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The group of endotrivial modules for symmetric and alternating groups

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Survey of two collaborations:
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Endotrivial modules are ubiquitous representations of modular group algebras of finite groups. Given a finite group G and a field k of prime characteristic p, a finitely generated kG-module M is *endotrivial* if the endomorphism algebra $\mathrm{End}_k M$ of k-linear transformations $M \to M$ splits as kG-module as the direct sum of the one-dimensional trivial module k and a projective module. These modules have been noticed long ago, and Everett Dade coined the expression in 1978. He was studying what he called *endo-permutation modules* over finite p-groups, and it turned out that the endotrivial modules constituted the building bricks of the endo-permutation modules. He remarked that the endo-permutation modules show up naturally as sources of simple modules for finite p-solvable groups. Thus, he started a classification program of these and hence also of endotrivial modules for finite p-groups. Both programs have been completed, respectively in 2004 and in 2006. Since the concept of endotrivial module extends well to arbitrary finite groups, we now aim at their classification in this broader context. Let us briefly outline a few motivations therefore. Recall that $\mathrm{End}_k M$ is isomorphic as kG-module to the tensor product $M^* \otimes_k M$, where $M^* = \mathrm{Hom}_k(M, k)$ is the k-linear dual of M. Hence, the endotrivial modules are invertible elements in the Green ring $\underline{\mathrm{A}}(G)$ of the stable module category $\underline{\mathrm{mod}}(kG)$ of G. They also induce self-equivalences of $\underline{\mathrm{mod}}(kG)$, and have several other "cohomological" properties.

A key fact is that the set $T(G)$ of isomorphism classes of endotrivial modules form a subgroup of the Picard group of $\underline{\mathrm{A}}(G)$, since the tensor product over the groud field of two endotrivial modules is also endotrivial. We call $T(G)$ the *group of endotrivial modules of* G. The composition law in $T(G)$ is induced by the tensor product "$\otimes_k$" with diagonal group action, that is, $[M] + [N] = [M \otimes_k N]$ for endotrivial modules M and N. Whence $T(G)$ is abelian and we have $0 = [k]$ and $-[M] = [M^*]$. The upshot of this point of view is that the classification of endotrivial kG-modules reduces to the computation of the structure of $T(G)$. Moreover, it has been shown that $T(G)$ is finitely generated and hence can be written as a direct sum $T(G) = TT(G) \oplus TF(G)$, where $TT(G)$ is the torsion subgroup and $TF(G)$ is torsion free. Thus, $TT(G)$ is a finite abelian group, whereas $TF(G)$ is isomorphic to a direct sum of n copies of $\mathbb{Z}$. In this notation, the torsion-free rank n is given by the following rule: if m is the number of conjugacy classes of maximal elementary abelian p-subgroups of G of order p^2, then $n = m$ if G has p-rank at most 2, and $n = m + 1$ otherwise. In particular, $T(G)$ is finite if G has cyclic Sylow p-subgroups, or possibly quaternion in case $p = 2$.

This decomposition of $T(G)$ allows one to split the problem of the classification of endotrivial modules into two parts: finding $TT(G)$ on one hand and determining $TF(G)$ on the other.

Endotrivial modules have been classified for several families of finite groups, including p-groups, groups with normal or cyclic Sylow p-subgroups, groups of Lie type in defining characteristic, and now also the symmetric and alternating groups, for all primes.

The paradigm of a non trivial endotrivial module is the *syzygy* of the trivial module. That is, the syzygy of k (also called the Heller translate) is the kernel $\Omega(k) = \Omega^1(k)$ of a projective cover $P_0 \to k$ of k. Iteratively, we then may construct other instances of endotrivial modules by taking $\Omega^n(k) = \Omega(\Omega^{n-1}(k))$, for all non negative integers n; also, we set $\Omega^n(k) = (\Omega^{-n}(k))^*$, for all $n < 0$, and $\Omega^0(k) = k$. It turns out that the set

$$\langle[\Omega(k)]\rangle = \{[\Omega^n(k)] \mid n \in \mathbb{Z}\} \quad \text{form a direct summand in } T(G).$$

In fact, often one has $\langle[\Omega(k)]\rangle = TF(G)$, because relatively few groups have maximal elementary abelian p-subgroups of order p^2. In any case the subgroup $\langle[\Omega(k)]\rangle$ of $T(G)$ contains a "large" part of $T(G)$, as the case of the symmetric and alternating groups demonstrate. It should also be pointed out that, except when $p = 2$ and the Sylow 2-subgroups are quaternion groups of order 8, then $T(G)$ is independent of the size of the field k, in the sense that the modules are defined over $\mathbb{F}_p$.

In view of the results obtained up to now, the strategy employed for the classification of endotrivial modules of a given finite group G, with Sylow p-subgroup P and normaliser $N = N_G(P)$, decomposes as follows.

(i) Using the classification, determine $T(P)$ and $T(N)$.
(ii) Compute the number of G-conjugacy classes of maximal elementary abelian p-subgroups of G of order p^2, whence get the isomorphism type of $TF(G)$.
(iii) Tackle $TT(G)$, using the fact that the restriction map $T(G) \to T(N)$ induced by the inclusion $N \hookrightarrow G$ is injective. A partial converse is given by the Green correspondence.

A key fact to keep in mind when working out the second half of point (i) is that $T(N)$ is generated by the isomorphism classes of N-stable endotrivial kP-modules. In other words, one only needs to determine which are the indecomposable endotrivial kP-modules that are invariant under N-conjugacy; these will extend to N and generate the whole group $T(N)$. In addition, from the classification of endotrivial kP-modules, $TT(P)$ is trivial, unless P is cyclic (of order at least 3), quaternion or semidihedral. Hence, in the (common) case that $TT(P) = \{[k]\}$, then $TT(G)$ is generated by the indecomposable trivial source modules whose restriction to P splits as the direct sum of k and some projective module. If G is the symmetric group, then these representations are among the so-called *Young modules.*

Part (ii) reduces to a "simple" computation of conjugacy classes, and by a previous observation, if $TF(G) \cong \mathbb{Z}$, then $TF(G) = \langle[\Omega(k)]\rangle$. Note that if the torsion-free rank of $T(G)$ is greater than 1, then the question of finding generators for a possible $TF(G)$ is still open, unless $G = N$ or G has a dihedral Sylow 2-subgroup.

Part (iii) is certainly the trickiest one, and there is no general approach to it. Nevertheless, knowing something about the representation theory of G in general might be helpful. In the case of the symmetric and alternating groups, several research articles were key in the determination of $T(G)$ (see reference list).

By applying this strategy to the case of the symmetric and alternating groups, and in addition, some subtler argument, voluntarily omitted in this report, we obtained the following results.

Theorem A: Let S_n be the symmetric group on n letters.

(a) If $p = 2$, then

$$T(S_n) \cong \begin{cases} \{0\} & \text{if } n \leq 3 , \\ \mathbb{Z}^2 & \text{if } n = 4,\ 5 , \\ \mathbb{Z} & \text{if } n \geq 6 . \end{cases}$$

(b) If $p \geq 3$ and $1 \leq n < 2p$, then

$$T(S_n) \cong \begin{cases} \{0\} & \text{if } n < p , \\ \mathbb{Z}/2(p-1)\mathbb{Z} & \text{if } n = p,\ p+1 , \\ \mathbb{Z}/2(p-1)\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } p+2 \leq n < 2p . \end{cases}$$

(c) If $p \geq 3$ and $2p \leq n < p^2$, then

$$T(S_n) \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } 2p \leq n < 3p , \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } 3p \leq n < p^2 . \end{cases}$$

(d) If $p \geq 3$ and $p^2 \leq n$, then

$$T(S_n) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p^2 \leq n < p^2 + p , \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p^2 + p \leq n . \end{cases}$$

Theorem B: Let A_n be the alternating group on n letters.

(a) If $p = 2$, then

$$T(A_n) \cong \begin{cases} \{0\} & \text{if } n \leq 3 , \\ \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z}) & \text{if } n = 4,\ 5 , \\ \mathbb{Z}^2 & \text{if } n = 6,\ 7 , \\ \mathbb{Z} & \text{if } n \geq 8 . \end{cases}$$

(b) If $p \geq 3$ and $1 \leq n < 2p$, then

$$T(A_n) \cong \begin{cases} \{0\} & \text{if } n < p , \\ \mathbb{Z}/(p-1)\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } n = p, p+1 , \\ \mathbb{Z}/2(p-1)\mathbb{Z} & \text{if } p+2 \leq n < 2p. \end{cases}$$

(c) If $p \geq 3$ and $2p \leq n < p^2$, then

$$T(A_n) \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z}) & \text{if } p = 3 \text{ and } n = 6,\ 7\ , \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } p > 3 \text{ and } n = 2p,\ 2p+1\ , \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } 2p+2 \leq n < 3p\ , \\ \mathbb{Z} & \text{if } 3p \leq n < p^2\ . \end{cases}$$

(d) If $p \geq 3$ and $p^2 \leq n$, then

$$T(A_n) \cong \begin{cases} \mathbb{Z}^2 & \text{if } p^2 \leq n < p^2 + p\ , \\ \mathbb{Z} & \text{if } p^2 + p \leq n\ . \end{cases}$$

Rather than a technical outline of the proof of this result, we end this report with a few remarks tying the endotrivial modules with the well-known representation theory of the symmetric groups. First, it is easy to see that the factor $\mathbb{Z}/2\mathbb{Z}$ showing up everywhere in the description of $T(S_n)$ and nowhere in $T(A_n)$ is the class of the sign representation (which is of course an endotrivial module). Another fact to point out is that the case $p = 2$ turns out to be easier than the other situations. The reason is that, except for some small degrees, the Sylow 2-subgroups are self-normalising and $T(P)$ is torsion-free. Incidentally, we obtain that the Specht module $S^{(3,1)}$ corresponding to the natural representation of S_4 is an endotrivial $\mathbb{F}_2S_4$-module and extends an $\mathbb{F}_2D_8$-module (where D_8 is a Sylow 2-subgroup of S_4, hence dihedral of order 8). More precisely, $S^{(3,1)}$ extends the kernel of the augmentation map $\Omega_{D_8/X}(k) = \ker(\mathbb{F}_2[D_8/X] \to \mathbb{F}_2)$, also called *relative syzygy*, for X a non-central subgroup of D_8 of order 2. From S_4 to S_5, a "branching game" yields the corresponding modules. Let us also outline the fact that S_4 and S_5 are the unique instances of arbitrary finite groups G with non-normal Sylow p-subgroups and torsion-free groups $TF(G)$ of rank greater than 1 for which explicit generators for $T(G)$ are known. Last but not least, we appealed to the algebra software Magma. It lead us to discover the main surprising fact to us: for $2p \leq n < p^2$, and $p > 2$, the group $TT(S_n)$ is generated by the classes of the sign representation AND that of an endotrivial module Y of dimension > 1. Explicitly, if $n = 2p + a$, with $0 \leq a < p$, then Y is isomorphic to the Young module corresponding to the partition $(p + a, p)$.

Full details of the results summarised in this report are in the articles:

- J. Carlson, N. Mazza and D. Nakano, *Endotrivial modules for the symmetric and alternating groups*, Proc. Edinburgh Math. Soc., to appear.
- J. Carlson, D. Hemmer and N. Mazza, *The group of endotrivial modules for the symmetric and alternating groups*, Proc. Edinburgh Math. Soc., to appear.

Finally, a few suggested related papers for further reading

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