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Geodesics on subriemannian manifolds

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1 Introduction

A subriemannian structure on a manifold M is a pair (D,g) such that D is a smooth distribution on M and g is a riemannian metric on D. A subriemannian manifold is a triple (M,D,g) such that M is a manifold and (D,g) is a subriemannian structure on M. In particular, if $D=TM$ then (M,D,g) is nothing but a riemannian manifold (M,g).

Riemannian geometry tells us that a minimizer (i.e., a shortest path) between two points of a riemannian manifold (M,g) is a geodesic, provided that the curve is parametrized by arc-length, and the geodesics are characterized to be the curves satisfying the geodesic equation expressed in local coordinates as:

$$\ddot{x}^i+\sum\Gamma^i_{jk}\dot{x}^j\dot{x}^k=0,$$

where Γ^i_{jk} denotes the Christoffel symbol. Conversely, every geodesic is locally length minimizing.

In the formulation of symplectic geometry, the geodesics $x(t)$ are the projections to the base manifold M of the integral curves $(x(t),p(t))$ of the Hamiltonian vector field $\overrightarrow{E}$ defined on the cotangent bundle T^*M, where E is the energy function associated to the metric g.

Now in subriemannian geometry, it is also of fundamental importance to study minimizers between two points of a subriemannian manifold (M,D,g). Since the metric g is defined only on the subbundle D of TM in this subriemannian case, there is no canonical means to define the length of a general curve $\gamma : [a, b] \to M$. But we can well speak of the length of γ if γ is an integral curve of D, that is, if $\dot\gamma(t) \in D_{\gamma(t)}$ for all t.

On the other hand Chow's theorem tells that if M is connected and if D is nonholonomic (in other word, bracket-generating), then any two points of M can be joined by a piecewise smooth integral curve of D.

Hence, especially for a nonholonomic subriemannian manifold (M, D, g), it makes sense and is important to study the minimizers (length minimizing piecewise smooth integral curves) between two points of the subriemannian manifold (M, D, g). However, contrary to the riemannian case, this problem is very subtle, mainly because the space $C_D(p, q)$ of all integral curves of D joining p and q may have singularities, while the space $C(p, q)$ of all curves joining p and q has no singularity and is a smooth infinite dimensional manifold, which makes difficult to apply directly the method of variation to the subriemannian case.

For a subriemannian manifold (M, D, g) we define a *normal biextremal* to be an integral curve of the Hamiltonian vector field $\overrightarrow{E}$ associated to the Hamiltonian function $E : T^*M \to \mathbf{R}$, where E is the energy function associated with the subriemannian metric g. We then define a *normal extremal* to be the projection to M of a normal biextremal. Then, as in riemannian geometry, a normal extremal is locally a minimizer.

However, R. Montgomery ([5], [6]) and I. Kupka [3] discovered that there exists a minimizer which is not a normal extremal, and hence called it abnormal. The appearance of abnormal minimizers is a surprising phenomenon never arising in riemannian geometry but peculiar to subriemannian geometry.

If D is a distribution on M, then the annihilator bundle $D^\perp$, considered as a submanifold of the symplectic manifold T^*M, carries a (singular) characteristic distribution $\mathrm{Ch}(D^\perp)$. An integral curve of this characteristic system $\mathrm{Ch}(D^\perp)$ contained in $D^\perp \backslash \{\text{zero section}\}$ is called an *abnormal biextremal*, of which the projection to M is called an *abnormal extremal*.

A rigorous application of the Pontryagin Maximum Principle of Optimal Control Theory to subriemannian geometry shows that a minimizer of subriemannian manifold (M, D, g) is either a normal extremal of (D, g) or an abnormal extremal of D.

This settled the long discussions that had been made until 1990's by many mathematicians with erroneous statements, and gave a right way to treat the problem of length-minimizing paths in subriemannian geometry.

In this paper we will give a survey on the problem of length-minimizing paths mainly following Liu and Sussmann [4]. We then consider this problem in a concrete case of the standard Cartan distribution. Referring to [8], we will carry out detailed computation of extremals, which will well illustrate how normal and abnormal extremals appear in subriemannian geometry.

2 Nonholonomic distributions

Let M be a differentiable manifold. A subbudle D of its tangent bundle TM of M of rank r is alternatively called a distribution on M of dimension r, since it gives a law which assigns to every point $p \in M$ an r-dimensional subspace D_p of the tangent space T_pM. A section of D on an open set $U \subset M$ is a local vector field X defined on U such that $X_p \in D_p$ for all $p \in U$. A local basis of D on U is a system of sections $X_1, \ldots, X_r$ of D defined on U such that $\{(X_1)_p, \ldots, (X_r)_p\}$ forms a basis of D_p for all $p \in U$. It is clear that for any point $p_0 \in M$ there is an local basis of D defined on a neighbourhood of p_0. If $\{X_1, \ldots, X_r\}$ is a local basis of D on U, then any section X of D on U is uniquely written:

$$X = f_1X_1 + \cdots + f_rX_r$$

with some functions $f_1, \ldots, f_r$ on U, and we say that D is locally generated, or defined, by $X_1, \ldots, X_r$.

Let $D^\perp$ denotes the annihilators of D, that is, $D = \bigcup\limits_{p\in M} D_p^\perp$ with

$$D_p^\perp = \{\alpha \in T_p^*M; \langle \alpha, v\rangle = 0 \text{ for all } v \in D_p\}.$$

Clealy $D^\perp$ is a subbundle of the cotangent bundle T^*M of rank s, where $s = \dim M - r$. If $\{\omega^1, \ldots, \omega^s\}$ is a local basis of $D^\perp$, we say that D is locally defined by the Pfaff system $\{\omega^1, \ldots, \omega^r\}$ or by the Pfaff equations:

$$\omega^1 = \cdots = \omega^s = 0.$$

In this sense, a distribution is also called a differential system or a Pfaff system.

Given an r-dimensional distribution D on M, one of the most important problems that has been studied since the nineteenth century is to study integral manifolds of D. An immersed submanifold $f : S \to M$ is called an integral manifold of D if

$$f_* T_s S \subset D_{f(s)} \quad \text{for all } s \in S.$$

Evidently the dimension of an integral manifold is $\leq r$. However, it is not always the case that there exists an r-dimensional integral manifold.

Definition 1 *A distribution D of dimension r on M is called completely integrable if about every point $p_0 \in M$ there is a coordinate system $(U, (x^1, \dots, x^n))$ such that all the submanifolds of U given by $x^{r+1} = \text{const}, x^{r+2} = \text{const}, \dots,$ $x^n = \text{const}$ are integral manifolds of D.*

As is well-known, the Frobenius theorem gives a criterion for D to be completely integrable:

Theorem 1 (Frobenius) *A distribution D on M is completely integrable if and only if D is involutive, that is, D satisfies the condition: "For any open set $U \subset M$, the Lie bracket $[X, Y]$ of sections X, Y of D on U is also a section of D." Moreover, if D is completely integrable then the manifold M is a disjoint union $\bigcup_\lambda L_\lambda$ of the maximal connected r-dimensional integral manifolds L_λ of D, each L_λ being called a leaf of D.*

The problem of finding integral manifolds of distributions which are not completely integrable are treated by Cartan-Kähler theory.

Now let us proceed to consider integral curves of D. In order to well analyse the length functional we had better expand the class of curves to consider to that of the absolutely continuous curves: A continuous curve $\gamma : I \to M$, I being an interval $[a, b]$ of $\mathbf{R}$, is absolutely continuous if it has a derivative for almost all t, and if in any coordinate system the components of this derivative are measurable functions. We then define an integrable curve of D to be an absolutely continuous curve $\gamma : I \to M$ such that $\dot\gamma \in D_{\gamma(t)}$ for almost all $t \in I$. An integral curve of D is also called integral path, D-arc, or horizontal curve.

If $\{X_1, \ldots, X_r\}$ is a local basis of D defined on an open set $U \subset M$, then a curve $\gamma : I \to U$ is an integral curve of D if

$$(*)\quad \dot\gamma(t) = c_1(t)(X_1)_{\gamma(t)} + \cdots + c_r(t)(X_r)_{\gamma(t)}$$

for some functions $c_1(t), \ldots, c_r(t)$. Conversely if the function $c_1(t), \ldots, c_r(t)$ are assigned then the curve $\gamma(t)$ is determined by the ordinary differential equation $(*)$. In control theory $c_1, \ldots, c_r$ are interpreted as control parameters and D (or $X_1, \ldots, X_r$) is regarded as a control system.

If two points $p, q \in M$ can be joined by an integral curve of D, we say that q is reachable from p, If D is completely integrable then the set of all points reachable from p is the leaf passing through p.

Let us now introduce a class of distributions which are in a sense at the opposite end from the completely integrable distributions.

Definition 2 *A distribution D on M is called nonholonomic or bracket-generating if for any local basis $X_1, \ldots, X_r$ of D on U the collection of all vector fields $\{X_i, [X_i, X_j], [X_i, [X_j, X_k]], \ldots\}$ generated by Lie brackets of the X_i spans the whole tangent bundle TU.*

This definition can be rephrased as follows: Let $\underline{D}$ denote the sheaf of germs of section of D. Define the sheaves $\{\mathcal{D}^k\}_{k\geq 1}$ inductively by setting first $\mathcal{D}^1 = \underline{D}$ and then

$$\mathcal{D}^{k+1} = \mathcal{D}^k + [\mathcal{D}^1, \mathcal{D}^k] \quad (k \geq 1).$$

Then D is completely integrable if $\mathcal{D}^1 = \mathcal{D}^2$, and nonholonomic if $\bigcup \mathcal{D}^k = \underline{TM}$.

The following theorem of Chow [2] is fundamental.

Theorem 2 (Chow) *Let M be a connected manifold and D a nonholonomic distribution on M, then there exists for any two points $p, q \in M$ a piecewise smooth integral curve by which p and q can be joined.*

A detailed proof can be also found in [11], or in [7].

3 Subriemannian distance

If (M, D, g) is a subriemannian manifold, and $p \in M$, $v \in D_p$, we define the *length* $\|v\|_g$ of v by

$$\|v\|_g = g_p(v, v)^{\frac{1}{2}}$$

If $\gamma : [a, b] \to M$ is an integral curve of D, then we define the *length* of γ by

$$\|\gamma\|_g = \int_a^b \|\dot{\gamma}(t)\|_g dt.$$

If γ is not an integral curve, we agree to define $\|\gamma\|_g = +\infty$. We then define a function $d_g : M \times M \to \mathbf{R} \cup \{\infty\}$ by

$$d_g(p, q) = \inf\{\|\gamma\|_g; \partial\gamma = (p, q)\},$$

where we denote $\partial\gamma = (\gamma(a), \gamma(b))$.

If M is connected and D is bracket-generating, then $d_g : M \times M \to \mathbf{R}$ is a metric function on M and the topology on M that the metric determines coincides with the original manifold topology of M. The first assertion follows from Chow's theorem and the second assertion follows from the Ball-Box Theorem ([9], See p.29). The distance $d_g : M \times M \to \mathbf{R}$ is called *subriemannian distance* or Carnot-Caratheodory metric.

If an integral curve $\gamma : [a, b] \to M$ of D satisfies

$$d_g(\gamma(a), \gamma(b)) = \|\gamma\|_g,$$

γ is called *a minimizer.* Concerning minimizers, here we cite the following two theorems ([7], p.10):

Theorem 3 (Local existence) *If D is a nonholonomic distribution on a manifold M, then any point p of M is contained in a neighbourhood U such that every q in U can be connected to p by a minimimizer.*

Theorem 4 (Global Existence) *Let M be a connected manifold and D a nonholonomic smooth distribution on M, and suppose that M is complete relative to the subriemannian distance function. Then any two points of M can be joined by a minimizer.*

4 Hamiltonian formalism

If M is a manifold and $k \in \{0, 1, \cdots, \} \cup \{\infty\}$, we use $C^k(M)$ to denote the set of all real-valued functions on M that are class C^k, and $V^k(M)$ to denote the set of all vector fields of class C^k on M.

If N is a symplectic manifold with symplectic 2-form Ω, and $H \in C^1(N)$, we use $\overrightarrow{H}$ to denote *the Hamiltonian vector field* associated to H. $\overrightarrow{H}$ is the vector field V on N such that $\Omega(X, V) = \langle dH, X\rangle$ for every vector field X on N. If $H \in C^k(N)$ and $k \geq 1$, then vector field $\overrightarrow{H}$ is of class C^{k-1}. If $H, K \in C^1(N)$, then *the Poisson bracket* $\{H, K\}$ is the directional derivative of K in the direction of $\overrightarrow{H}$, i.e.,

$$\{H, K\} = \langle dK, \overrightarrow{H}\rangle = \Omega(\overrightarrow{H}, \overrightarrow{K}).$$

The we have the following formulas

$$\{H, KL\} = \{H, K\}L + \{H, L\}K,$$

$$\{H, \{K, L\}\} + \{K, \{L, H\}\} + \{L, \{H, K\}\} = 0,$$

and

$$\overrightarrow{HK} = \overrightarrow{H}K + K\overrightarrow{H}.$$

Note also the fact that the map $H \to \overrightarrow{H}$ is a Lie algebra homomorphism from $(C^\infty(N), \{,\})$ to $(V^\infty(N), [,])$.

The cotangent bundle T^*M of a manifold M has a natural symplectic structure determined by the 2-form $\Omega_M = d\omega_M$, where ω_M is the Liouville form given by

$$\omega_M(x, \lambda)(v) = \langle \lambda, d\pi_M^*(v)\rangle \quad \text{for } v \in T_{(x,\lambda)}(T^*M),$$

π_M^* being the projection $T^*M \to M$. Relative to a coordinate chart $T^*\kappa = (x^1, \ldots, x^n, \lambda_1, \ldots, \lambda_n)$ induced by a chart $\kappa = (x^1, \ldots, x^n)$ on M, we have the formulas

$$\omega_M = \sum_j \lambda_j dx^j,$$

$$\Omega_M = \sum_j d\lambda_j \wedge dx_j,$$

$$\vec{H} = \sum_j \left(\frac{\partial H}{\partial \lambda_j} \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial}{\partial \lambda_j} \right),$$

$$\{H, K\} = \sum_j \left(\frac{\partial H}{\partial \lambda_j} \frac{\partial K}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial K}{\partial \lambda_j} \right).$$

To each vector field X on M we associated the function $H_X : T^*M \to \mathbf{R}$ given by

$$H_X(q, \lambda) = \langle \lambda, X(q) \rangle \quad \text{for } \lambda \in T_q^*M.$$

Then H_X is of class C^k if and only if X is. Moreover,

$$d\pi_M^*(\vec{H}_X(x, \lambda)) = X(x) \quad \text{for all } (x, \lambda) \in T^*M$$

The identify

$$\{H_X, H_Y\} = H_{[X,Y]}$$

holds for $X, Y \in V^1(M)$, and therefore the map $X \to H_X$ is a Lie algebra homomorphism from $(V^\infty(M), [,])$ to $(C^\infty(N), \{,\})$.

If $X \in V^1(M)$ then the vector field $\vec{H}_X$ is called the *Hamiltonian lift* of X.

5 Normal extremals

Let (M, D, g) be a subriemannian manifold. If $(p, \lambda) \in T^*M$, then the restriction $\lambda|_{D_p}$ of λ to the subspace D_p of T_pM has well-defined norm, since D_p is an inner product space. We will use $\|\lambda\|_g$ to denote this norm. The function $E : T^*M \to \mathbf{R}$ given by

$$E(x, \lambda) = -\frac{1}{2}\|\lambda\|_g^2$$

is the energy function of the subriemannian structure (D, g).

Definition 3 *A normal biextremal of a subriemannian structure (D, g) is a curve $\Gamma : I \to T^*M$ such that*

(i) *Γ is an integral curve of the Hamiltonian vector field $\vec{E}$, namely*

$$\dot{\Gamma}(t) = \vec{E}_{\Gamma(t)}$$

(ii) *E does not vanish along* Γ.

A normal extremal is a curve in M *which is a projection of a normal biextremal.*

Theorem 5 *Let* (M, D, g) *be a subriemannian manifold. Then every normal extremal is locally length minimizing.*

This theorem is non-trivial, but the proof is similar to that of riemannian case. However, contrary to the riemannian case, the converse of the theorem does not hold. There appeared several papers asserting that every minimizer of a subriemannian manifold is a normal extremal. But Kupka [3] and Montgomery [5] proved that there exists a subriemannian manifold and a minimizer of the subriemannian manifold which is not a normal extremal. Such a minimizer is called an abnormal minimizer. In the following sections we will give a characterization of the abnormal minimizers.

6 Characteristic system

Let (N, Ω) be a symplectic manifold. For a submanifold S of N we define the characteristic system (bundle) $\mathrm{Ch}(S)$ of by

$$\mathrm{Ch}(S) = TS \cap (TS)^{\perp},$$

that is, the fibre $\mathrm{Ch}(S)_s$ on $s \in S$ is given by

$$\mathrm{Ch}(S)_s = T_sS \cap (T_sS)^{\perp},$$

where

$$(T_sS)^{\perp} = \{v \in T_sN; \Omega(v, u) = 0 \text{ for all } u \in T_sS\}.$$

Let $F_1, \ldots, F_r$ be local defining equations of S, say, defined on a neighbourhood U of $s_0 \in S$ such that $(dF_1)_s, \ldots, (dF_r)_s$ are linearly independent for $s \in U$ and

$$U \cap S = \{F_1 = \cdots = F_r = 0\}.$$

From the very definition of Hamiltonian vector field we see immediately that $\{(\overrightarrow{F_1})_s, \ldots, (\overrightarrow{F_r})_s\}$ forms a basis of $(T_sS)^\perp$ for $s \in U$. Hence we have

$$\mathrm{Ch}(S)_s = T_sS \cap \langle (\overrightarrow{F_1})_s, \ldots, (\overrightarrow{F_r})_s \rangle.$$

Let $\Omega_S = \iota_S^* \Omega$, where $\iota_S : S \to N$ is the canonical inclusion, and let:

$$\mathrm{Null}_s(\Omega_S) = \{v \in T_sS; \Omega_S(v, u) = 0 \text{ for all } u \in T_sS\}.$$

Then it is clear that

$$\mathrm{Ch}(S)_s = \mathrm{Null}_s(\Omega_S).$$

We then have:

Proposition 1 *For a submanifold S of a symplectic manifold (N, Ω), the characteristic system* $\mathrm{Ch}(S) = \bigcap_{s \in S} \mathrm{Ch}(S)_s \subset TS$ *is given by:*

$$\begin{aligned} \mathrm{Ch}(S)_s &= T_sS \cap (T_sS)^\perp \\ &= (T_sS) \cap \langle (\overrightarrow{F_1})_s, \ldots, (\overrightarrow{F_r})_s \rangle \\ &= \mathrm{Null}_s(\Omega_S) \end{aligned}$$

If $\dim \mathrm{Ch}(S)_s$ *is constant, then* $\mathrm{Ch}(S)$ *is a completely integrable subbundle of TS.*

The last assertion of the proposition follows from the exactness of the symplectic form.

7 Abnormal extremals

Let (M, D, g) be a subriemannian manifold. We denote by $D^\perp$ the annihilator bundle of D and by $\mathrm{Ch}(D^\perp)$ its characteristic system.

Definition 4 *An abnormal biextremal of (M, D, g) is an curve $\Gamma : I \to D^\perp \backslash \{O\}$ (O denoting the zero section) such that $\dot{\Gamma}(t) \in \mathrm{Ch}(D^\perp)_{\Gamma(t)}$ for almost all $t \in I$. An abnormal extremal of (M, D, g) is a curve in M which is a projection of an abnormal biextremal.*

It should be remark that the above definition does not depend on the metric g but depends only on (M, D).

If $\{X_1, \dots, X_r\}$ is a local basis of D defined on $U \subset M$, then $H_{X_1}, \dots, H_{X_r}$ give defining equations of $D^\perp$ on $\pi_M^* U$. Hence by Proposition 5 , we have

$$\mathrm{Ch}(D^\perp)_z = T_z D^\perp \cap \langle (\overrightarrow{H_{X_1}})_z, \dots, (\overrightarrow{H_{X_r}})_z \rangle.$$

Therefore a curve $\Gamma : I \to (\pi_M^*)^{-1} U \backslash \{O\}$ is an abnormal biextremal of (M, D) if and only if

$$\begin{cases} \text{(i) } H_{X_i}(\Gamma(t)) = 0 & \text{for all } t \in I \text{ and } i = 1, \dots, r \\ \text{(ii) } \dot{\Gamma}(t) \in \langle (\overrightarrow{H_{X_1}})_{\Gamma(t)}, \dots, (\overrightarrow{H_{X_r}})_{\Gamma(t)} \rangle & \text{for almost all } t \in I \end{cases}$$

By using the Pontryagin Maximam Principle on Control system, it is shown that the following theorem holds (see [4], p.81, Appendix B).

Theorem 6 *Let (M, D, g) be a subriemannian manifold, and let $\gamma : [a, b] \to M$ be length-minimizer parametrized by arc-length. Then γ is a normal extremal or an abnormal extremal.*

8 Extremals on the standard Cartan distribution

As was shown by Cartan[1], a generic Pfaff system defined by three Pfaff equations in the space of five variables, that is, a tangent distribution D of rank 2 on $\mathbf{R}^5$ enjoys interesting properties: Its automorphism group makes a Lie group of dimension not greater than 14, and if the maximal dimension is attained, then the automorphism group is locally isomorphic to the exceptional simple Lie group G_2 and the tangent distribution D is locally isomorphic to the standard Cartan distribution defined as follows: Let $(x^1, x^2, x^3, x^4, x^5)$ be the standard coordinates of $\mathbf{R}^5$ and let the vector fields $X_1, \dots, X_5$ be given by:

$$X_1 = \frac{\partial}{\partial x^1} - \frac{1}{2} x^2 \frac{\partial}{\partial x^3} - (x^3 - \frac{1}{2} x^1 x^2) \frac{\partial}{\partial x^4}$$

$$X_2 = \frac{\partial}{\partial x^2} + \frac{1}{2}x^1\frac{\partial}{\partial x^3} - (x^3 + \frac{1}{2}x^1x^2)\frac{\partial}{\partial x^5}$$

$$X_3 = \frac{\partial}{\partial x^3},\ X_4 = \frac{\partial}{\partial x^4},\ X_5 = \frac{\partial}{\partial x^5}.$$

These vector fields satisfy the following bracket relations:

$$\begin{cases} [X_1, X_2] = X_3 \\ [X_1, X_3] = X_4 \\ [X_2, X_3] = X_5 \\ \text{The others are trivial} \end{cases}$$

The dual basis $\omega^1, \ldots, \omega^5$ of $X_1, \ldots, X_5$ is given by:

$$\begin{cases} \omega^1 = dx^1 \\ \omega^2 = dx^2 \\ \omega^3 = dx^3 - \frac{1}{2}(x^1dx^2 - x^2dx^1) \\ \omega^4 = dx^4 + (x^3 - \frac{1}{2}x^1x^2)dx^1 \\ \omega^5 = dx^5 + (x^3 + \frac{1}{2}x^1x^2)dx^2. \end{cases}$$

Then we have the following the structure equations:

$$\begin{cases} d\omega^1 = 0 \\ d\omega^2 = 0 \\ d\omega^3 + \omega^1 \wedge \omega^2 = 0 \\ d\omega^4 + \omega^1 \wedge \omega^3 = 0 \\ d\omega^5 + \omega^2 \wedge \omega^3 = 0. \end{cases}$$

Let us take D to be the tangent distribution spanned by X_1 and X_2, that is,

$$\Gamma(D) = \langle X_1, X_2 \rangle = \{\omega^3 = \omega^4 = \omega^5 = 0\}.$$

Then, choosing a subriemannian metric g on D so that $\{X_1(p), X_2(p)\}$ forms an orthonormal basis of D_p, we consider the subriemannian manifold $(\mathbf{R}^5, D, g)$.

Let us determine the normal extremals and the abnormal extremals of this subriemannian manifold.

If $(x^1, x^2, x^3, x^4, x^5, p_1, p_2, p_3, p_4, p_5)$ are the local coordinates in $T^*\mathbf{R}^5$, the energy function E of (D, g) is given by

$$\begin{aligned} E &= -\frac{1}{2}[\{p_1 - \frac{1}{2}x^2 p_3 - (x^3 - \frac{1}{2}x^1x^2)p_4\}^2 \\ &+ \{p_2 + \frac{1}{2}x^1 p_3 - (x^3 + \frac{1}{2}x^1x^2)p_5\}^2]. \end{aligned}$$

Then the Hamiltonian vector field $\overrightarrow{E}$ is given by

$$\begin{aligned} \overrightarrow{E} &= -A\frac{\partial}{\partial x^1} - B\frac{\partial}{\partial x^2} + (\frac{1}{2}x^2 A - \frac{1}{2}x^1 B)\frac{\partial}{\partial x^3} + (x^3 - \frac{1}{2}x^1x^2)A\frac{\partial}{\partial x^4} \\ &+ (x^3 + \frac{1}{2}x^1x^2)B\frac{\partial}{\partial x^5} + \{\frac{1}{2}x^2 p_4 A + (\frac{1}{2}p_3 - \frac{1}{2}x^2 p_5)B\}\frac{\partial}{\partial p_1} \\ &+ \{(\frac{1}{2}x^1 p_4 - \frac{1}{2}p_3)A - \frac{1}{2}x^1 p_5 B\}\frac{\partial}{\partial p_2} + (-p_4 A - p_5 B)\frac{\partial}{\partial p_3}, \end{aligned}$$

where

$$A = p_1 - \frac{1}{2}x^2 p_3 - (x^3 - \frac{1}{2}x^1x^2)p_4 \tag{1}$$

$$B = p_2 + \frac{1}{2}x^1 p_3 - (x^3 + \frac{1}{2}x^1x^2)p_5. \tag{2}$$

Then we see that a normal biextremal of (D, g) satisfies

$$\dot{x^1} = -A \tag{3}$$

$$\dot{x^2} = -B \tag{4}$$

$$\dot{x^3} = \frac{1}{2}x^2 A - \frac{1}{2}x^1 B \tag{5}$$

$$\dot{x^4} = (x^3 - \frac{1}{2}x^1x^2)A \tag{6}$$

$$\dot{x^5} = (x^3 + \frac{1}{2}x^1x^2)B \tag{7}$$

$$\dot{p_1} = \frac{1}{2}x^2 p_4 A + (\frac{1}{2}p_3 - \frac{1}{2}x^2 p_5)B \tag{8}$$

$$\dot{p_2} = (-\frac{1}{2}p_3 + \frac{1}{2}x^1 p_4)A - \frac{1}{2}x^1 p_5 B \tag{9}$$

$$\begin{aligned}
&(10) \qquad & \dot{p_3} &= -p_4 A - p_5 B \\
&(11) \qquad & \dot{p_4} &= 0 \\
&(12) \qquad & \dot{p_5} &= 0
\end{aligned}$$

Differentiating the equation (1), and substituting (3),(4),(5),(8),(10),(11) into it, we have

$$(13) \qquad \ddot{x^1} = p_3 \dot{x^2}.$$

Similarly differentiating (2), and substituting (3),(4),(5),(9),(10),(12) into it, we have

$$(14) \qquad \ddot{x^2} = -p_3 \dot{x^1}.$$

On the other hand, p_4, p_5 are constant by (11), (12). Then integrating (10), we have:

$$(15) \qquad p_3 = p_4 x^1 + p_5 x^2 + C,$$

where C is a constant. Therefore the second order differential equations with respect to x^1 and x^2 are given in the formulae (13), (14) and (15). These equations for (x^1, x^2) can be written in the following form:

$$\begin{pmatrix} \ddot{x^1} \\ \ddot{x^2} \end{pmatrix} = p_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{x^1} \\ \dot{x^2} \end{pmatrix},$$

where p_3 is a linear function given by (15). Since the acceleration vector $\begin{pmatrix} \ddot{x^1} \\ \ddot{x^2} \end{pmatrix}$ is obtained by the rotation of $\frac{\pi}{2}$ of the velocity vector $\begin{pmatrix} \dot{x^1} \\ \dot{x^2} \end{pmatrix}$ with the scalar multiplication of p_3, this equation represents the equation of motion of an electron moving in a plane under a magnetic field whose direction is perpendicular to the plane and whose magnitude is given by the linear function $p_3 = p_4 x^1 + p_5 x^2 + C$. By a change of local coordinates:

$$\begin{cases} x = p_4 x^1 + p_5 x^2 + C \\ y = p_5 x^1 - p_4 x^2, \end{cases}$$

we have

$$\begin{cases} \ddot{x} = -x\dot{y} \\ \ddot{y} = x\dot{x}. \end{cases}$$

Then we also have

$$\dot{y} = \frac{1}{2}x^2 + k,$$

where k is a constant. By substituting this equation into $\ddot{x} = -x\dot{y}$, we have

$$\ddot{x} = -\frac{1}{2}x^3 - kx.$$

So we have

$$\frac{1}{2}\{\dot{x}\}^2 = -\frac{1}{8}x^4 - \frac{1}{2}kx^2,$$

and

$$\dot{x} = \pm\sqrt{-\frac{1}{4}x^4 - kx^2}.$$

Since

$$-\frac{1}{4}x^2(x^2 + 4k) \geq 0$$

we see $k \leq 0$. If $k = 0$, we have $x = \dot{x} = 0$. Therefore x^1 and x^2 run along the line

$$p_4x^1 + p_5x^2 + C = 0.$$

If $k < 0$,

$$p_4x^1 + p_5x^2 + C$$

moves periodically between $-2\sqrt{-k}$ and $2\sqrt{-k}$.

Now we will give the differential equations that an abnormal extremal $\Gamma : I(= [\alpha, \beta]) \to T^*\mathbf{R}^5\backslash\{O\}$ of D must satisfy. If we choose the local coordinates $(x^1, x^2, x^3, x^4, x^5, p_1, p_2, p_3, p_4, p_5)$ in $T^*\mathbf{R}^5$, the Hamiltonian function H_{X_1} and H_{X_2} can be expressed as

$$\begin{aligned} H_{X_1} &= p_1 - \frac{1}{2}x^2p_3 - (x^3 - \frac{1}{2}x^1x^2)p_4, \\ H_{X_2} &= p_2 + \frac{1}{2}x^1p_3 - (x^3 + \frac{1}{2}x^1x^2)p_5. \end{aligned}$$

By the definition of an abnormal extremal of (M,D), H_{X_1} and H_{X_2} should vanish along the curve Γ. Hence we have:

$$p_1 - \frac{1}{2}x^2p_3 - (x^3 - \frac{1}{2}x^1x^2)p_4 = 0,$$

$$p_2 + \frac{1}{2}x^1p_3 - (x^3 + \frac{1}{2}x^1x^2)p_5 = 0.$$

Now the Hamiltonian lift of $\vec{H}_{X_1}$ of X_1 and $\vec{H}_{X_2}$ of X_2 can be expressed as:

$$\begin{aligned}\vec{H}_{X_1} &= \frac{\partial}{\partial x^1} - \frac{1}{2}x^2\frac{\partial}{\partial x^3} - (x^3 - \frac{1}{2}x^1x^2)\frac{\partial}{\partial x^4} \\ &- \frac{1}{2}x^2p_4\frac{\partial}{\partial p_1} - (\frac{1}{2}x^1p_4 - \frac{1}{2}p_3)\frac{\partial}{\partial p_2} + p_4\frac{\partial}{\partial p_3},\end{aligned}$$

$$\begin{aligned}\vec{H}_{X_2} &= \frac{\partial}{\partial x^2} + \frac{1}{2}x^1\frac{\partial}{\partial x^3} - (x^3 + \frac{1}{2}x^1x^2)\frac{\partial}{\partial x^5} \\ &- (\frac{1}{2}p_3 - \frac{1}{2}x^2p_5)\frac{\partial}{\partial p_1} + \frac{1}{2}x^1p_5\frac{\partial}{\partial p_2} + p_5\frac{\partial}{\partial p_3}.\end{aligned}$$

Then the following conditions must be satisfied:

$$\dot{\Gamma}(t) = a^1(t)(\vec{H}_{X_1})_{\Gamma(t)} + a^2(t)(\vec{H}_{X_2})_{\Gamma(t)},$$

where $a^1(t)$ and $a^2(t)$ are some functions on I.

Therefore if $\Gamma(t) = (x(t), p(t))$ is an abnormal extremal of (M,D) then

$(x(t), p(t))$ satisfies the following equations,

$$
\begin{aligned}
&(16) && \dot{x}^1 = a^1\\
&(17) && \dot{x}^2 = a^2\\
&(18) && \dot{x}^3 = -\frac{1}{2}a^1x^2 + \frac{1}{2}a^2x^1\\
&(19) && \dot{x}^4 = -a^1(x^3 - \frac{1}{2}x^1x^2)\\
&(20) && \dot{x}^5 = -a^2(x^3 + \frac{1}{2}x^1x^2)\\
&(21) && \dot{p}_1 = -\frac{1}{2}a^1x^2p_4 - a^2(\frac{1}{2}p_3 - \frac{1}{2}x^2p_5)\\
&(22) && \dot{p}_2 = a^1(-\frac{1}{2}x^1p_4 + \frac{1}{2}p_3) + \frac{1}{2}a^2x^1p_5\\
&(23) && \dot{p}_3 = a^1p_4 + a^2p_5\\
&(24) && \dot{p}_4 = 0\\
&(25) && \dot{p}_5 = 0
\end{aligned}
$$

$$
\begin{aligned}
&(26) && p_1 - \frac{1}{2}x^2p_3 - (x^3 - \frac{1}{2}x^1x^2)p_4 = 0\\
&(27) && p_2 + \frac{1}{2}x^1p_3 - (x^3 + \frac{1}{2}x^1x^2)p_5 = 0
\end{aligned}
$$

Differentiating the equation (27), and substituting (16), (17), (18), (22), (23), (25) into it, we have

$$p_3\dot{x}^1 = 0.$$

Similarly differentiating (26), and substituting (16), (17), (18), (21), (23), (24) into it, we have

$$p_3\dot{x}^2 = 0.$$

From these equations on account of (16), (17), it follows that if $(a^1(t_0), a^2(t_0)) \neq 0$ at t_0, then $p_3 = 0$ around t_0. Therefore we may assume $p_3 \equiv 0$, and we have

$$\dot{p}_3 = (a^1(t)p_4 + a^2(t)p_5) = 0.$$

Hence we have

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \varphi \begin{pmatrix} p_5 \\ -p_4 \end{pmatrix},$$

where φ is a function along the abnormal biextremal. If we set

$$\psi = \int_{\alpha}^{t} \varphi(s)ds,$$

we have

$$\begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} = \psi(t) \begin{pmatrix} p_5 \\ -p_4 \end{pmatrix} + \begin{pmatrix} q^1 \\ q^2 \end{pmatrix},$$

where $q^1 = x^1(\alpha)$, $q^2 = x^2(\alpha)$. Then x^3, x^4, x^5 are obtained by integrating (18), (19), (20). Thus the lines in (x^1, x^2)-space give rise to the abnormal extremals.

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