamma)} + j\, \rho^{(\beta)} * \rho^{(\gamma)} * \rho^{(\alpha)} + j^2\, \rho^{(\gamma)} * \rho^{(\alpha)} * \rho^{(\beta)} \tag{12}$$

Because of the symmetry of such ternary *j-bracket* one has

$$\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\}_{ABC} = j\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\}_{BCA},$$

so that it becomes obvious that with respect to the *j-bracket* composition law the matrices $\rho^{(\alpha)}$ form a ternary subalgebra. Indeed, we have

$$\{\rho^{(1)}, \rho^{(2)}, \rho^{(1)}\} = -\rho^{(2)}; \quad \{\rho^{(2)}, \rho^{(1)}, \rho^{(2)}\} = -\rho^{(1)}; \tag{13}$$

all other combinations being proportional to the above ones with a factor j or j^2, whereas the j-brackets of three identical matrices obviously vanish.

Let us find the simplest representation of this ternary algebra in terms of a *j-commutator* defined in an associative algebra of matrices $M_2(\mathbf{C})$ as follows:

$$[A, B, C] := ABC + j\, BCA + j^2\, CAB \tag{14}$$

It is easy to see that the trace of any j-bracket of three matrices must vanish; therefore, the matrices that would represent the cubic matrices $\rho^{(\alpha)}$ must be traceless. Then it is a matter of simple exercise to show that any two of the three Pauli sigma-matrices divided by $\sqrt{2}$ provide a representation of the ternary j-skew algebra of the ρ-matrices; e.g.

$$\sigma^1\sigma^2\sigma^1 + j\,\sigma^2\sigma^1\sigma^1 + j^2\,\sigma^1\sigma^1\sigma^2 = -2\,\sigma^2,$$

$$\sigma^2\sigma^1\sigma^2 + j\,\sigma^1\sigma^2\sigma^2 + j^2\,\sigma^2\sigma^2\sigma^1 = -2\,\sigma^1$$

Thus, it is possible to find a representation in the associative algebra of finite matrices for the non-associative j-bracket ternary algebra. A similar representation can be found for the two cubic matrices $r\bar{h}o^{(\dot\alpha)}$ with the j^2-skew bracket.

The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.

Let $U^{A'}_{A}$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U^{B'}_{B}\,\theta^B$.

We are looking for the solution of the covariance condition for the ρ-matrices:

$$\Lambda^{\alpha'}_{\beta}\,\rho^{\beta}_{ABC} = U^{A'}_{A}\,U^{B'}_{B}\,U^{C'}_{C}\,\rho^{\alpha'}_{A'B'C'}. \tag{15}$$

Now, $\rho^1_{121} = 1$, and we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ-matrix on the right-hand side is $\rho^{1'}_{A'B'C'}$, which has only three components,

$$\rho^{1'}_{1'2'1'} = 1, \quad \rho^{1'}_{2'1'1'} = j^2, \quad \rho^{1'}_{1'1'2'} = j,$$

which leads to the following equation:

$$\Lambda^{1'}_{1} = U^{1'}_{1}\,U^{2'}_{2}\,U^{1'}_{1} + j^2\,U^{2'}_{1}\,U^{1'}_{2}\,U^{1'}_{1} + j\,U^{1'}_{1}\,U^{1'}_{2}\,U^{2'}_{1} = U^{1'}_{1}\,(U^{2'}_{2}\,U^{1'}_{1} - U^{2'}_{1}\,U^{1'}_{2}),$$

because $j^2 + j = -1$.

For the alternative choice $\alpha' = 2'$ the ρ-matrix on the right-hand side is $\rho^{2'}_{A'B'C'}$, whose three non-vanishing components are

$$\rho^{2'}_{2'1'2'} = 1, \quad \rho^{2'}_{1'2'2'} = j^2, \quad \rho^{2'}_{2'2'1'} = j.$$

The corresponding equation becomes now:

$$\Lambda^{2'}_{1} = U^{2'}_{1}\,U^{1'}_{2}\,U^{2'}_{1} + j^2\,U^{1'}_{1}\,U^{2'}_{2}\,U^{2'}_{1} + j\,U^{2'}_{1}\,U^{2'}_{2}\,U^{1'}_{1} = U^{2'}_{1}\,(U^{1'}_{2}\,U^{2'}_{1} - U^{1'}_{1}\,U^{2'}_{2}),$$

The remaining two equations are obtained in a similar manner. We choose now the three lower indices on the left-hand side equal to another independent combination, (212). Then the ρ-matrix on the left hand side must be ρ^2 whose component ρ^2_{212} is equal to 1. This leads to the following equation when $\alpha' = 1'$:

$$\Lambda^{1'}_{2} = U^{1'}_{2}\,U^{2'}_{1}\,U^{1'}_{2} + j^2\,U^{2'}_{2}\,U^{1'}_{1}\,U^{1'}_{2} + j\,U^{1'}_{2}\,U^{1'}_{1}\,U^{2'}_{2} = U^{1'}_{2}\,(U^{1'}_{2}\,U^{2'}_{1} - U^{1'}_{1}\,U^{2'}_{2}),$$

and the fourth equation corresponding to $\alpha' = 2'$ is:

$$\Lambda^{2'}_{2} = U^{2'}_{2} U^{1'}_{1} U^{2'}_{2} + j^2 U^{1'}_{2} U^{2'}_{1} U^{2'}_{2} + j\, U^{2'}_{2} U^{2'}_{1} U^{1'}_{2} = U^{2'}_{2} \,(U^{1'}_{1} U^{2'}_{2} - U^{2'}_{1} U^{1'}_{2}).$$

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears everywhere on the right-hand side. The obvious solution relating *linearly* the matrices $\Lambda^{\alpha'}_{\beta}$ to the matrices $U^{A'}_{B}$ is to impose

$$det\,(U^{A'}_{B}) = U^{1'}_{1} U^{2'}_{2} - U^{2'}_{1} U^{1'}_{2} = 1 \tag{16}$$

Then we have

$$\Lambda^{1'}_{1} = U^{1'}_{1}, \quad \Lambda^{2'}_{2} = U^{2'}_{2}, \quad \Lambda^{1'}_{2} = -U^{1'}_{2}, \quad \Lambda^{2'}_{1} = -U^{2'}_{1}, \tag{17}$$

from which it follows immediately that also $det\,\Lambda = 1$,

$$det\,(\Lambda^{\alpha'}_{\beta}) = \Lambda^{1'}_{1} \Lambda^{2'}_{2} - \Lambda^{2'}_{1} \Lambda^{1'}_{2} = 1 \tag{18}$$

Both conditions (16) and (18) define the $SL(2, \mathbf{C})$ group, the covering group of the Lorentz group.

$$\Lambda^{2'}_{1} = -U^{2'}_{1}\,[det(U)], \tag{19}$$

The remaining two equations are obtained in a similar manner, resulting in the following:

$$\Lambda^{1'}_{2} = -U^{1'}_{2}\,[det(U)], \qquad \Lambda^{2'}_{2} = U^{2'}_{2}\,[det(U)]. \tag{20}$$

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears everywhere on the right-hand side. Taking the determinant of the matrix $\Lambda^{\alpha'}_{\beta}$ one gets immediately

$$det\,(\Lambda) = [det\,(U)]^3. \tag{21}$$

Taking into account that the inverse transformation should exist and have the same properties, we arrive at the conclusion that $det\,\Lambda = 1$,

$$det\,(\Lambda^{\alpha'}_{\beta}) = \Lambda^{1'}_{1} \Lambda^{2'}_{2} - \Lambda^{2'}_{1} \Lambda^{1'}_{2} = 1 \tag{22}$$

which defines the $SL(2, \mathbf{C})$ group, the covering group of the Lorentz group.

However, the U-matrices on the right-hand side are defined only up to the phase, which due to the cubic character of the covariance relations (3 - 20), and they can take on three different values, 1, j or j^2, i.e. the matrices $j\, U^{A'}_{B}$ or $j^2\, U^{A'}_{B}$ satisfy the same relations as the matrices $U^{A'}_{B}$ defined above. The determinant of U can take on the values 1, j or j^2 while $det(\Lambda) = 1$

Let us then choose the matrices $\Lambda^{\alpha'}_{\beta}$ to be the usual spinor representation of the $SL(2, \mathbf{C})$ group, while the matrices $U^{A'}_{B}$ will be defined as follows:

$$U^{1'}_{1} = j\Lambda^{1'}_{1}, U^{1'}_{2} = -j\Lambda^{1'}_{2}, U^{2'}_{1} = -j\Lambda^{2'}_{1}, U^{2'}_{2} = j\Lambda^{2'}_{2}, \tag{23}$$

the determinant of U being equal to j^2.

Obviously, the same reasoning leads to the conjugate cubic representation of $SL(2,\mathbf{C})$ if we require the covariance of the conjugate tensor

$$\bar{\rho}^{\dot{\beta}}_{\dot{D}\dot{E}\dot{F}} = j\,\bar{\rho}^{\dot{\beta}}_{\dot{E}\dot{F}\dot{D}} = j^2\,\bar{\rho}^{\dot{\beta}}_{\dot{F}\dot{D}\dot{E}},$$

by imposing the equation similar to (15)

$$\Lambda^{\dot{\alpha}'}_{\dot{\beta}}\,\bar{\rho}^{\dot{\beta}}_{\dot{A}\dot{B}\dot{C}} = \bar{\rho}^{\dot{\alpha}'}_{\dot{A}'\dot{B}'\dot{C}'}\bar{U}^{\dot{A}'}_{\dot{A}}\,\bar{U}^{\dot{B}'}_{\dot{B}}\,\bar{U}^{\dot{C}'}_{\dot{C}}. \tag{24}$$

The matrix $\bar{U}$ is the complex conjugate of the matrix U whose determinant is equal to j.

Moreover, the two-component entities obtained as images of cubic combinations of quarks, $\psi^\alpha = \rho^\alpha_{ABC}\theta^A\theta^B\theta^C$ and $\bar{\psi}^{\dot{\beta}} = \bar{\rho}^{\dot{\beta}}_{\dot{D}\dot{E}\dot{F}}\,\bar{\theta}^{\dot{D}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}}$ should anti-commute, because their arguments do so, by virtue of (4):

$$(\theta^A\theta^B\theta^C)(\bar{\theta}^{\dot{D}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}}) = -(\bar{\theta}^{\dot{D}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}})(\theta^A\theta^B\theta^C)$$

We have found the way to derive the covering group of the Lorentz group acting on spinors via the usual spinorial representation. The spinors are obtained as the homomorphic image of tri-linear combination of three quarks (or anti-quarks). The quarks transform with matrices U (or $\bar{U}$ for the anti-quarks), but these matrices are not unitary: their determinants are equal to j^2 or j, respectively. So, quarks cannot be put on the same footing as classical spinors; they transform under a Z_3-covering of the $SL(2,\mathbf{C})$ group.

A similar covariance requirement can be formulated with respect to the set of 2-forms mapping the quadratic quark-anti-quark combinations into a four-dimensional linear real space. As we already saw, the symmetry (4) imposed on these expressions reduces their number to four. Let us define two quadratic forms, $\pi^\mu_{A\dot{B}}$ and its conjugate $\bar{\pi}^\mu_{\dot{B}A}$ with the following symmetry requirement

$$\pi^\mu_{A\dot{B}}\,\theta^A\bar{\theta}^{\dot{B}} = \bar{\pi}^\mu_{\dot{B}A}\,\bar{\theta}^{\dot{B}}\theta^A. \tag{25}$$

The Greek indices $\mu,\nu...$ take on four values, and we shall label them $0,1,2,3$.

It follows immediately from (4) that

$$\pi^\mu_{A\dot{B}} = -j^2\,\bar{\pi}^\mu_{\dot{B}A}. \tag{26}$$

Such matrices are non-hermitian, and they can be realized by the following substitution:

$$\pi^\mu_{A\dot{B}} = j^2\,i\,\sigma^\mu_{A\dot{B}}, \quad \bar{\pi}^\mu_{\dot{B}A} = -j\,i\,\sigma^\mu_{\dot{B}A} \tag{27}$$

where $\sigma^\mu_{A\dot{B}}$ are the unit 2 matrix for $\mu = 0$, and the three hermitian Pauli matrices for $\mu = 1,2,3$.

Again, we want to get the same form of these four matrices in another basis. Knowing that the lower indices A and $\dot{B}$ undergo the transformation with matrices $U^{A'}_B$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$, we demand that there exist some 4×4 matrices $\Lambda^{\mu'}_{\nu}$ representing the transformation of lower indices by the matrices U and $\bar{U}$:

$$\Lambda^{\mu'}_{\nu}\, \pi^{\nu}_{A\dot{B}} = U^{A'}_{A}\, \bar{U}^{\dot{B}'}_{\dot{B}}\, \pi^{\mu'}_{A'\dot{B}'}, \tag{28}$$

and this defines the vector (4×4) representation of the Lorentz group.

With the invariant "spinorial metric" in two complex dimensions, ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$ such that $\varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}}$, we can define the contravariant components $\pi^{\nu\, A\dot{B}}$. It is easy to show that the Minkowskian space-time metric, invariant under the Lorentz transformations, can be defined as

$$g^{\mu\nu} = \frac{1}{2}\left[\pi^{\mu}_{A\dot{B}}\, \pi^{\nu\, A\dot{B}}\right] = diag(+,-,-,-) \tag{29}$$

Together with the anti-commuting spinors ψ^{α} the four real coefficients defining a Lorentz vector, $x_{\mu}\, \pi^{\mu}_{A\dot{B}}$, can generate now the supersymmetry via standard definitions of super-derivations.

4 The Z_3 graded matrices and third order equations

If the observed strongly interacting fermions (nucleons) are well described by wave functions belonging to the spinorial representation of the Lorentz group, and if we suppose that they are obtained as *products* of three quark states, then it is natural to ask what is the generalization of Dirac's equation (which is of the first order in partial derivatives) that would require taking its *third power* in order to become diagonalized ?

Let us suppose that ternary combinations leading to the observed fermionic states are tensor products of three quark states, but also the spinorial wave functions of nucleons can be obtained as ternary products of wave functions describing the quarks. Therefore we must introduce a Schrödinger-like equation for quark wave functions linear in the momentum operator, only the *third power* of which would become diagonal.

Moreover, as we shall see, general solutions of the third-order differential equation resulting from diagonalization will necessarily contain complex wave-vectors and complex frequencies, thus enhancing free propagation of such solutions. However, with a proper choice of three factors, one can produce exponential functions with pure imaginary exponents, like the freely propagating solutions of Dirac or Klein-Gordon equations. Similar phenomenon is observed in superconductors, where certain products of two electron wave functions, with specially coupled momenta, behave like a freely propagating boson, while the electrons themselves do not propagate freely being trapped in the surrounding atomic lattice.

This leads naturally to the ternary generalization of Clifford algebras. Instead of the usual binary relation defining the usual Clifford algebra,

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\, g^{\mu\nu}\, \mathbf{1}, \quad \text{with } \; g^{\mu\nu} - g^{\nu\mu} = 0$$

we should introduce its ternary generalization, which is quite obvious:

$$Q^a\,Q^b\,Q^c + Q^b\,Q^c\,Q^a + Q^cQ^aQ^b = 3\,\eta^{abc}\,\mathbf{1}, \tag{30}$$

where the tensor η^{abc} must satisfy $\eta^{abc} = \eta^{bca} = \eta^{cab}$ The lowest-dimensional representation of such an algebra is given by complex 3×3 matrices:

$$Q^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \quad Q^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

These matrices are given the Z_3-grade 1; their hermitian conjugates $Q^{*a} = (Q^a)^\dagger$ are of Z_3-grade 2, whereas the diagonal matrices are of Z_3-grade 0; it is easy to check that the grades add up modulo 3. The matrices Q^a, $(a = 1,2,3)$ satisfy the ternary relations (30) with η^{abc} a totally symmetric tensor, whose only non-vanishing components are

$$\eta^{111} = \eta^{222} = \eta^{333} = 1, \quad \eta^{123} = \eta^{231} = \eta^{321} = j^2, \quad \text{and } \eta^{321} = \eta^{213} = \eta^{132} = j.$$

Therefore, the Z_3-graded generalization of Dirac's equation should read:

$$\frac{\partial\,\psi}{\partial\,t} = Q^1\,\frac{\partial\psi}{\partial\,x} + Q^2\,\frac{\partial\psi}{\partial y} + Q^3\,\frac{\partial\psi}{\partial z} + B\,m\,\psi \tag{31}$$

where ψ stands for a triplet of wave functions, which can be considered either as a column, or as a grade 1 matrix with three non-vanishing entries $u\,v\,w$, and B is the diagonal 3×3 matrix with the eigenvalues 1, j and j^2. It is interesting to note that this is possible only with *three* spatial coordinates.

In order to diagonalize this equation, we must act three times with the same operator, which will lead to the same equation of *third order*, satisfied by each of the three components u, v, w, e.g.:

$$\frac{\partial^3\,u}{\partial t^3} = \left[\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial\,y^3} + \frac{\partial^3}{\partial z^3} - 3\,\frac{\partial^3}{\partial x \partial y \partial z}\right] u + m^3\,u \tag{32}$$

This equation can be solved by separation of variables; the time-dependent and the space-dependent factors have the same structure:

$$A_1\,e^{\omega\,t} + A_2\,e^{j\,\omega\,t} + A_3 e^{j^2\,\omega\,t}, \qquad B_1\,e^{\mathbf{k}.\mathbf{r}} + B_2\,e^{j\,\mathbf{k}.\mathbf{r}} + B_3\,e^{j^2\,\mathbf{k}.\mathbf{r}}$$

The independent *real* solutions of the third-order equation can be arranged in a 3×3 matrix as follows:

$$\begin{pmatrix} A_{11}\,e^{\omega\,t+\mathbf{k}.\mathbf{r}} & A_{12}\,e^{\omega\,t-\frac{\mathbf{k}.\mathbf{r}}{2}}\cos\xi & A_{13}\,e^{\omega\,t-\frac{\mathbf{k}.\mathbf{r}}{2}}\sin\xi \\ A_{21}\,e^{-\frac{\omega\,t}{2}+\mathbf{k}.\mathbf{r}}\cos\tau & A_{22}\,e^{-\frac{\omega\,t}{2}-\frac{\mathbf{k}.\mathbf{r}}{2}}\cos\tau\,\cos\xi & A_{23}\,e^{-\frac{\omega\,t}{2}-\frac{\mathbf{k}.\mathbf{r}}{2}}\cos\tau\sin\xi \\ A_{31}\,e^{-\frac{\omega\,t}{2}+\mathbf{k}.\mathbf{r}}\sin\tau & A_{32}\,e^{-\frac{\omega\,t}{2}-\frac{\mathbf{k}.\mathbf{r}}{2}}\sin\tau\,\cos\xi & A_{33}\,e^{-\frac{\omega\,t}{2}-\frac{\mathbf{k}.\mathbf{r}}{2}}\sin\tau\,\sin\xi \end{pmatrix}$$

where $\tau = \frac{\sqrt{3}}{2}\omega\,t$ and $\xi = \frac{\sqrt{3}}{2}\mathbf{kr}$.

The parameters $\omega, \mathbf{k}$ and m must satisfy the cubic dispersion relation:

$$\omega^3 = k_x^3 + k_y^3 + k_z^3 - 3\, k_x k_y k_z + m^3 \tag{33}$$

The relation (33) is invariant under the simultaneous change of sign of ω, $\mathbf{k}$ and m, which suggests the introduction of another set of solutions constructed in the same manner, but with minus sign in front of ω and $\mathbf{k}$, which we shall call *conjugate* solutions.

Although neither of these functions belongs to the space of tempered distributions, on which a Fourier transform can be performed, their ternary skew-symmetric products contain only trigonometric functions, depending on the combinations

$$2\,(\tau - \xi) \text{ and } 2\,(\tau + \xi).$$

As a matter of fact, the *determinant* of the above matrix is a combination of trigonometric functions only. The same is true for the binary products of "conjugate" solutions, with the opposite signs of ωt and $\mathbf{k}.\mathbf{r}$ in the exponentials.

This fact suggests that it is possible to obtain via linear combinations of these products the solutions of *second* or *first order* differential equations, like the Klein-Gordon or the Dirac equation. Indeed, the determinant of the matrix of elementary combinations (4) yields the following result:

$$det \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \; [\cos(2\tau - 2\xi) + \cos(2\tau + 2\xi)]$$

Still, the parameters ω and $\mathbf{k}$ do not satisfy the proper mass shell relations; however, it is possible to find new parameters, which are linear combinations of these, that will satisfy quadratic relations which may be interpreted as a mass shell equation. We can more readily see this if we use the following parametrisation: let us put

$$\zeta = (k_x + k_y + k_z), \qquad \chi = Re(jk_x + j^2 k_y + k_z), \quad \eta = Im(jk_x + j^2 k_y + k_z),$$

$$\text{and} \quad s^2 = \chi^2 + \eta^2 \qquad \phi = Arctg(\eta/\chi).$$

In these coordinates the cubic mass hyperboloid equation becomes

$$\omega^3 - \zeta\, s^2 = m^3 \tag{34}$$

We can define a one-dimensional subset of the above 3-dimensional hypersurface by requiering

$$\omega^2 - s^2 = [\,2\,m^3 - (\omega - \zeta)\,(\omega^2 + s^2)\,]/(\omega + \zeta) = M^2 = Const.$$

If we have *three* cubic relations of the same type, corresponding to the dispersion relations of three quarks satisfying the 3-rd order differential equation), then the ordinary mass hyperboloid can be produced if we require the following coupling in the k-space:

$$\omega_1^2 + \omega_2^2 + \omega_3^2 - s_1^2 - s_2^2 - s_3^2 = \Omega^2 - s_1^2 - s_2^2 - s_3^2 = 3\,M^2$$

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