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Spacelike hypersurfaces and submanifolds in de Sitter space

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1 Introduction

This note is the announcement of [9]. We also give some related remarks.

De Sitter space is defined as a pseudo-sphere in Minkowski space, and there is a pseudo-Riemannian metric on de Sitter space. Submanifolds on de Sitter space are separated by spacelike, timelike and lightlike parts. We studied the differential geometry of spacelike parts of submanifolds in de Sitter space.

In [7] we studied the differential geometry of spacelike hypersurfaces by using an analogous tool of [3], which is called a lightcone Gauss image. Izumiya, Pei, Romero Fuster and Takahashi [6] introduced the notion of canal hypersurfaces and horospherical hypersurfaces to study the differential geometry of submanifolds in the hyperbolic space. In [9] we use analogous notions of [6], which is called a spacelike canal hypersurfaces CM_θ and horospherical hypersurfaces, to study the case of spacelike submanifolds M of codimension $r \geq 2$ in de Sitter space by applying the theory of singularity. In this note we mainly argue the relations with spacelike canal hypersurfaces and spacelike submanifolds. We observe that lightcone parabolic points of CM_θ correspond to horospherical points of M, and the lightcone Gauss images and horospherical hypersurfaces have singularities.

In §2 we review the differential geometry of spacelike submanifolds. In §3 we construct spacelike canal hypersurfaces from the timelike parallel unit orthonormal sections. In §4 we define the notion of horospherical hypersurfaces of spacelike submanifolds, and argue the geometric relations between spacelike submanifolds and spacelike canal hypersurfaces. In §5 we apply the theory of contacts of submanifolds to our situation. In §6 we pick up the results on [9].

2 Spacelike submanifolds in de Sitter space

In this section we review the differential geometry of spacelike submanifolds of codimension at least two in de Sitter space.

Let $\mathbb{R}^{n+1} = \{\mathbf{x} = (x_0, \ldots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, \ldots, n)\}$ be an $(n+1)$-dimensional vector space. For any vectors $\mathbf{x} = (x_0, \ldots, x_n)$, $\mathbf{y} = (y_0, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo*

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scalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ a *Minkowski* $(n+1)$*-space* and write $\mathbb{R}_1^{n+1}$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$. We say that a vector $\mathbf{x} \in \mathbb{R}_1^{n+1} \setminus \{\mathbf{0}\}$ is *spacelike*, *timelike* or *lightlike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}_1^{n+1}$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. For a vector $\mathbf{v} \in \mathbb{R}_1^{n+1} \setminus \{\mathbf{0}\}$ and a real number c, we define a *hyperplane with pseudo normal* $\mathbf{v}$ in the Minkowski space by $\mathrm{HP}(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}$. We say that a hyperplane $HP(\mathbf{v}, c)$ is spacelike, timelike or lightlike if the vector $\mathbf{v}$ is timelike, spacelike or lightlike.

We respectively define *hyperbolic* n*-space* and *de Sitter* n*-space* by

$$\begin{aligned} H_{\pm}^n(-1) &= \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, sgn(x_0) = \pm 1\}, \\ S_1^n &= \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}, \end{aligned}$$

and we write $H^n(-1) = H_+^n(-1) \cup H_-^n(-1)$. For any $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}_1^{n+1}$, we define a vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_n$ with the property $\langle \mathbf{x}, \mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_n \rangle = \det(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n)$, so that $\mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_n$ is pseudo-orthogonal to any $\mathbf{x}_i$ for $i = 1, \ldots, n$. We also define *future* (resp. *past*) *lightcone* at the origin by

$$\begin{aligned} LC_+^* &= \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0,\ x_0 > 0\ \}, \\ LC_-^* &= \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0,\ x_0 < 0\ \}, \end{aligned}$$

and we write $LC^* = LC_+^* \cap LC_-^*$.

We now define spacelike submanifolds of codimension at least two in de Sitter space, and review the differential geometry of them. Let r be an integer at least two and $U \subset \mathbb{R}^{n-r}$ be an open subset. We say that an embedding map $\mathbf{X} : U \longrightarrow S_1^n$ is *spacelike* if every non zero vector generated by $\{\mathbf{X}_{u_i}(\mathbf{u})\}_{i=1}^{n-r}$ is spacelike, where $\mathbf{u} \in U$ and $\mathbf{X}_{u_i} = \partial \mathbf{X}/\partial u_i$. We identify $M = \mathbf{X}(U)$ with U through the embedding $\mathbf{X}$ and call M a *spacelike submanifold of codimension* r in de Sitter space.

Let $p = \mathbf{X}(\mathbf{u})$, we write T_pM as a tangent space of $\mathbf{X}$ at p, and N_pM as a pseudo-normal space of $\mathbf{X}$ at p in $\mathbb{R}_1^{n+1}$. We define $N_p^*(M) = N_pM \cap T_pS_1^n$. Let $\mathbf{n} : U \longrightarrow H^n(-1)$ be a timelike unit normal vector field on M with the property $\mathbf{n}(\mathbf{u}) \in N_p^*M$ for all $p = \mathbf{X}(\mathbf{u})$. We say that the timelike unit normal vector field $\mathbf{n}$ is *parallel on* M if $\mathrm{Im}(d_u\mathbf{n}) \subset T_pM$ for all $\mathbf{u} \in U$. We call the linear transformation $S_p(\mathbf{n}) = -(\mathrm{id}_{T_pM} + d_p\mathbf{n})$ a *horospherical* $\mathbf{n}$*-shape operator* of M at $p = \mathbf{X}(\mathbf{u})$. In [9] we also defined an $\mathbf{n}$*-shape operator* $A_p(\mathbf{n}) = -d_p\mathbf{n}^T$, but in this note we omit it.

We denote eigenvalues of $S_p(\mathbf{n})$ and $\det S_p(\mathbf{n})$ by $\bar{\kappa}_p(\mathbf{n})$ and $K_h(\mathbf{n})(\mathbf{u})$, which we respectively call *horospherical principal curvatures* and a *horospherical Gauss-Kronecker curvature* with respect to $\mathbf{n}$. We say that a point $p_0 = \mathbf{X}(\mathbf{u}_0)$ is $\mathbf{n}$*-umbilic* if $S_{p_0}(\mathbf{n}) = \bar{\kappa}_{p_0}(\mathbf{n})\mathrm{id}_{T_{p_0}M}$. We also say that the spacelike submanifold M is *totally* $\mathbf{n}$*-umbilic* if every point on M is $\mathbf{n}$-umbilic.

We say that $HP(\mathbf{v}, c) \cap S_1^n$ is an *elliptic* hyperquadric (resp. a *hyperbolic* hyperquadric) if $HP(\mathbf{v}, c)$ is spacelike (resp. timelike). We say that $HP(\mathbf{v}, c) \cap S_1^n$ is a *de Sitter hyperhorosphere* if $c \neq 0$ and $HP(\mathbf{v}, c)$ is lightlike. We have the following result for the totally umbilic spacelike hypersurfaces, which is analogous to ([6], Proposition 3.1).

Proposition 2.1. ([9]) Let $\mathbf{X} : U \longrightarrow S_1^n$ be a spacelike submanifold of codimension $r \geq 2$ and $\mathbf{n}$ be a timelike parallel unit normal vector field on $M = \mathbf{X}(U)$. Suppose that $M = \mathbf{X}(U)$ is totally $\mathbf{n}$-umbilic, then the horospherical $\mathbf{n}$-principal curvatures are constant $\bar{\kappa}(\mathbf{n})$, and M is a part of a hyperquadric $HP(\mathbf{v}, c) \cap S_1^n$ for some $\mathbf{v} \in \mathbb{R}_1^{n+1}$ and $c \in \mathbb{R}$. Under this condition we have following cases:

(1) If $1 < |\bar{\kappa}(\mathbf{n}) + 1|$ then M is a part of a hyperbolic hyperquadric $HP(\mathbf{v}, +1)$.
(2) If $0 < |\bar{\kappa}(\mathbf{n}) + 1| < 1$ then M is a part of an elliptic hyperquadric $HP(\mathbf{v}, +1)$.
(3) If $\bar{\kappa}(\mathbf{n}) = -1$ then M is a part of an elliptic hyperquadric $HP(\mathbf{v}, 0)$.
(4) If $\bar{\kappa}(\mathbf{n}) = 0$ then M is a part of a de Sitter hyperhorosphere $HP(\mathbf{v}, +1)$.

We remark that the case $\bar{\kappa}(\mathbf{n}) = -2$ is not occurred.

We induce a Riemannian metric (the *horospherical first fundamental form*) on M by $ds^2 = \sum_{i,j=1}^{n-r} g_{ij} du_i du_j$ on $M = \mathbf{X}(U)$, where $g_{ij} = \langle \mathbf{X}_{u_i}, \mathbf{X}_{u_j} \rangle$. Let $\mathbf{n}$ be a timelike parallel normal vector field, we define the *horospherical second fundamental invariant* with respect to $\mathbf{n}$ by $\bar{h}_{ij}(\mathbf{n}) = -\langle \mathbf{X}_{u_i} + \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle$. Then we have the following Weingarten type formula

$$(\mathbf{X} + \mathbf{n})_{u_i} = -\sum_{k=1}^{n-r} \bar{h}_i^j(\mathbf{n}) \mathbf{X}_{u_j},$$

where $(\bar{h}_i^j(\mathbf{n}))_{ij} = (\bar{h}_{ik}(\mathbf{n}))_{ik} (g^{kj})_{kj}$ and $(g^{kj}) = (g_{kj})^{-1}$. Therefore, the horospherical Gauss-Kronecker curvature with respect to $\mathbf{n}$ is given by

$$K_h(\mathbf{n}) = \det(\bar{h}_{ik}(\mathbf{n})) / \det(g_{kj}).$$

Since the coefficients of the second fundamental invariant with respect to $\mathbf{n}$ is expressed by $\langle \mathbf{X} + \mathbf{n}, \mathbf{X}_{u_i u_j} \rangle$. So that we have a following remark.

Remark 2.2. Let $\mathbf{n}$ and $\mathbf{n}'$ be timelike parallel unit normal vector fields on M. If $\mathbf{n}_0 = \mathbf{n}'(\mathbf{u}_0) = \mathbf{n}(\mathbf{u}_0)$, then $\bar{h}_{ik}(\mathbf{n})(\mathbf{u}_0) = \bar{h}_{ik}(\mathbf{n}')(\mathbf{u}_0)$.

Let $p_0 = \mathbf{X}(\mathbf{u}_0)$ and $\mathbf{n}_0$ be a timelike unit normal vector at p_0 on M. We say that a point $p_0 = \mathbf{X}(\mathbf{u}_0)$ is an $\mathbf{n}_0$-*parabolic point* (resp. $\mathbf{n}_0$-*umbilic point*) of M if $K_h(\mathbf{n})(\mathbf{u}_0) = 0$ ($S_{p_0}(\mathbf{n}) = \bar{\kappa}_{p_0}(\mathbf{n}) \mathrm{id}_{T_{p_0}M}$) for some timelike parallel unit normal vector field $\mathbf{n}$ with $\mathbf{n}(\mathbf{u}_0) = \mathbf{n}_0$. We also say that p_0 is an $\mathbf{n}_0$-*horospherical point* if $S_{p_0}(\mathbf{n}) = O_{T_pM}$.

3 Spacelike canal hypersurfaces

In this section we construct spacelike canal hypersurfaces of spacelike submanifolds in de Sitter space and argue the differential geometry of them. In [7] we have studied the differential geometry of spacelike hypersurfaces in de Sitter space.

Let $r \geq 2$ and $\mathbf{X}$ be a spacelike submanifold of codimension r in de Sitter space. We assume that there are unit orthonormal sections $\mathbf{n}_0, \ldots, \mathbf{n}_{r-1}$ on M, where $\mathbf{n}_0(\mathbf{u})$ is a timelike unit normal vector and $\mathbf{n}_i(\mathbf{u})$ for $i = 1, \dots, r-1$ are spacelike unit normal vectors. We define a map $\mathbf{e} : U \times H^{r-1}(-1) \longrightarrow H^n(-1)$ by

$$\mathbf{e}(\mathbf{u}, \bar{\mu}) = \mu_0 \mathbf{n}_0(\mathbf{u}) + \sum_{i=1}^{r-1} \mu_i \mathbf{n}_i(\mathbf{u}),$$

where $\bar{\mu} = (\mu_0, \dots, \mu_{r-1})$. Let $\theta > 0$, we define a *spacelike canal hypersurface of M* by

$$\bar{\mathbf{X}}_\theta : U \times H^{r-1}(-1) \longrightarrow S^n_1, \quad \bar{\mathbf{X}}_\theta(\mathbf{u}, \bar{\mu}) = \cosh\theta \mathbf{X}(\mathbf{u}) + \sinh\theta \mathbf{e}(\mathbf{u}, \bar{\mu}),$$

We now observe the condition that the spacelike canal hypersurfaces degenerates. Let $(\mu_1, \dots, \mu_{r-1})$ be a coordinate of $H^{r-1}(-1)$ where $\bar{\mu} = (\mu_0, \dots, \mu_{r-1})$. The derivatives of $\bar{\mathbf{X}}_\theta$ at $(\mathbf{u}, \bar{\mu})$ is

$$\begin{aligned} (\bar{\mathbf{X}}_\theta)_{u_i}(\mathbf{u}, \bar{\mu}) &= \cosh\theta \mathbf{X}_{u_i}(\mathbf{u}) + \sinh\theta \mathbf{e}_{u_i}(\mathbf{u}, \bar{\mu}), \\ (\bar{\mathbf{X}}_\theta)_{\mu_j}(\mathbf{u}, \bar{\mu}) &= \frac{\mu_j}{\mu_0}\mathbf{n}_0(\mathbf{u}) + \mathbf{n}_j(\mathbf{u}), \end{aligned}$$

for $i = 1, \dots, n-r$ and $j = 1, \dots, r-1$. Since $\{\mathbf{n}_j(\mathbf{u})\}_{j=1}^{r-1}$ are linearly independent, so that $\bar{\mathbf{X}}_\theta$ is degenerate at $\mathbf{e}(\mathbf{u}, \bar{\mu})$ if and only if a linear transformation on T_pM

$$d_{\mathbf{u}}\bar{\mathbf{X}}_\theta = \cosh\theta \, \mathrm{id}_{T_pM} + \sinh\theta S_p(\mathbf{e}(\mathbf{u}, \bar{\mu})),$$

is degenerate, where $S_p(\mathbf{e}(\mathbf{u}, \bar{\mu}))$ is the horospherical $\mathbf{e}(\mathbf{u}, \bar{\mu})$-shape operator at $p = \mathbf{X}(\mathbf{u})$ of M. Therefore we have the following proposition.

Proposition 3.1. Let M be a spacelike submanifold of codimension $r \geq 2$ and $\bar{\mathbf{X}}_\theta$ is a spacelike canal hypersurface of M. Then a point $(\mathbf{u}, \bar{\mu})$ is the singular point of $\bar{\mathbf{X}}_\theta$ if and only if $-\cosh\theta/\sinh\theta$ is an eigenvalue of $S_p(\mathbf{e}(\mathbf{u}, \bar{\mu}))$.

From now on, we assume that $\theta > 0$ is sufficiently small and V is an open subset of $U \times H^{r-1}_{\pm}(-1)$ such that $\bar{\mathbf{X}}_\theta$ is an embedding map on V. We write the image of spacelike canal hypersurfaces as $CM_\theta = \bar{\mathbf{X}}(V)$. According to [7], a timelike unit normal vector field $\bar{\mathbf{e}} : V \longrightarrow H^n(-1)$ is given by

$$\bar{\mathbf{e}}(\mathbf{u}, \bar{\mu}) = \sinh\theta \mathbf{X}(\mathbf{u}) + \cosh\theta \mathbf{e}(\mathbf{u}, \bar{\mu}).$$

Therefore a *positive lightcone Gauss image* $\mathbb{L}_{CM_\theta} : V \longrightarrow LC^*$ is defined by

$$\mathbb{L}_{CM_\theta}(\mathbf{u}, \bar{\mu}) = \bar{\mathbf{X}}_\theta(\mathbf{u}) + \bar{\mathbf{e}}(\mathbf{u}, \bar{\mu}) = (\cosh\theta + \sinh\theta)(\mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})).$$

We may identify V as CM_θ, and the differential map $d\mathbb{L}(\mathbf{u}, \bar{\mu})$ is a linear transformation on $T_{\bar{p}}CM_\theta$, where $\bar{p} = \bar{\mathbf{X}}_\theta(\mathbf{u}, \bar{\mu})$. We call $\bar{S}_{\bar{p}} = -d\mathbb{L}(\mathbf{u}, \bar{\mu})$ a *lightcone shape operator* of CM_θ at $\bar{p}$. The *lightcone Gauss-Kronecker curvature* of CM_θ is defined to be the determinant of the lightcone shape operator $\bar{S}_{\bar{p}}$, and we denote by $K_\ell(\mathbf{u}, \bar{\mu})$. We say that $\bar{p} = \bar{\mathbf{X}}_\theta(\mathbf{u}, \bar{\mu})$ is a *lightcone parabolic point* of CM_θ if $K_\ell(\mathbf{u}, \bar{\mu}) = 0$.

We also define a *lightcone height function* $\bar{H} : V \times LC^* \longrightarrow \mathbb{R}$ of the spacelike hypersurface $\bar{\mathbf{X}}_\theta$ by

$$\bar{H}((\mathbf{u},\bar{\mu}),\mathbf{v}) = \langle \bar{\mathbf{X}}_\theta(\mathbf{u},\bar{\mu}),\mathbf{v}\rangle - 1.$$

We denote $\bar{h}_{\mathbf{v}}(\mathbf{u},\bar{\mu}) = \bar{H}((\mathbf{u},\bar{\mu}),\mathbf{v})$ for any $\mathbf{v} \in LC^*$. We have showed the following relations between the lightcone height functions and lightcone Gauss images. (See Proposition 3.1 and 3.2 in [7])

(1) $H((\mathbf{u},\bar{\mu}),\mathbf{v}) = 0$ and $\partial H((\mathbf{u},\bar{\mu}),\mathbf{v})/\partial u_i = \partial H((\mathbf{u},\bar{\mu}),\mathbf{v})/\partial \mu_i = 0$ (for $i = 1,\dots,n-r$ and $j = 1,\dots,r-1$) if and only if $\mathbf{v} = \mathbb{L}(\mathbf{u},\bar{\mu})$.
(2) If $\mathbf{v} = \mathbb{L}(\mathbf{u},\bar{\mu})$, then $\bar{p} = \mathbf{X}(\mathbf{u},\bar{\mu})$ is a lightcone parabolic point if and only if the Hessian matrix of $\bar{h}_{\mathbf{v}}$ degenerates at $(\mathbf{u},\bar{\mu})$, that is $\det \mathrm{Hess}\, \bar{h}_{\mathbf{v}}(\mathbf{u},\bar{\mu}) = 0$.

In [7] we also applied the theory of Legendrian singularities to the differential geometry of spacelike hypersurfaces in de Sitter space, which is an analogous argument to [3]. For any spacelike hypersurfaces in de Sitter space, the corresponding lightcone height function is a Morse family of hypersurfaces. The discriminant set of the lightcone height function is the image of lightcone Gauss image. We can construct the Legendrian immersion germ whose generating family is the lightcone height function.

4 Horospherical points and lightcone parabolic points

In this section we discuss relations between spacelike canal hypersurfaces and spacelike submanifolds in de Sitter space.

Let $\mathbf{X}$ be a spacelike submanifold of codimension $r \geq 2$ and $\mathbf{n}_0,\dots,\mathbf{n}_{r-1}$ be unit orthonormal sections as above. We define the family of functions $H : U \times LC^* \longrightarrow \mathbb{R}$ by

$$H(\mathbf{u},\mathbf{v}) = \langle \mathbf{X}(\mathbf{u}),\mathbf{v}\rangle - 1,$$

and we call H a *horospherical height function on* M. For $\mathbf{v}_0 \in LC^*$ we denote $h_{\mathbf{v}_0}(\mathbf{u}) = \langle \mathbf{X}(\mathbf{u}),\mathbf{v}_0\rangle - 1$.

Proposition 4.1. ([9]) Let $H : U \times LC^* \longrightarrow \mathbb{R}$ be a horospherical height function of a spacelike submanifold $X : U \longrightarrow S^n_1$ of codimension r. Then $H(\mathbf{u},\mathbf{v}) = \partial H(\mathbf{u},\mathbf{v})/\partial u_i = 0$ for $i = 1,\dots,n-r$ if and only if $\mathbf{v} = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u},\bar{\mu})$ for some $\bar{\mu} \in H^{r-1}(-1)$.

We define a map $HS_{\mathbf{X}} : U \times H^{r-1}(-1) \longrightarrow LC^*$ by

$$HS_{\mathbf{X}}(\mathbf{u},\bar{\mu}) = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u},\bar{\mu}),$$

which we call a *horospherical hypersurface of* M. We remark that $HS_{\mathbf{X}}$ is independent to the choice of orthonormal frames of $N(M)$ up to the diffeomorphic parametrization. The following proposition is analogous to ([6], Proposition 3.5).

Proposition 4.2. ([9]) Let $\mathbf{X} : U \longrightarrow S_1^n$ be a spacelike hypersurface of codimension $r \geq 2$ in de Sitter space, then $HS_{\mathbf{X}}(\mathbf{u}, \bar{\mu}) = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})$ is a constant map for some smooth map $\bar{\mu} : U \longrightarrow H^{r-1}(-1)$ if and only if M is a part of de Sitter hyperhorosphere $HP(\mathbf{v}, 1) \cap S_1^n$. By Proposition 2.1, if M is totally $\mathbf{e}(\mathbf{u}, \bar{\mu}(\mathbf{u}))$-umbilic for some parallel normal vector field $\mathbf{e}(\mathbf{u}, \bar{\mu}(\mathbf{u}))$ and $K_h(\mathbf{e}(\mathbf{u}, \bar{\mu}(\mathbf{u})))(\mathbf{u}) = 0$, then the above assertion holds.

Let $\mathrm{Hess}\, h_{\mathbf{v}_0}(\mathbf{u}_0)$ be the Hessian matrix of $h_{\mathbf{v}_0}(\mathbf{u})$ at $\mathbf{u} = \mathbf{u}_0$. In [9] we have the following relation

$$\mathrm{rank}\, \mathrm{Hess}\, h_{\mathbf{v}_0}(\mathbf{u}_0) = \mathrm{rank}\, (\bar{h}_{ij}(\mathbf{v}_0)(\mathbf{u}_0))_{ij}.$$

Therefore the $\mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$-horospherical point (i.e. singular point of $HS_{\mathbf{X}}$) corresponds to the point with $\mathrm{Hess}\, h_{\mathbf{v}_0}(\mathbf{u}_0) = O$.

Proposition 4.3. ([9]) Let $\mathbf{X}$ be a spacelike submanifold of codimension $r \geq 2$. The corresponding horospherical height function H is a Morse family of hypersurfaces.

The above proposition enables us to apply the theory of Legendre singularities. By Proposition 4.1, the discriminant set of the horospherical height function H is the image of horospherical hypersurface $HS_{\mathbf{X}}$. We can construct the Legendrian immersion germs whose generating family is the horospherical height function.

We remark that there are relations between the horospherical points of M and the lightcone parabolic points of CM_θ. We have the following relation

$$\mathcal{M}_{e^\theta} \circ HS_{\mathbf{X}}(\mathbf{u}, \bar{\mu}) = \mathbb{L}_{CM_\theta}(\mathbf{u}, \bar{\mu}),$$

where $\mathcal{M}_c : LC^* \longrightarrow LC^*$ is a diffeomorphism on LC^* which is defined by $M_c(\mathbf{v}) = c\mathbf{v}$. Since the singular points of lightcone Gauss images (resp. horospherical hypersurfaces) correspond to the lightcone parabolic points (resp. horospherical points), we have the following remark.

Remark 4.4. Let p is a point on M. Then p is an $\mathbf{e}(\mathbf{u}, \bar{\mu})$-horospherical point on M if and only if $\bar{\mathbf{X}}_\theta(p, \mathbf{e}(\mathbf{u}, \bar{\mu}))$ is a lightcone parabolic point on CM_θ.

Therefore the regularity of the lightcone Gauss image is not depend on the parameter θ on the regular part of the spacelike canal hypersurface CM_θ.

5 Tangent de Sitter hyperhorospheres

In this section we use the theory of contacts of submanifolds due to Montaldi [10].

Let X_i and Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, $\dim Y_1 = \dim Y_2$ and $y_i \in X_i \cap Y_i$ for $i = 1, 2$. We say that *the contact* of X_1 and Y_1 at y_1 is *the same type* as *the contact* of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi((X_1, y_1)) = (X_2, y_2)$ and $\Phi((Y_1, y_1)) = (Y_2, y_2)$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. Two function germs $g_1, g_2 : (\mathbb{R}^n, a_i) \longrightarrow (\mathbb{R}, 0)$ $(i = 1, 2)$ are $\mathcal{K}$*-equivalent* if there are a diffeomorphism germ $\Phi : (\mathbb{R}^n, a_1) \longrightarrow (\mathbb{R}^n, a_2)$ and a function germ $\lambda : (\mathbb{R}^n, a_1) \longrightarrow \mathbb{R}$ with $\lambda(a_1) \neq 0$ such that $f_1 = \lambda \cdot (g_2 \circ \Phi)$. In [10] Montaldi has shown the following theorem.

Theorem 5.1. ([10]) Let X_i and Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, $\dim Y_1 = \dim Y_2$ and $y_i = X_i \cap Y_i$ for $i = 1, 2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

We now apply this theory to our situation. Given $\mathbf{v}_0 \in LC^*$, we define a submersion $\mathfrak{h}_{\mathbf{v}_0} : S_1^n \longrightarrow \mathbb{R}$ by $\mathfrak{h}_{\mathbf{v}_0}(x) = \langle x, \mathbf{v}_0 \rangle - 1$. So that $\mathfrak{h}_{\mathbf{v}_0}^{-1}(0) = HP(\mathbf{v}_0, +1) \cap S_1^n$ is a de Sitter hyperhorosphere. If $\mathbf{v}_0 = HS(\mathbf{u}_0, \bar{\mu}_0)$ for some $(\mathbf{u}_0, \mu_0)$, then we have

$$(\mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{X})(\mathbf{u}_0) = 0, \qquad \frac{\partial(\mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{X})}{\partial u_i}(\mathbf{u}_0) = 0.$$

This means that the de Sitter hyperhorosphere $\mathfrak{h}_{\mathbf{v}_0}^{-1}(0) = HP(\mathbf{v}_0, +1) \cap S_1^n$ is tangent to M at $p_0 = \mathbf{X}(\mathbf{u}_0)$. In this case we call $HP(\mathbf{v}_0, +1) \cap S_1^n$ a *tangent de Sitter hyperhorosphere* of M at $\mathbf{X}(\mathbf{u}_0)$. By Theorem 5.1 the contact type between the spacelike submanifold and its tangent de Sitter hyperhorosphere is determined by the $\mathcal{K}$-equivalence class of the horospherical height function $h_{v_0} = \mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{X}$.

We applied this theory to the contacts between the spacelike canal hypersurface and its tangent de Sitter hyperhorosphere (See [7]). Let $\bar{\mathbf{v}}_0 = \mathbb{L}(\mathbf{u}_0, \bar{\mu}_0)$, then the contact type of them is determined by the $\mathcal{K}$-equivalence class of the lightcone height function $\bar{h}_{\bar{\mathbf{v}}_0}$.

6 Classification

In this section we argue the classification of singularities appeared on horospherical hypersurfaces and lightcone Gauss images.

We assume that the corresponding Legendrian immersion germs generated by the horospherical height functions are Legendrian stable, then we have the following correspondence list of classes. Further details are written in a main theorem in [9].

(1) $\mathcal{A}$-equivalence class of horospherical hypersurface germs.
(2) Legendrian equivalence class of Legendrian immersion germs.
(3) $\mathcal{P}$-$\mathcal{K}$-equivalence class of horospherical height function germs H.
(4) $\mathcal{K}$-equivalence class of horospherical height function germs $h_{\mathbf{v}}$.
(5) Contact types between spacelike submanifolds and their tangent de Sitter hyperhorospheres.
(6) $\mathcal{A}$-equivalence class of lightcone Gauss image germs.
(7) Legendrian equivalence class of Legendrian immersion germs.
(8) $\mathcal{P}$-$\mathcal{K}$-equivalence class of lightcone height function germs $\bar{H}$.
(9) $\mathcal{K}$-equivalence class of lightcone height function germs $\bar{h}_{\bar{\mathbf{v}}}$.
(10) Contact types between spacelike hypersurfaces and their tangent de Sitter hyperhorospheres.

Since the horospherical hypersurface and the lightcone Gauss image are similar, the corank of horospherical height function is up to $n - r$. So that the singular types of lightcone Gauss images are restricted.

We now consider a simple case $n = 4$ and $r = 2$. M is a spacelike surface in de Sitter four space and CM_θ is a spacelike three-manifold. The horospherical height function h_{v_0} is a two parameter function germ. By the list of singularities of generic function germs. We have following singularities of generic horospherical hypersurfaces:

(1) HS_X has $\mathcal{A}_2$-type (h_{v_0} is $\mathcal{K}$-equivalent to $g(u_1, u_2) = u_1^2 - u_2^3$).
(2) HS_X has $\mathcal{A}_3$-type (h_{v_0} is $\mathcal{K}$-equivalent to $g(u_1, u_2) = u_1^2 \pm u_2^4$).

Both of the singularities correspond to the parabolic points on CM_θ, but only one principal curvature vanishes. The lightcone height function $\bar{h}_{\bar{v}_0}$ is $\mathcal{K}$-equivalent to $g(u_1, u_2, \mu_1) = \pm\mu_1^2 + u_1^2 \pm u_2^{k+1}$ for $(k = 2, 3)$.

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