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UNDERSTANDING QUOTIENT SINGULARITIES THROUGH NONCOMMUTATIVE ALGEBRA

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This is a write-up of my lecture delivered at RIMS, Kyoto, in November 2008. The full technical details of this work can be found in the series of papers [Wem07], [Wem08], [Wem09a] and [Wem09b]. For a slightly more geometrical interpretation of the results presented here please consult the lecture notes [Wem09c].

1. Introduction

Put simply, given a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ the McKay Correspondence relates the geometry of the minimal resolution of the singularity $\mathbb{C}^2/G$ to the representation theory of G. Since the group is inside $\mathrm{SL}(2,\mathbb{C})$ things are particularly nice, for example there is a 1-1 correspondence

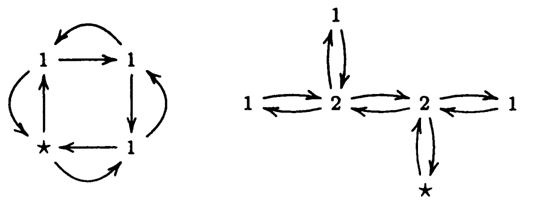
$$\{\text{exceptional curves}\} \leftrightarrow \{\text{non-trivial irreducible representations}\}.$$

We can add a little more structure to the right hand side:

Definition 1.1. *For given finite G acting on $\mathbb{C}^2 = V$, the McKay quiver is defined to be the quiver with vertices corresponding to the isomorphism classes of indecomposable representations, and the number of arrows from ρ_1 to ρ_2 is defined to be*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(\rho_1, \rho_2 \otimes V)$$

Example 1.2. For the groups $\frac{1}{4}(1,3)$ and $BD_{4\cdot 3}$ inside $\mathrm{SL}(2,\mathbb{C})$ the McKay quivers are

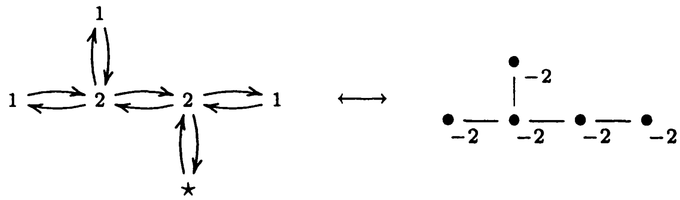


respectively, where the number on a vertex is the dimension of the representation at that vertex.

Equipped with this extra structure, McKay observed that

$$\{\text{dual graph of the minimal resolution}\} \rightleftarrows \text{McKay quiver}$$

where we go from one side to the other by deleting (or adding) the vertex corresponding to the trivial representation. Furthermore it is precisely the ADE Dynkin diagrams which appear. For example



The author is a JSPS Postdoctoral Fellow.

In this lecture I shall explain how the above generalizes (with some changes) to all finite subgroups of $\mathrm{GL}(2,\mathbb{C})$. The key to this generalization is not to try and build a quiver in such a simple way from the representation theory, but instead to obtain it as the quiver of an algebra obtained as the endomorphism ring of a certain module. The algebra will contain more information, which we can then ask to provide us with derived equivalences and moduli spaces.

2. The GL(2, $\mathbb{C}$) McKay Correspondence

Throughout this section we assume that all groups are small, that is contain no pseudoreflections except the identity. Before continuing we should firstly point out that when $G \not\leq SL(2,\mathbb{C})$ there are more representations than exceptional divisors, so such a simple picture as above cannot be true. However work by Wunram [Wun88] in the 80's gives us a 1-1 correspondence

$$\{\text{exceptional curves}\} \leftrightarrow \{\text{non-trivial } \textit{special} \text{ irreducible representations}\}.$$

To define what we mean by special, for a representation ρ denote $M_\rho = (\mathbb{C}[[x,y]] \otimes_{\mathbb{C}} \rho)^G$ where G acts on both sides of the tensor. Denoting the minimal resolution by $f : \widetilde{X} \to \mathbb{C}^2/G$ we may consider the vector bundle $\mathcal{M}_\rho := f^*(M_\rho)/tors$ on $\widetilde{X}$. The representation ρ is said to be special if $H^1(\mathcal{M}_\rho^\vee) = 0$.

This is not the easiest definition to work with and for a long time it was an open question to explicitly write down the specials for non-cyclic groups. This problem has now been solved [IW08] and we have a full classification, although this shall not be needed in this lecture.

We arrive at the main definition:

Definition 2.1. *The ring* $\mathrm{End}(\oplus M_\rho)$*, where the sum is over all special representations, is called the reconstruction algebra.*

The reason for the name is twofold - firstly (as we shall see below) the quiver of $\mathrm{End}_R(\oplus M_\rho)$ can be reconstructed combinatorially from the dual graph of the minimal resolution. Secondly, the geometry can be reconstructed from the algebra by considering a certain moduli space of representations.

To describe the reconstruction, we need to introduce a piece of combinatorics.

Definition 2.2. [Art66] *For the dual graph* $\{E_i\}$*, define the fundamental cycle* $Z_f = \sum_i r_i E_i$ *(with each* $r_i \geq 1$*) to be the unique smallest element such that* $Z_f \cdot E_i \leq 0$ *for all vertices* i*.*

Note that given the data of a dual graph, Z_f is very quick to calculate - its an entirely combinatorial property of the dual graph. In the case of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ these numbers are what you expect from Lie theory.

Theorem 2.3 (The $GL(2,\mathbb{C})$ McKay Correspondence). *Let G be a finite subgroup of* $\mathrm{GL}(2,\mathbb{C})$ *and denote the minimal resolution by* $\widetilde{X} \to \mathbb{C}^2/G$*. Then the reconstruction algebra* $\mathrm{End}(\oplus M_\rho)$ *can be written as a quiver with relations as follows: for every special representation* ρ_i *(corresponding to the exceptional curve* E_i*) associate a vertex labelled* i*, and also associate a vertex* $\star$ *corresponding to the trivial representation. Then the number of arrows and relations between the vertices is given as follows:*

	Number of arrows	Number of relations
$i \to j$	$(E_i \cdot E_j)_+$	$(-1 - E_i \cdot E_j)_+$
$\star \to \star$	0	$-1 - Z_f \cdot Z_f$
$i \to \star$	$-E_i \cdot Z_f$	0
$\star \to i$	$(E_i^2 + 2 - Z_f \cdot E_i)_+$	$(E_i^2 + 2 - Z_f \cdot E_i)_-$

where for any integer $a \in \mathbb{Z}$ *denote* $a_+ = \begin{cases} a & a \geq 0 \\ 0 & a < 0 \end{cases}$ *and* $a_- = \begin{cases} 0 & a \geq 0 \\ -a & a < 0 \end{cases}$.

There are many more subgroups of $\mathrm{GL}(2,\mathbb{C})$ than $\mathrm{SL}(2,\mathbb{C})$ and so the above theorem provides us with a much larger class of singularities on which noncommutative methods can be deployed to help understand the geometry. The following is an easy corollary to the above, and reduces the calculation to that of certain base cases:

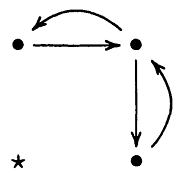
Lemma 2.4. *Suppose two curve systems $E = \{E_i\}$ and $F = \{F_i\}$ have the same dual graph and fundamental cycle, such that $-F_i^2 \leq -E_i^2$ for all i. Then the quiver for the curve system E is obtained from the quiver of the curve system F by adding $-E_i^2 + F_i^2$ extra arrows $i \to \star$ for every curve E_i.*

The correspondence is perhaps best understood via examples.

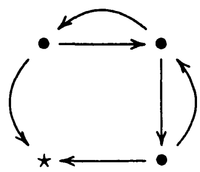
Example 2.5. Consider the group $\frac{1}{4}(1,3)$. For this example the dual graph is

$$\underset{-2}{\bullet} \text{---} \underset{-2}{\bullet} \text{---} \underset{-2}{\bullet}$$

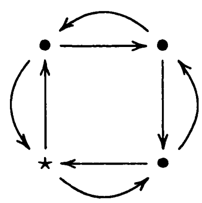
After the $i \to j$ and $\star \to \star$ steps in the theorem, we have the following picture



Now to calculate how to connect $\star$, we need to know the fundamental cycle. But here $Z_f = \begin{matrix}1 & 1 & 1\end{matrix}$ and so in matrix from $(-E_i \cdot Z_f)_{i\in I} = \begin{matrix}1 & 0 & 1\end{matrix}$. Thus after the $i \to \star$ step:



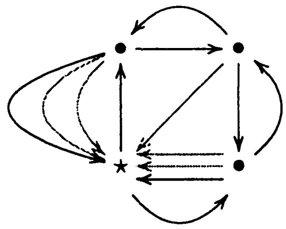
For the $\star \to i$ step notice that since all curves are (-2)-curves the number of arrows $\star \to i$ is equal to the number of arrows $i \to \star$. Consequently the quiver of the reconstruction algebra is



Example 2.6. Consider now the dual graph

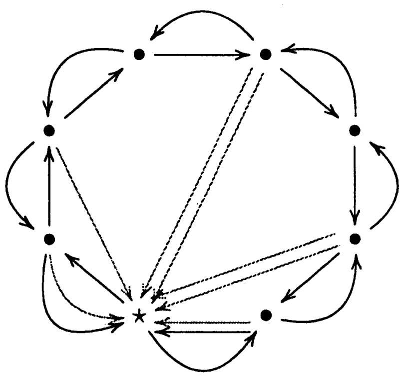
$$\underset{-4}{\bullet} \text{---} \underset{-3}{\bullet} \text{---} \underset{-4}{\bullet}$$

corresponding to the group $\frac{1}{40}(1,11)$. Now Z_f is the same as in the previous example, so by Lemma 2.4 we just have to add extra arrows to the above; we thus deduce that the reconstruction algebra is



All other cyclic group cases are identical, and follow easily. For example

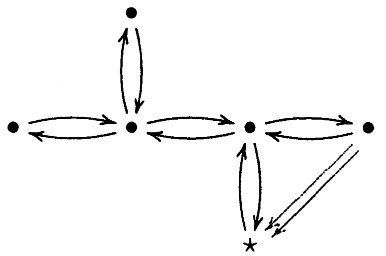
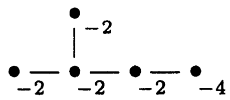
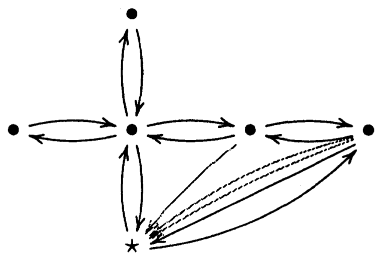
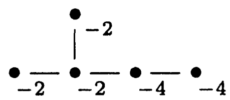
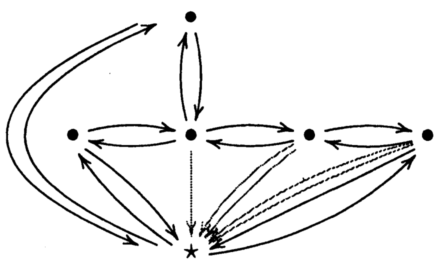
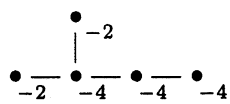
Example 2.7. For the group $\frac{1}{693}(1, 256)$, the reconstruction algebra is



corresponding to the dual graph

Lastly, we consider some non-abelian groups.

Example 2.8. Some dihedral groups.

Reconstruction Algebra	dual graph	Z_f	group
		$\begin{smallmatrix} & 1 & & \\ 1 & 2 & 2 & 1 \end{smallmatrix}$	$\mathbb{D}_{10,7}$
		$\begin{smallmatrix} & 1 & & \\ 1 & 2 & 1 & 1 \end{smallmatrix}$	$\mathbb{D}_{26,15}$
		$\begin{smallmatrix} & 1 & & \\ 1 & 1 & 1 & 1 \end{smallmatrix}$	$\mathbb{D}_{56,15}$

In fact reconstruction algebras exist for more than just quotient singularities, and are built in an identical way. Also it is possible to reconstruct on non-minimal resolutions, but the combinatorics change slightly.

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[Wem09b] ———, *Reconstruction algebras of type D (II)*, arXiv:0905.1155 (2009).

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