数理解析研究所講究録
第 1707 巻 2010 年 37-55

Vassiliev-type invariants revisited

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Dec. 8, 2009 at Nihon Univ. Sakura-Jousui, Tokyo.

In this talk, we revisit an 'old and new' theme on *the topology of bifurcation locus* Γ *in the infinite dimensional space* $\mathcal{M}$ *of smooth mappings from a m-fold* M *to Euclidean space*, initiated by R. Thom [27],

$$\mathcal{M} := C^\infty(M, \mathbb{R}^n) \supset \Gamma := \{C^\infty \text{ unstable maps}\}.$$

(A) Study $H^0(\mathcal{M}-\Gamma)$ as the space of all isotopy invariants of C^∞ structural stable maps $\Rightarrow$ **Vassiliev-type invariants ...**

(B) Study $\mathcal{M}$ as a representation space of the diffeomorphism group $\mathrm{Diff}(M) \Rightarrow$ **Thom polynomials ...** .

As for (A), I will describe a general framework based on the Thom-Mather theory, and state an elementary observation (Theorem 3.3 below): *for generic maps* $M \to \mathbb{R}^n$ *where* $m = \dim M \geq 2$, *a naïve analog to finite type knot invariants is not so fruitful.* The order-one invariants were studied in several cases (recent works are, e.g., [30], [31], [22]), however we have still missed a proper definition of *Vassiliev-type invariants of higher order* for such maps. As for (B), I will only comment about a few examples and propose a further direction.

All spaces and mappings are of class C^∞ thoughout.

1 Mapping space

1.1 $\mathcal{A}$-equivalence and invariant stratification

Let M be a compact manifold of dimension m without boundary, and N a manifold of dimension n without boundary. Denote the space of smooth maps, equipped with C^∞ topology, by $\mathcal{M} := C^\infty(M, N)$.

A map $\varphi : U \to \mathcal{M}$ (U being finite dimensional) is called of type C^∞ if the evaluation map $M \times U \to N$ is a C^∞ map (Frechet manifold structure on $\mathcal{M}$).

The *$\mathcal{A}$-equivalence group* or *right-left group* is the direct product of diffeomorphism groups

$$\mathcal{A}_{M,N} := \mathrm{Diff}(M) \times \mathrm{Diff}(N) \;\text{ acting on } \mathcal{M} \text{ by } \; (\varphi, \tau).f := \tau \circ f \circ \varphi^{-1}.$$

Also put $\mathcal{A}^0_{M,N}$:= the connected component containing (id_M, id_N).

Definition 1.1 $f, g : M \to N$ $(\in \mathcal{M})$ are $\mathcal{A}$-equivalent if $\mathcal{A}_{M,N}.f = \mathcal{A}_{M,N}.g$ i.e., $\exists(\varphi,\tau) \in \mathcal{A}_{M,N}$ s.t. $g = (\varphi,\tau).f$

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \varphi \downarrow \cong & & \cong \downarrow \tau \\ M & \xrightarrow{g} & N \end{array}$$

We say f is a *C^∞-structurally stable map*, if the orbit of $\mathcal{A}_{M,N}.f$ is an open set in $\mathcal{M}$. If (m,n) is in so-called *Mather's nice range*, structurally stable maps form an open dense subset in $\mathcal{M}$ (*C^∞-Structural Stability Theorem*, proved by John Mather). Often say *generic maps* for short.

Any orbits $\mathcal{A}_{M,N}.f$ become $\mathcal{A}_{M,N}$-invariant Frechet submanifolds of $\mathcal{M}$. We are interested in *finite codimensional* orbits (or families of orbits).

Definition 1.2 In this note, a *multi-singularity type* means a $\mathcal{A}$-type of germs

$$f : M, S \to N, f(S) \quad (S \text{ and } f(S) \text{ are finite}),$$

i.e., an unordered l-tuple $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_l)$ of $\mathcal{A}$-classes of multi-germs $\alpha_j : \mathbb{R}^m, S_j \to \mathbb{R}^n, 0$ where $l = |f(S)|$ and $|S| = \sum |S_j|$. Here α_j may be a family (moduli) of $\mathcal{A}$-classes of multi-germs. The ($\mathcal{A}_e$-)codimension of $\boldsymbol{a}$ is defined by $s = |\boldsymbol{a}| := \sum \operatorname{codim} \alpha_j$ (codim means $\mathcal{A}_e$-codimension).

Assume that $|\boldsymbol{a}| = s < \infty$. We put

$$\Gamma(\boldsymbol{a}) := \text{Closure}\{\, f \in \Gamma_s \mid f \text{ has multi-singularities of type } \boldsymbol{a} \,\} \subset \Gamma_s.$$

By definition, $\Gamma(\boldsymbol{a}) = \Gamma(\alpha_1) \cap \cdots \cap \Gamma(\alpha_l)$ in $\mathcal{M}$, and if $\boldsymbol{a}$ is a multi-singularity type adjacent to $\boldsymbol{b}$, then $\Gamma(\boldsymbol{b})$ is contained in $\Gamma(\boldsymbol{a})$. The set $\Gamma(\boldsymbol{a})$ becomes a *pseudo-algebraic* subset in $\mathcal{M}$ in the sense of [17]: In particular, its smooth part is a Frechet submanifold of codimension s.

Put $\Gamma_\infty := \{\, f \in \mathcal{M} \mid \operatorname{codim} \mathcal{A}_{M,N}.f = \infty \,\}$ and $\mathcal{U}_0 := \mathcal{M} - \Gamma_\infty$.

Lemma 1.3 codim $\mathcal{A}_{M,N}.f < \infty$ *(i.e., $f \in \mathcal{U}_0$) if and only if there exists a finite subset S in M such that*
1) the germ $f : M, S \to N, f(S)$ is of $\mathcal{A}_e$-finite codimension;
2) each point $x \in f^{-1}(f(S)) - S$ is not critical;
3) For any finite S' away from S, the germ of f at S' is stable.

Theorem 1.4 (Mather [17])
Assume that (m,n) belongs to the nice range. Then Γ_∞ has infinite codimension, and there exists a filtration

$$\mathcal{M} \supset (\Gamma =) \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_s \supset \cdots \supset \Gamma_\infty, \qquad \text{codim } \Gamma_s = s$$

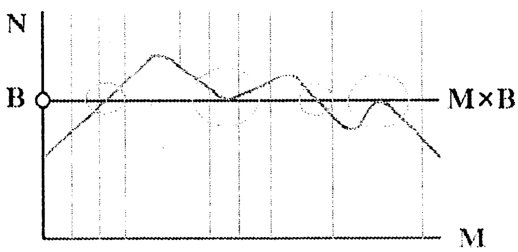
by $\mathcal{A}_{M,N}$-invariant closed pseudo-algebraic subsets Γ_s such that there is a topologically locally trival fibration $\pi_s : \Gamma_s - \Gamma_{s+1} \to Y_s$ where each fibre is an $\mathcal{A}_{M,N}$-orbit and Y_s is a finite dimensional manifold.

Remark 1.5 1) $H^0(\mathcal{M} - \Gamma)$ classifies all C^∞-structurally stable maps up to $\mathcal{A}^0_{M,N}$.
2) The rank of $H^0(\mathcal{M} - \Gamma)$ (also $H^0(\Gamma_s - \Gamma_{s+1})$) is at most countable.
3) Each component of Y_s corresponds to the moduli space of certain multi-singularities.
4) For a multi-singularity type $\boldsymbol{a}$ of codimension s, $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ becomes an $\mathcal{A}_{M,N}$-invariant Frechet submanifold of codimension s in $\mathcal{M}$; $\Gamma_s - \Gamma_{s+1}$ is a union of (countably many) such Frechet submanifolds.
5) If (m,n) is out of the nice range, Γ_∞ may have finite codimension. In this case, Theorem 1.4 holds when replacing $\mathcal{M}$ by $\mathcal{U} := \mathcal{M} - \Gamma_\infty$.

1.2 Contact equivalence for $\mathcal{M}$

Let $B \subset N$ be a p-dimensional closed submanifold and $p : M \times N \to M$ the projection. Put

$$\mathcal{K}_{M,N,B} := \{ H \in \mathrm{Diff}(M \times N) \mid H \text{ preserves } M \times B \text{ and fibers of } p \}.$$



Definition 1.6 $f, g : M \to N$ are $\mathcal{K}_B$-equivalent if

$$\exists H \in \mathcal{K}_B \ \text{ s.t. } \ H(\mathrm{graph}(f)) = \mathrm{graph}(g).$$

Theorem 1.7 $\widehat{\Gamma}_\infty := \{f \in \mathcal{M}, \text{codim } \mathcal{K}_B.f = \infty\}$ *has infinite codimension in $\mathcal{M}$. Moreover, there exists a filtration*

$$\mathcal{M} \supset (\widehat{\Gamma} =) \widehat{\Gamma}_1 \supset \widehat{\Gamma}_2 \supset \cdots \supset \widehat{\Gamma}_s \supset \cdots \supset \widehat{\Gamma}_\infty, \qquad \text{codim } \widehat{\Gamma}_s = s$$

by $\mathcal{K}_B$-invariant closed pseudo-algebraic subsets such that there is a topologically locally trival fibration $\pi_s : \widehat{\Gamma}_s - \widehat{\Gamma}_{s+1} \to Z_s$ where each fibre is an $\mathcal{K}_B$-orbit and Z_s is a finite dimensional manifold.

Notice that $f \in \mathcal{M} - \widehat{\Gamma}$ iff f is transverse to B. So, $H^0(\mathcal{M} - \widehat{\Gamma})$ classifies diffeo-types of submanifolds $f^{-1}(B)$ in M.

2 Vassiliev complex

2.1 Spectral sequence

Take Mather's filtration of $\mathcal{U}$ (Theorem [17]) and its 'dual filtration': Put

$$\mathcal{U}_s := \mathcal{M} - \Gamma_{s+1},$$

then we have invariant open subsets of $\mathcal{V}$:

$$\mathcal{M} - \Gamma = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots \subset \mathcal{U}_s \subset \cdots \subset \bigcup_{s=0}^{\infty} \mathcal{U}_s \subset \mathcal{M}.$$

We think of a spectral sequence associated to this filtration: the first E_1-terms are

$$E_1^{s,t} := H^{s+t}(\mathcal{U}_s, \mathcal{U}_{s-1}) \leftarrow H^t(\Gamma_s - \Gamma_{s+1})$$

with a coefficient ring R. The arrow indicates the *Alexander duality for functional spaces* in the sense of Eells [7], that is, the Thom isomorphism for *coorientable* components of the s-codimensional manifold $\Gamma_s - \Gamma_{s+1}$ (for a non-coorientable connected component the Thom class within integer coefficients vanishes, but it works within $\mathbb{Z}_2$-coefficients). Thus $E_1^{s,0}$ is the R-module generalted by coorientable components in $\Gamma_s - \Gamma_{s+1}$, especially,

$$E_1^{0,0} = H^0(\mathcal{U}_0) = H^0(\mathcal{M} - \Gamma).$$

The first cochain complex is

$$0 \to E_1^{0,t} \to E_1^{1,t} \to E_1^{2,t} \to \cdots$$

where the operator $d_1 : E_1^{s,t} \to E_1^{s+1,t}$ is the connection homomorphism ∂ of cohomology exact sequence for the triple $(\mathcal{U}_{s+1}, \mathcal{U}_s, \mathcal{U}_{s-1})$. As usual, we put for $r \geq 1$,

$$E_{r+1}^{s,t} := \frac{\ker[d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}]}{\mathrm{Im}\,[d_r : E_r^{s-r,t+r-1} \to E_r^{s,t}]}$$

with d_{r+1}, and we have a natural homomorphism $E_\infty^{s,t} \to H^*(\mathcal{M})$.

Instead, we may take

$$0 \to E_1^{1,t} \to E_1^{2,t} \to E_1^{3,t} \to \cdots$$

and $E_r^{s,t}$ ($s \geq 1$), that approximates $H^*(\mathcal{U}, \mathcal{U}_0) = H^*(\mathcal{M}, \mathcal{M} - \Gamma)$, the cohomology with support on Γ.

2.2 Vassiliev complex

Define $C^0(\mathcal{A}) = 0$ and for $s \geq 1$,

$$C^s(\mathcal{A}) \ = \oplus R \cdot \boldsymbol{a}$$

the vector space generated by *coorientable* $\mathcal{A}$-classes $\boldsymbol{a}$ of multi-singularities of codimension s (precisely saying, it is defined as an inductive limit of vector spaces generated by coorientable strata of codimension s in some invariant Whitney stratifications of multi-jet spaces, cf. [28], [19]). The coboundary ∂ is defined by using versal unfoldings. $C^s(\mathcal{A})$ is regarded as a submodule of $E_1^{s,0}$ by a simple identification

$$C^s(\mathcal{A}) \subset E_1^{s,0}, \quad \boldsymbol{a} \mapsto \Gamma(\boldsymbol{a}).$$

The coboundary $\partial : C^s(\mathcal{A}) \to C^{s+1}(\mathcal{A})$ is induced from d_1.

Definition 2.1 The cochain complex $(C^*(\mathcal{A}), \partial)$ is called the *local Vassiliev complex for $\mathcal{A}$-classification of multi-singularities.*

The operator $\partial : C^s(\mathcal{A}) \to C^{s+1}(\mathcal{A})$ can explicitly be written down as follows. Let $\boldsymbol{a} \in C^s(\mathcal{A})$ and $\boldsymbol{b} \in C^{s+1}(\mathcal{A})$. Take a versal deformation of $\boldsymbol{b}$. On the parameter space, the bifurcation diagram $\Psi(\boldsymbol{a})$ of type $\boldsymbol{a}$ is defined: It is either empty or 1-dimensional semi-algebraic curves approaching the origin. Count the *incidence coefficient* $[\boldsymbol{a}; \boldsymbol{b}]$, defined by the algebraic intersection number of $\Psi(\boldsymbol{a})$ with a small oriented sphere centered at the origin: Then $\partial \boldsymbol{a} = \sum [\boldsymbol{a}; \boldsymbol{b}]\ \boldsymbol{b}$, the sum taken over all generators $\boldsymbol{b}$. An example is shown in subsection 2.4 described below.

Remark 2.2 Notice that the Vassiliev complex is determined only by the local classification of singularities. In fact, although there are possibly many connected components in each $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$, they are regarded as just 'one stratum' in the complex $C(\mathcal{A})$. A more finer subcomplex, say an *enriched Vassiliev complex*,

$$C_{en}^s(\mathcal{A}) \subset E_1^{s,0}$$

may be obtained by some additional 'non-local' data to $C^*(\mathcal{A})$. Some choices are to input: the data of configurations of $S_1, \cdots, S_k$ on M at which a multi-signularity occurs (when $m = 1$, these data are called *cord-diagrams* or *weight systems*), the placement of the singular point locus in M (if $m \geq 2$), the topological types of singular fibers (if $m \geq n$), and so on.

2.3 Local invariants for generic maps

There is a natural homomorphism, for $s \geq 1$,

$$H^s(C(\mathcal{A}))(= \ker \partial) \to E_2^{s,0} \to E_\infty^{s,0} \to H^s(\mathcal{M}, \mathcal{M} - \Gamma) \to H^s(\mathcal{M}).$$

In particular, if $H^1(\mathcal{M}) = 0$, we have $H^1(C(\mathcal{A})) \to H^0(\mathcal{M} - \Gamma)$.

Definition 2.3 A function $v \in H^0(\mathcal{M} - \Gamma)$ is called a *local invariant for generic maps* if it comes from $H^1(C(\mathcal{A}))$ (cf. Goryunov [6]) . If we take some enriched complex instead, then we say v is *semi-local* or *enriched-local.*

Local invariants should be " *Euler characteristic associated to stable objects* " which are determined only by local modifications and initial data, e.g.

- number of individual singular points, $\sharp\, \Sigma(f)$, or more generally,
Euler characteristics of singular point sets of several types $\chi(\Sigma(f))$,
Euler characteristics of images $\chi(f(M))$, $\chi(f(\Sigma(f)))$, etc,
- Whitney index (rotation number), normal Euler numbers, Smale invariants for generic immersions;
- total linking number for oriented links; Bennequin invariants for critical value sets (a sort of linking numbers) ... ...

Some related works are listed below:

(m,n)	object/singularities	
$(1,1)$	functions on circle bundles	Kazarian [13]
$(1,2)$	generic curves, wavefronts	Arnold [2], [3], Aicardi [1]
$(1,3)$	knots and links	Vassiliev [29]
$(2,1)$	Morse function/singular fibres	Saeki-T.Yamamoto [24]
$(2,2)$	generic maps	Ohmoto-Aicardi [20]
$(2,3)$	generic maps	Goryunov [6]
$(2,4)$	generic immersions and 2-knots	Kamada [12], Habiro et al [16]
$(3,2)$	generic maps	M.Yamamoto [30]
$(3,3)$	geneirc maps	Oset-Sinha [22]
$(3,4)$	generic immersions	Ekholm [8]
$(3,5)$	generic immersions	Ekholm [8], Saeki-Szűcs-Takase [26]
$(4,3)$	generic maps/singular fibres	Saeki [23], T.Yamamoto [31]

2.4 Examples of local invariants

- $(m, n) = (1, 2)$ (Arnold [2])

$\Longrightarrow$ Basic invariants J^+, J^-, St for generic immersed plane curves (Precisely saying, St is not local in the above sense, but semi-local; $J^\pm$ are local).

- $(m, n) = (2, 4)$

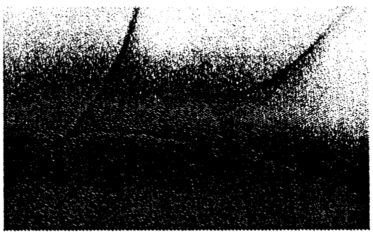
Codim. 0: immersions with transverse double pts (=generic immersion).
Codim. 1: A_1-singularity, and tangency of two sheets.
Codim. 2: A_2-singularity, and triple points
$\Longrightarrow$ Local invariants of generic immersion $f : M^2 \to \mathbb{R}^4$ are $\sharp$ of double points $d(f)$ and normal Euler number $e(f)$.

- $(m, n) = (2, 3)$ (Goryunov [6])

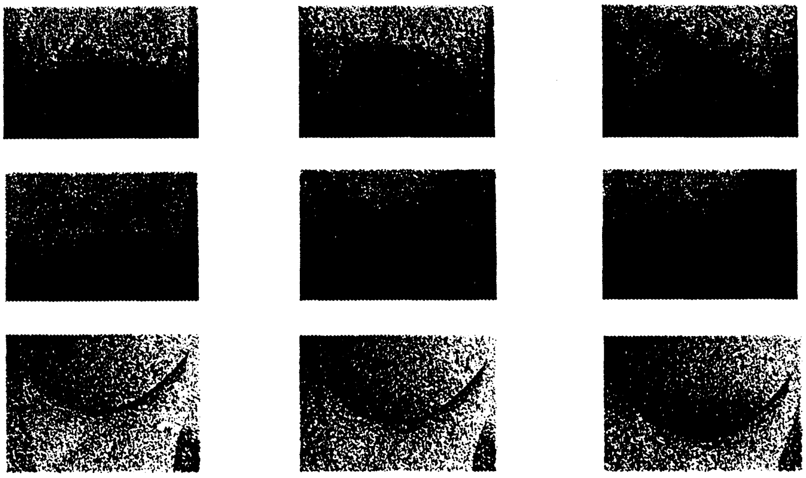
Codim. 0: maps with Cross-cap and transverse double/triple pts (=generic maps).
Codim. 1: There are 12 types
$\Longrightarrow$ Local invariants of generic maps $M^2 \to \mathbb{R}^3$ are $\sharp$ of triple pts, $\sharp$ of Cross-caps, and a new invariant (relating inverse self-tangency)

- $(m, n) = (2, 2)$ (Ohmoto-Aicardi [20], [18])

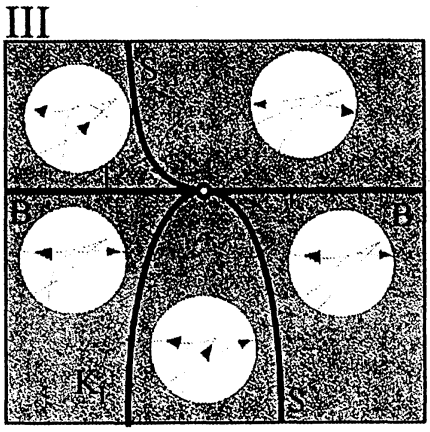
Codim. 0: maps with fold, cusps and double folds (=generic maps).
The apparent contour (= critical value set (discriminant)) of a generic map $M^2 \to \mathbb{R}^2$ looks like

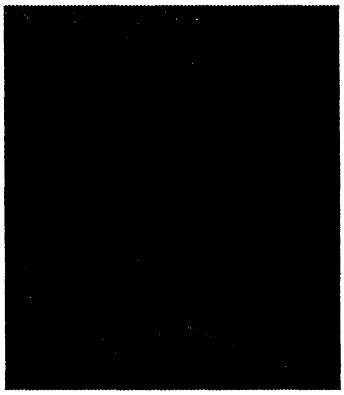


Codim. 1: there are 10 types $\alpha = L, B, \cdots, C_1$ of (multi-)germs: Here three examples named by S, B, K_0 are depicted below (These are *coorientable*: an orientation of parameter is defined as the number of double pts (or cusps) increases – from the picture on the left to the picture on the right).



Codim. 2: there are 20 types $\beta = I, II, III, A_1^4, \cdots$ of (multi-)germs: The type III has the following bifurcation diagram (2-parameter):





Then incidence coefficients $[\boldsymbol{a}; \boldsymbol{b}]$ $(\boldsymbol{b} = III)$ are counted as follows:

$$[S; III] = 2, \quad [B; III] = -2. \quad [K_1; III] = -1.$$

The same computations for any $\boldsymbol{a}, \boldsymbol{b}$ can be done, and then we have

$$0 \longrightarrow C^1(\mathcal{A}_{2,2}) \simeq \mathbb{Z}^{10} \xrightarrow{\partial} C^2(\mathcal{A}_{2,2}) \simeq \mathbb{Z}^{20} \xrightarrow{\partial}$$

$\Longrightarrow$ rank $H^1(C(\mathcal{A}); \mathbb{Z}) = 3$ and local invariants (over $R \ni \frac{1}{4}$) are generated modulo constants by

$\Delta c = 2\Delta I_1$:	$\sharp$ of cusps;
$\Delta d = \Delta I_2 + \Delta I_3$:	$\sharp$ of double folds;
$\Delta \nu = \Delta I_1 - \Delta_2 + \Delta I_3$:	projective Bennequin invariant.

Those generates local invariants of generic maps $M^2 \to \mathbb{R}^2$. Further, note that type B (beak-to-beak) can be separated into two types according to how components of contour curves are mutually connected, that yields an *semi-local* invariant, see Hacon-Mendes-Romero Fuster [10]:

$$\Delta I_4 = \Delta l - \Delta b_1 + \Delta b_2 : \ \sharp \text{ of components of critical set } C(f).$$

Remark 2.4 The projective Bennequin number itself is an interesting invariant for apparent contours. Although it is not easily computed, there is a nice algorithm of Bellettini et al [4].

3 Finite type invariants of mappings

3.1 Finite type invariants of mappings: Global $\mathcal{A}$-classification

Let $\mathcal{M} = C^\infty(M, N)$. For $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_k)$ where all α_j are of codimension 1, a normal slice to $\Gamma(\boldsymbol{a})$ is denoted by

$$\Xi^{\boldsymbol{a}} : [-1,1]^k \to \mathcal{M}, \quad (t_1, \cdots, t_k) \mapsto \Xi^{\boldsymbol{a}}_{t_1 \cdots t_k}, \quad \Xi^{\boldsymbol{a}}_0 \in \Gamma_k$$

We define *finite type invariants in a naïve sense* as follows. Let R be a commutative ring.

Definition 3.1 A function $v : \mathcal{M} - \Gamma \to R$ is an *invariant of order r* if the following equality holds for any $k \geq r+1$ and for any k-parameter family $\Xi^{\boldsymbol{a}}$ having type $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_k)$ with $\operatorname{codim} \alpha_j = 1$,

$$\sum_{\epsilon} \epsilon_1 \cdots \epsilon_k \, v(\Xi^{\boldsymbol{a}}_{\epsilon_1 \cdots \epsilon_k}) = 0 \qquad (*)$$

where the sum is taken over 2^k combinations of $\epsilon_i = \pm 1$.

Let V_r denote the R-module generated by invariants of order r. By definition, we have a filtration

$$V_0 \subset V_1 \subset \cdots \subset V_\infty := \bigcup_{r=0}^{\infty} V_r \subset H^0(\mathcal{M} - \Gamma).$$

Obviously $V_0 = H^0(\mathcal{M})$, constant functions over connected components of $\mathcal{M}$. In a natural way, V_∞ becomes a graded ring: the multiplication $V_r \times V_l \to V_{r+l}$ is defined by $v_1 v_2(f) := v_1(f) v_2(f)$ $(f \in \mathcal{M} - \Gamma)$.

Remark 3.2 In case of $M = S^1$ and $N = \mathbb{R}^3$, Vassiliev constructed a simplicial resolution of Γ in $\mathcal{M} \times \mathbb{R}^\infty$, and introduced a spectral sequence for the resolution space, which produces $\{V_s\}_{s=0}^{\infty}$ of knot invariants (For simplicity, let the coefficient ring $R = \mathbb{Q}$). In our terminology there is some enriched Vassiliev complex so that $H^s(C_{en}(\mathcal{A})) \simeq V_s/V_{s-1}$ and 'initial data' (called an *actuality table*) gives an inductive construction of an injective homomorphism $H^s(C_{en}(\mathcal{A})) \to V_s$ for each $s = 0, 1, 2, \cdots$.

Theorem 3.3 *Let (m, n) be in the nice range. Let the source manifold M be closed, without boundary and of dimension $m \geq 2$. Assume that $H^1(\mathcal{M}) = 0$ (for example, $N = \mathbb{R}^n$).*

1. *If $m + 1 < n$, any finite type invariants $v : \mathcal{M} - \Gamma \to R$ are polynomials in local invariants modulo constants, i.e.*

$$V_r/V_0 = \oplus_{k=1}^{r} \mathrm{Sym}^k(V_1/V_0), \quad V_1/V_0 = H^1(C(\mathcal{A})).$$

2. *If $m + 1 \geq n$, finite type invariants $v : \mathcal{M} - \Gamma \to R$, whose value do not change along any "non-transverse strata" in Γ, are polynomials in local invariants modulo constants.*

Theorem says that the above naïve definition is somehow irrelevant for generic maps with source dimension greater than 1. A similar statement as the claim 1 in Theorem has already been known in particular dimension: $n = 2m$ (Kamada [12], Januszkiewicz-Swiatkowski [11]) and $n = 2m - 1$ ($m \geq 4$) (Ekholm [9]).
Sketch of the proof: Let $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_s)$ where $\operatorname{codim} \alpha_j = 1$, and recall

$$\Gamma(\boldsymbol{a}) = \bigcap \Gamma(\alpha_j) := \{\, f \in \Gamma_s \mid f \text{ has multi-singularities of type } \boldsymbol{a} \,\}.$$

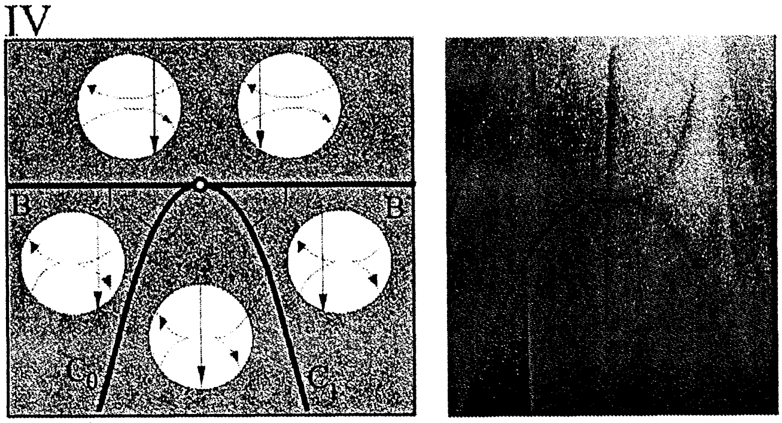
- <u>*The self-transverse locus $\Gamma(\boldsymbol{a})$ is irreducible*</u>, that is, any $f, g \in \Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ can be joined by some generic path γ in $\Gamma(\boldsymbol{a})$: $\gamma(0) = f$, $\gamma(1) = g$.

This is because $m \geq 2$ and M is connected. (Remark that the case of $m = 1$ is completely different: $\Gamma(\boldsymbol{a})$ has many irred. components labeled by 'non-local' data called *cord-diagrams*.)

- If $m + 1 < n$, γ meets only self-transverse loci $\Gamma(\boldsymbol{a}) \cap \Gamma(\beta)$ for β of codimension 1.

Roughly saying, codimension one invariant cycles $\sum a_i\beta_i$ in $\Gamma(\boldsymbol{a})$ are determined by the same type coherent system for local invariants, and an estimate about the dimension of invariants of order s leads us to conclude the assertion stated in the theorem.

• If $m + 1 \geq n$, γ also meets *'non-transverse loci'* $\Gamma(\alpha) \cap \Gamma(\beta)$ for β of codimension 2. For example, in case of $m = n = 2$, there is a *self-tangency locus* $\Gamma(\mathrm{B}) \cap \Gamma(\mathrm{IV})$ ($\alpha = (\mathrm{B})$, $\beta = (\mathrm{IV})$):



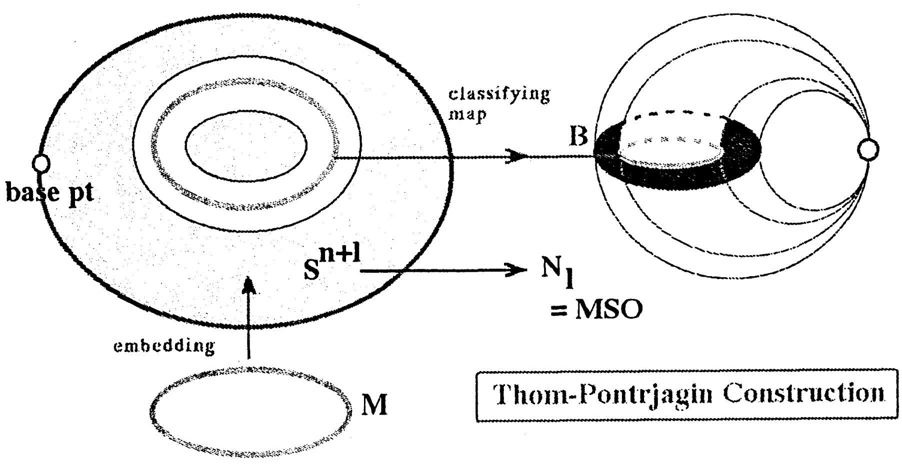
When our curve $\gamma(t)$ (red sheet) passes through the beaks point, our invariant v may change. In order to define 'non trivial higher order' Vassiliev-type invariants, we can not ignore these kinds of jumps of invariants.

3.2 Finite type invariants for closed n-folds : Global $\mathcal{K}$-classification

Recall the classical Thom-Pontjagin construction. Let M be a compact oriented n-manifold, and embed it in $\mathbb{R}^{n+\ell}$ ($\ell \gg n$): The classifying map of the normal bundle of rank ℓ is $\rho : M \to B$ where $B := BSO(\ell)$ is the Grassmanian of oriented ℓ-planes in $\mathbb{R}^\infty$; it is naturally extended to a map preserving base points to the Thom space $N_\ell := MSO(\ell)$):

$$f : S^{n+\ell} \to N_\ell, \qquad f(p_0) = \infty \in N_\ell.$$

Take ρ to be generic: *f is transverse to B and $M = f^{-1}(B)$.*



The cobordism group of oriented n-manifolds is

$$\Omega^{ori}(n) \simeq \pi_0\left(\lim_{\ell\to\infty} C^\infty(S^{n+\ell}, N_\ell)_{base}\right) = \pi_0(\Omega^\infty N_\infty).$$

We put

$$\widehat{\mathcal{M}} := C^\infty(S^{n+\ell}, N_\ell)_{base}, \quad B := BSO(\ell) \subset N_\ell \qquad (\ell \gg 0)$$

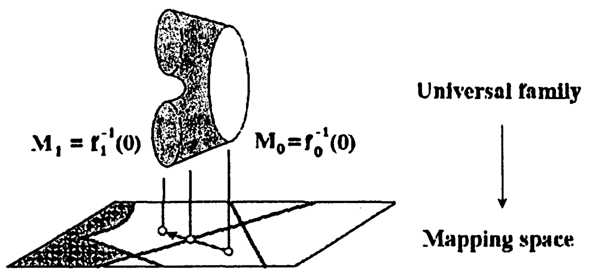
Note that $\widehat{\mathcal{M}} - \widehat{\Gamma}$ is the space of closed oriented n-mfds (see R. Thom, section 3 in [27]). It follows from Theorem 1.7 that there is a $\mathcal{K}_B$-invariant stratification

$$\widehat{\mathcal{M}} \supset \widehat{\Gamma} = \widehat{\Gamma}_1 \supset \widehat{\Gamma}_2 \supset \cdots.$$

A global version of *Martinet's versality theorem* is stated as follows (Kazarian [15]): Let $e : S^{n+\ell} \times \widehat{\mathcal{M}} \to N_\ell$ be the evaluation map $e(p,f) := f(p)$, and denote by π_B the restriction to $e^{-1}(B)$ of the projection to the second factor $\widehat{\mathcal{M}}$. Then, we may regard π_B as the "universal C^∞ stable map", and $\widehat{\Gamma}$ as the "discriminant set of π_B":

$$\begin{array}{ccccc} Q^{n+p} & \longrightarrow & e^{-1}(B) & \subset & S^{n+\ell} \times \widehat{\mathcal{M}} \xrightarrow{e} N_\ell \\ \downarrow & & \downarrow \pi_B & & \\ P^p & \longrightarrow & \widehat{\mathcal{M}} & & \end{array}$$

In fact, any C^∞-structural stable maps $Q \to P$ with $\dim Q - \dim P = n$ can be obtained as a fiber product of π_B and a smooth map $P \to \widehat{\mathcal{M}}$ transverse to $\widehat{\Gamma}$ (uniquely up to isotopies).



Let $\mathcal{M}$ be a connected component of $\widehat{\mathcal{M}}$. Note that $H^0(\mathcal{M} - \widehat{\Gamma})$ *classifies all compact oriented n-manifolds belonging to a fixed cobordism class".*

Remark 3.4
The set of diffeomorphism types of *compact* n-dimensional C^∞-manifolds is at

most countable (classical, Milnor etc).
(2) The 'null-cobordant' component in $\widehat{\mathcal{M}}$ is

$$\mathcal{M}_0 = C^\infty(S^{n+\ell}, S^\ell)_{base}, \qquad B = \{0\} \subset S^\ell, \quad \ell \gg 0.$$

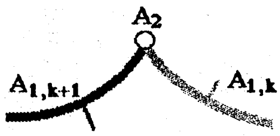
Note $\Omega^{ori}(1) = \Omega^{ori}(2) = \Omega^{ori}(3) = 0$. So, in these cases, $\widehat{\mathcal{M}} = \mathcal{M}_0$.
($n = 1$) Saeki [23] introduced a cochain complex for topological types of singular fibre, which is equivalent to a (enriched) Vassiliev complex for $\mathcal{K}_B$-inv. filtration of $\widehat{\Gamma}$.
($n = 3$) Sirokova [25] dealt with "the space of closed ori. 3-mfds".

Remark 3.5
1) $\widehat{\Gamma}_1 - \widehat{\Gamma}_2$ consists of maps f having one Morse singularity A_1 on $f^{-1}(B)$, i.e., a handle surgery: for $0 \leq k \leq [\frac{n+1}{2}]$,

$$A_{1,k} : (x_1, \cdots, x_{n+1}, z) \mapsto (-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2, z).$$

This is coorientable, except for n odd, $k = [\frac{n+1}{2}]$.
2) $\widehat{\Gamma}_2 - \widehat{\Gamma}_3$ consists of maps f having either of two Morse singularities, or - one A_2-singularity (=cancelation of surgeries) :

$$A_{2,k} : (x_1, \cdots, x_{n+1}, y, z) \mapsto (x_1^3 + yx_1 \pm x_2^2 \pm \cdots \pm x_{n+1}^2, y, z).$$



3) The 'self-tangential locus' belongs to $\widehat{\Gamma}_3$.

Naïve finite type invariants are defined in the same way as before:

Definition 3.6 A locally constant function $v : \mathcal{M} - \widehat{\Gamma} \to R$ (R being a commutative ring) is *of order r* if

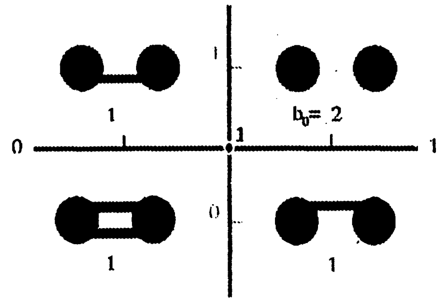
$$\sum_{\epsilon} \epsilon_1 \cdots \epsilon_k\, v(\Xi^{A_1^k}_{\epsilon_1 \cdots \epsilon_k}) = 0$$

for any k-tuple self-intersection ($k \geq r + 1$), i.e., any connected components of $\widehat{\Gamma}(A_1^k) - \widehat{\Gamma}_{k+1}$.

Theorem 3.7 (Folklore)
1) In case of n even, finite type invariants are generated only by the Euler characteristics $\chi : \mathcal{M} - \widehat{\Gamma} \to \mathbb{Z}$, $f \mapsto \chi(f^{-1}(B))$, modulo constants.
2) In case of n (> 1) odd, finite type invariants are generated only by the semi-Euler characteristics $\chi_2 : \mathcal{M} - \widehat{\Gamma} \to \mathbb{Z}_2$, $f \mapsto \chi_2(f^{-1}(B))$, modulo constants.

This is almost trivial and well-known perhaps. The proof is the same as before: *the locus $\Gamma(\boldsymbol{a})$ is irreducible for any $\boldsymbol{a} = (A_{1,k_1}, \cdots, A_{1,k_s})$,* that means that the above naïve definition allows us to forget any information about glueing maps of a specified surgery of type $\boldsymbol{a}$.

Remark 3.8 In particular, Betti number functions $f \mapsto b_k(f^{-1}(B))$ are not finite type invariants in the naïve sense.



Any n-tuple self-intersection of this type has non-zero values.

In order to keep the information of glueing maps of surgeries, we need more restrictions, i.e., not to be allowed to make other surgeries freely. A way to make such a restriction is to consider smaller mapping space: For instance, take an open subset of $\mathcal{M}$ with *fixed Betti numbers*, e.g., *the space of homology spheres*. That is the case as the theory of finite type invariants for homology 3-spheres (Ohtsuki [21]) and its generalization (Cochran-Melvin [5]).

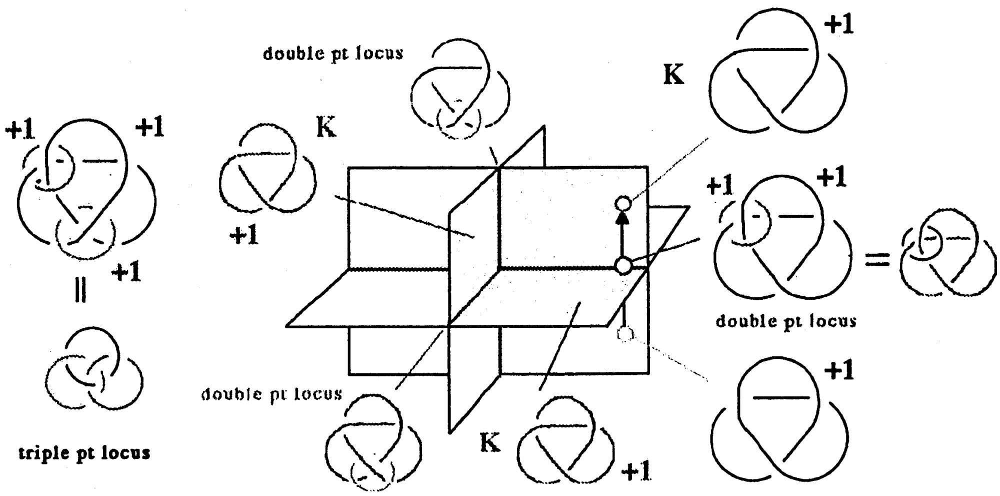
Let $\mathcal{M}_{ZHS}$ be the space of $\mathbb{Z}$-homology 3-sphere

$$\mathcal{M}_{ZHS} \subset \mathcal{M} = C^\infty(S^{3+\ell}, S^\ell)_{base}.$$

A codimension 1 stratum of $\widehat{\Gamma}(A_{1,2})$ in $\mathcal{M}_{HS}$ corresponds to the Dehn surgery along a framed knot with framing coefficient ± 1.

The self-transverse locus $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ in $\mathcal{M}_{ZHS}$ has quite many connected components, each of which is labeled by an 'algebraically split' framed link. Further, many components become coorientable.

The picture below dipicts strata adjacent to the stratum labeled by the Borromean link (a component of the triple-point locus $\widehat{\Gamma}(A_1A_1A_1)$ of the discriminant $\widehat{\Gamma}$ in $\mathcal{M}_{ZHS}$). White walls (labeled by the trivial knot) are non-coorientable; On the other hand, colored walls (labeled by the trefoil knot) form a coorientable cycle in $\mathcal{M}_{ZHS}$, which distinguishes the Poincaré 3-fold from the standard 3-sphere S^3.



4 Characteristic classes for fiber bundles

Let M be a compact, connected oriented manifold. We regard the affine space $\mathcal{M} = C^\infty(M, \mathbb{R}^\ell)$ *as a representation of the diffeomorphism group* $G = \operatorname{Diff} M$.

First, recall the classifying space of the topological group $G = \operatorname{Diff} M$ of orientation preserving diffeomorphisms. If n is quite high, $C^\infty(M, \mathbb{R}^\ell) - \Gamma = \operatorname{Emb}(M, \mathbb{R}^\ell)$, the space of all embeddings of M in $\mathbb{R}^\ell$. Sending $\ell \to \infty$, we may identify the classifying space of G with the topological quotient

$$B \operatorname{Diff} M = \operatorname{Emb}(M, \mathbb{R}^\infty)/\operatorname{Diff} M.$$

Denote it by BG for short and put $EG = \operatorname{Emb}(M, \mathbb{R}^\infty)$. Since EG is highly connected, the canonical map $EG \to BG$ gives the universal principal bundle for the group G. Let $BM := (EG \times M)/G$, the associated bundle with fibre M, then any smooth fiber bundle $E \to B$ (B paracompact), with fiber M and structure group $G = \operatorname{Diff} M$, can be obtained, up to isomorphisms, from the universal bundle $BM \to BG$ via the classifying map $\rho : B \to BG$. Any element of $H^*(BG)$ is called a universal G-characteristic class: G-characteristic classes of $E \to B$ are defined by their ρ^*-image in $H^*(B)$.

Now think of the composition of an embedding of M and a fixed 'projection' onto $\mathbb{R}^n$ for some small n,

$$\begin{tikzcd} M \arrow[r, "incl"] \arrow[dr, "f"'] & \mathbb{R}^\infty \arrow[d, "proj"] \\ & \mathbb{R}^n \end{tikzcd}$$

Then the map must admit unavoidable (structurally stable) singularities and by using these data let us try to characterize the topology of M, that was an idea of R. Thom.

So we put $\mathcal{M} = C^\infty(M, \mathbb{R}^n)$ and $B\mathcal{M} \to BG$ to be the associated bundle with fibre $\mathcal{M}$. Now $\mathcal{M}$ is a contractible space, hence the Borel cohomology $H^*_G(\mathcal{M}) := H^*(B\mathcal{M})$ is isomorphic to $H^*_G(pt) = H^*(BG)$.

The Vassiliev complex has much meanings in this equivarant setting: We then have (under assumption $N = \mathbb{R}^n$),

$$H^s(C(\mathcal{A})) \to E_\infty^{s,0} \to H^s_G(\mathcal{M}) \simeq H^*(BG).$$

We denote by $Tp_c \in H^*(BG)$ the G-characteristic class associated to a cocycle $c = \sum \lambda_i \boldsymbol{a}_i \in C^s(\mathcal{A})$. In fact, it holds (Kazarian [14]) that Tp_c is written as a universal polynomial in the relative Novikov-Landweber classes

$$\pi_* cl^I(T_\pi) = \pi_*(cl_1^{i_1}(T_\pi) \cdots cl_k^{i_k}(T_\pi))$$

where T_π is the relative tangent bundle of $\pi : E \to B$ (see below) and cl means Pontrjagin class, Euler class (with rational coefficients) or Stiefel-Whitney class (coefficients in $\mathbb{Z}_2$). This means the following: Suppose that we are given a fiber bundle $\pi : E \to B$ with fiber M over a manifold B and a smooth map $f : E \to \mathbb{R}^n$ over the total space of the bundle.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & \mathbb{R}^n \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} & B\mathcal{M} \arrow[d] \\ B \arrow[ur, "f"] \arrow[r, "\rho"'] & BG \end{tikzcd}$$
$$B \qquad\qquad\qquad\qquad$$

To any multi-singularity type $\boldsymbol{a}$ and appropriately generic $f : E \to \mathbb{R}^n$, we associate the *bifurcation locus* $B\boldsymbol{a}(f)$ ($\subset B$), which is a locally closed submanifold consisting of points $b \in B$ over which the map $f_b : E_b \simeq M \to \mathbb{R}^n$ admits the multi-singularity of type $\boldsymbol{a}$ at some finite points of E_b. Given a Vassiliev cocylce $c := \sum \lambda_i \boldsymbol{a}_i \in C^s(\mathcal{A})$ and a generic f, we define the *bifurcation cycle* $B_c(f)$ to be the geometric cycle $\sum \lambda_i B\boldsymbol{a}_i(f)$ in B: It is a geometric presentation of the G-characteristic class

$$\mathrm{Dual}\,[\overline{B_c(f)}] = \rho^* Tp_c.$$

Here n should be reasonably small: For if we take n to ∞, cocycles of BG live in $\mathcal{M} - \Gamma$.

Thus interesting problems from this singularity approach would be:
- Find the precise forms of universal polynomials Tp_c for given classes

$[c] \in H^*(C(\mathcal{A}))$,
- Find nontrivial relations among those G-characteristic classes Tp_c's,
- Find elements in $H^*(C(\mathcal{A}))$ representing torsion parts of G-characteristic classes (as geometric realization), etc.

Example 4.1 For example, in case that M is oriented circle S^1,

$$H^*(BS^1) = H^*(BU(1)) = \mathbb{Z}[c_1]$$

where c_1 is the first Chern class of complex line bundles. In [13] Kazarian observed that the class c_1 can be realized by some bifurcation locus of functions $E \to \mathbb{R}$ or maps $E \to \mathbb{R}^2$ over total space E of S^1-bundles (Also, for a classification of singularities of bifurcation loci, he computed the corresponding universal polynomials in c_1). But if one takes $\mathbb{R}^3$ as the target space, c_1 can not be realized by any bifurcation points, i.e., c_1 lives in the space of knots (embeddings).

Example 4.2 Recall that for an oriented C^∞-surface bundle $\pi : E \to B$ with fibre a closed oriented surface M, the r-th *Morita-Miller-Munford class* $e_r(E) \in H^{2r}(B;\mathbb{Z})$ is defined to be the pushforward $\pi_* e(T_\pi)^{r+1}$ where T_π is the relative tangent bundle over the total space E and $e(T_\pi) \in H^2(E;\mathbb{Z})$ is the Euler class. It is obvious that the MMM class $e_r(E)$ is realized by the Σ^2-bifurcation locus of generic maps $E \to \mathbb{R}^{r+1}$ ($r \geq 1$), where Σ^2 means singularities $\varphi : \mathbb{R}^2, 0 \to \mathbb{R}^{r+1}, 0$ of $\dim \ker d\varphi = 2$:

$$[\overline{B_{\Sigma^2}(f)}] = \pi_*[\overline{\Sigma^2(f)}] = \pi_* e(T_\pi \otimes f^*\epsilon^{r+1}) = \pi_* e(T_\pi)^{r+1} = e_r(E).$$

In case of generic maps $f : E \to \mathbb{R}^2$ (i.e., $r = 1$) with $\dim B = 2$, $B_{\Sigma^2}(f)$ consists of discrete points b in B, over which there is a point $p \in E_b$ such that the germ $E, p \to \mathbb{R}^2$ of f at p is $\mathcal{A}$-equivalent to

$$I_{22} + II_{22} : (x^2 \pm y^2 + x^3 + ay, xy + bx)$$

where x, y are local coordinates of fibre and a, b are local coordinates of B (=deformation parameters). As another example, there is a work by Saeki-Yamamoto [24] which shows that $e_1(E)$ is realized by the codimension 2 bifurcation locus corresponding to a special topological type of singular fiber of generic functions $f : E \to \mathbb{R}$: The singular fiber consists of 3 circle components each two of which meet at 2 nodal points.

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