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ON THE CLEBSCH-GORDAN PROBLEM FOR QUIVER REPRESENTATIONS

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1. Introduction

This survey contains the results presented in my talk at the conference *Representation Theory of Finite Groups and Algebras, and Related Topics* in RIMS, Kyoto. It is based on a joint paper with Erik Darpö [3] where many of the theorems presented here are proved.

Let $\Bbbk$ be a field. For an arbitrary finite dimensional algebra A over $\Bbbk$, there is no known way of naturally defining a tensor product on the category of left A-modules. However, if A is for instance the group algebra over a finite group G, then the underlying structure provided by G yields a tensor product defined by diagonal action. For path algebras over quivers one can similarly define a tensor product point-wise and arrow-wise. Our aim is to study this tensor product.

Quivers were introduced by Gabriel [5] and have ever since played an important role in the representation theory of finite dimensional algebras. A quiver Q is an oriented graph and as such consist of a set of vertices Q_0 and a set of arrows Q_1 between the vertices. For example the following quiver has 3 vertices and 2 arrows:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

To each quiver Q we associate its path algebra, the modules over which can be interpreted as representations of Q. A representation V of Q assigns to each vertex $x \in Q_0$ a vector space V_x (over $\Bbbk$) and to each arrow $x \xrightarrow{\alpha} y$ a linear map $V(\alpha) : V_x \to V_y$. The direct sum of representations is defined point-wise, i.e. for representations V and W of Q, their direct sum $V \oplus W$ is defined by

$$(V \oplus W)_x = V_x \oplus W_x$$

for each $x \in Q_0$ and

$$(V \oplus W)(\alpha) = V(\alpha) \oplus W(\alpha)$$

for each $\alpha \in Q_1$. We define the tensor product $V \otimes W$ similarly: set

$$(V \otimes W)_x = V_x \otimes W_x$$

for each $x \in Q_0$ and

$$(V \otimes W)(\alpha) = V(\alpha) \otimes W(\alpha)$$

for each $\alpha \in Q_1$. Since the tensor product is defined point-wise it commutes with direct sums. To describe it completely it is therefore enough to solve the following problem: Given indecomposable representations V and W of a quiver Q, find the decomposition of $V \otimes W$ into indecomposables. This problem has a commonly studied analogy in group representation theory and is called the Clebsch-Gordan problem, as it originates from the invariant theory of Clebsch and Gordan [2].

A classical instance of the Clebsch-Gordan problem is for the loop quiver

$$\bullet \circlearrowleft \alpha$$

If $\Bbbk$ is algebraically closed, the indecomposable representations of Q are given by Jordan blocks

$$J_\lambda(l) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}$$

where l is the size of the matrix and $\lambda \in \Bbbk$ is the eigenvalue. The Clebsch-Gordan problem then amounts to finding the Jordan normal form of the Kronecker product of two Jordan blocks. In characteristic zero this problem was originally solved by Aitken [1], but has also been solved independently by Huppert [11] and Martsinkovsky-Vlassov [13]. The solution is given by the following Theorem.

Theorem 1. *For all $\lambda, \mu \in \Bbbk \setminus \{0\}$ and positive integers l, m the following formulae hold:*

(1) $J_\lambda(l) \otimes J_\mu(m) \sim \bigoplus_{i=0}^{l-1} J_{\lambda\mu}(l+m-2i-1)$ *if* $l \leqslant m$ *and* $\operatorname{char} \Bbbk = 0$,
(2) $J_\lambda(l) \otimes J_0(m) \sim lJ_0(m)$,
(3) $J_0(l) \otimes J_0(m) \sim (m-l+1)J_0(l) \oplus \bigoplus_{i=1}^{l-1} 2J_0(i)$ *if* $l \leqslant m$.

Here $A \sim B$ means that A is similar to B. The first formula in Theorem 1 fails in positive characteristic. An algorithm for determining the corresponding decomposition in characteristic $p > 0$ has been found Iima and Iwamatsu [12], but no explicit formula is known.

For Dynkin quivers of type $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}_6$, the solution is found in [10] and [9] over an arbitrary field. In tame type the solution has been found for extended Dynkin quivers of type $\tilde{\mathbb{A}}$ in [7] and for the double loop quiver

$$\alpha \circlearrowright \bullet \circlearrowleft \beta$$

with relations $\alpha^n = \beta^n = \alpha\beta = \beta\alpha = 0$ in [8]. However, these tame cases are only reduced to the loop case. And thus, one piece of the puzzle remains for fields that are not algebraically closed. In the sequel we will try to remedy this situation.

If the description of indecomposables is complicated, a solution to the Clebsch-Gordan problem along the lines of Theorem 1 becomes hard to digest. To obtain a more qualitative grasp of the solution we introduce the representation ring.

Let $S(Q)$ be the set of isomorphism classes of representations of the quiver Q. It has the structure of a semi-ring with addition and multiplication defined by

$$[V] + [W] = [V \oplus W]$$

and

$$[V][W] = [V \otimes W],$$

where $[V]$ denotes the isomorphism class of the representation V. The representation ring of Q is the Groethendieck ring associated to $S(Q)$ and is denoted $R(Q)$. As an abelian group $R(Q)$ is freely generated by the isomorphism classes of indecomposables, and the structure constants are given by the Clebsch-Gordan coefficients.

In the Dynkin case we have the following general description. For each $k \in \mathbb{N}$ set $R_k = \mathbb{Z}[T_1, \ldots T_k]/(T_iT_j \mid 1 \leq i, j \leq k)$.

Proposition 1. *If Q is of Dynkin type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}_6$, then there are natural numbers $k_r \in \mathbb{N}$ such that*

$$R(Q) \xrightarrow{\sim} \prod_{r=1}^{n} R_{k_r}$$

The precise numbers k_r depend on the type and orientation of Q, and can be found in [10] and [9].

2. The loop over a prefect field

As mentioned earlier the loop case plays an important role in all known solutions for quivers of tame type. We proceed to study this case under the assumption that $\Bbbk$ is perfect, i.e. every irreducible polynomial $f(x) \in \Bbbk[x]$ has distinct zeros in the algebraic closure $\bar{\Bbbk}$.

Assume that Q is the loop quiver

$$\bullet \circlearrowleft \alpha$$

A representation V of Q is completely determined by the linear operator $V(\alpha)$. We obtain a module over $\Bbbk[x]$ by declaring that the action of x should be given by $V(\alpha)$. In fact, this gives rise to an equivalence of categories

$$\mathrm{rep}_{\Bbbk} Q \xrightarrow{\sim} \Bbbk[x] - \mathrm{mod},$$

where $\mathrm{rep}_{\Bbbk} Q$ denotes the category of representations of Q and $\Bbbk[x] - \mathrm{mod}$ denotes the category of $\Bbbk[x]$-modules. We define the tensor product on $\Bbbk[x] - \mathrm{mod}$ via this equivalence. Moreover, let R be the representation ring of $\Bbbk[x]$ with respect to this tensor product. The following classification result for $\Bbbk[x] - \mathrm{mod}$ is well-known.

Theorem 2. *The modules $\Bbbk[x]/f(x)^s$, where s is a positive integer and $f(x) \in \Bbbk[x]$ is irreducible and monic, classify all indecomposable finite-dimensional $\Bbbk[x]$-modules up to isomorphism.*

Our aim is to decompose $\Bbbk[x]/f(x)^s \otimes \Bbbk[x]/g(x)^t$ for all $s, t > 0$ and irreducible polynomials $f(x), g(x)$. The two last formulae in Theorem 1 hold independent of the ground field and translating to our setting we obtain:

Proposition 2. *Let s and t be positive integers and $f(x) \in \Bbbk[x]$ irreducible with $f(0) \neq 0$. Then the following formulae hold.*

(1) $\Bbbk[x]/x^s \otimes \Bbbk[x]/f(x)^t \xrightarrow{\sim} t(\deg f)\Bbbk[x]/x^s$.

(2) $\Bbbk[x]/x^s \otimes \Bbbk[x]/x^t \xrightarrow{\sim} (t-s+1)\Bbbk[x]/x^s \oplus \bigoplus_{i=1}^{s-1} 2\Bbbk[x]/x^i$ *if* $s \leqslant t$.

A consequence of Proposition 2 is that the $\mathbb{Z}$-span of the elements $[\Bbbk[x]/x^s]$ in R forms an ideal I. Moreover, if V is a $\Bbbk[x]$-module on which x acts as an automorphism, then $[V]$ acts on I as multiplication by $\dim V$.

It remains to decompose $\Bbbk[x]/f(x)^s \otimes \Bbbk[x]/g(x)^t$ for all f, g satisfying $f(0) \neq 0 \neq g(0)$. This corresponds in the algebraically closed case to Jordan blocks of non-zero eigenvalue. To harness the results obtained for $\Bbbk$ algebraically closed we employ the following lemma due to Noether, see [4].

Lemma 1. *Let K be an algebraic field extension of $\Bbbk$ and A an associative $\Bbbk$-algebra with identity. Further let V and W be finite-dimensional A-modules. If $K \otimes V$ and $K \otimes W$ are isomorphic as $K \otimes A$-modules, then V and W are isomorphic as A-modules.*

Our strategy is now to take our problem to the algebraic closure by tensoring with $\overline{\Bbbk}$ and then applying Theorem 1. After that we use Lemma 1 to get back to the ground field $\Bbbk$.

Observe that $J_\lambda(1) \otimes J_1(l) = \lambda \mathbb{I}_l + \lambda J_0(l)$. If $\lambda \neq 0$, then $\lambda J_0(l)$ is nilpotent of degree l and thus $J_\lambda(1) \otimes J_1(l) \sim J_\lambda(l)$. Applying the strategy outlined above we obtain the following result.

Proposition 3. *For any positive integer s and irreducible polynomial $f(x) \in \Bbbk[x]$ with $f(0) \neq 0$, the $\Bbbk[x]$-modules $\Bbbk[x]/f(x)^s$ and $\Bbbk[x]/(x-1)^s \otimes \Bbbk[x]/f(x)$ are isomorphic.*

Let R' be the $\mathbb{Z}$-span of the elements $v_s := [\Bbbk[x]/(x-1)^s]$, where $s > 0$ and $\bar{R}$ the $\mathbb{Z}$-span of all elements of the form $[\Bbbk[x]/f(x)]$, such that $f(x) \in \Bbbk[x]$ is irreducible with $f(0) \neq 0$. Moreover, define a ring structure on $R' \otimes_{\mathbb{Z}} \bar{R} \oplus I$ by $(a \otimes b)w = \dim(a)\dim(b)w$ for all $a \in R'$, $b \in R'$ and $w \in I$. Using Proposition 3 one can show the following general description of R.

Theorem 3. *The $\mathbb{Z}$-linear map*

$$\phi : R' \otimes_{\mathbb{Z}} \bar{R} \oplus I \to R,$$

defined by $\phi(a \otimes b + w) = ab + w$ is a ring isomorphism.

By Proposition 2, the structure of I is independent of $\Bbbk$. Moreover, the action of $R' \otimes_{\mathbb{Z}} \bar{R}$ on I is given by dimension. Hence, it remains to describe the rings R' and $\bar{R}$. Using Galois theory we obtain the following result for of $\bar{R}$.

Proposition 4. *Let $G = \mathcal{G}(\overline{\Bbbk}/\Bbbk)$ be the absolute Galois group of $\Bbbk$ and $\overline{\Bbbk}^\iota$ the group of invertible elements in $\overline{\Bbbk}$. There is an isomorphism of rings:*

$$\bar{R} \xrightarrow{\sim} (\mathbb{Z}\overline{\Bbbk}^\iota)^G$$

Where $(\mathbb{Z}\overline{\Bbbk}^\iota)^G$ denotes the ring of invariants under G.

Proposition 4 can be made more explicit in case $\Bbbk$ is real or algebraically closed (see [3]). We proceed to describe the ring R', which turns out only to depend on the characteristic of $\Bbbk$. In characteristic zero we can apply Theorem 1 and obtain:

Theorem 4. *Assume that the characteristic of* $\Bbbk$ *is zero. The ring morphism*

$$\phi : \mathbb{Z}[T] \to R',$$

defined by $T \mapsto v_2$ *is an isomorphism.*

Assume that $\operatorname{char} \Bbbk = p > 0$ and let $\alpha \in \mathbb{N}$. Let $G_\alpha = \langle \sigma_\alpha \rangle$ be the cyclic group of order $q := p^\alpha$. Then there is an algebra isomorphism

$$\Bbbk G_\alpha \to \Bbbk[T]/T^q$$

defined by $\sigma_\alpha \mapsto T+1$. Hence the modules $\Bbbk G_\alpha/(\sigma_\alpha - 1)^s$, where $1 \leq s \leq q$ classify all indecomposable $\Bbbk G_\alpha$-modules. Let A_α be the representation ring of $\Bbbk G_\alpha$. Then we may view A_α as a subring of R' by identifying $[\Bbbk G_\alpha/(\sigma_\alpha - 1)^s]$ with v_s. This identification gives rise to chain of inclusions

$$A_0 \subset A_1 \subset \ldots \subset \bigcup_{\alpha \in \mathbb{N}} A_\alpha = R'.$$

The rings A_α have been described by Green in [6]. Set $w_\alpha = v_{p^\alpha+1} - v_{p^\alpha-1}$. Under our identification [6, Theorem 3] becomes the following:

Theorem 5. *Assume that* $\operatorname{char} \Bbbk = p > 0$ *and let* $\alpha \in \mathbb{N}$. *Set* $q = p^\alpha$. *Then*

$$w_\alpha v_r = \begin{cases} v_{r+q} - v_{q-r} & \text{if} \quad 1 \leqslant r \leqslant q \\ v_{r+q} + v_{r-q} & \text{if} \quad q < r \leqslant (p-1)q \\ v_{r-q} + 2v_{pq} - v_{(2p-1)q-r} & \text{if} \quad (p-1)q < r \leqslant pq \end{cases}$$

Moreover this equation defines the multiplicative structure of R'.

Thus we have described the rings R' and $\bar{R}$. Together our results on the ideal I, this completes our description of the representation ring R.

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