数理解析研究所講究録
第 1709 巻 2010 年 38-50



# SOME PROPERTIES OF SPIN MODULES OF THE SYMMETRIC GROUPS

北海道教育大学旭川校 奥山 哲郎 (TETSURO OKUYAMA)
HOKKAIDO UNIVERSITY OF EDUCATION, ASAHIKAWA CAMPUS

## 1. INTRODUCTION

In the lecture, we investigated the cohomological properties of the spin module of the symmetric group over a field of characteristic 2. This is a joint work with K. Uno.

Let $\Sigma_n$ be the symmetric group of degree $n$ and $k$ be an algebraically closed field of characteristic 2. Simple $k\Sigma_n$-modules are parametrized by the set of 2-regular partitions of $n$. For $n = 2(m+1)$, the simple module $D^{(m+2,m)}$ corresponding to the partition $(m+2, m)$ is called the spin module of $\Sigma_{2(m+1)}$. $D^{(m+2,m)}\downarrow_{\Sigma_{2m+1}}$ remains to be simple and isomorphic to $D^{(m+1,m)}$, the simple $k\Sigma_{2m+1}$-module corresponding to the partition $(m+1, m)$ of $2m+1$. $D^{(m+1,m)}$ is also called the spin module of $\Sigma_{2m+1}$. The spin module has many interesting properties and investigated by several authors (see for example, Benson [5], Gow-Kleshchev [8], Danz-Külshammer [7]). For representations of the symmetric group, we refer to a book of James [9]. The following theorem is due to Nagai [10] and Uno [14] and we discussed its proof at 2007 RIMS meeting [11].

**Theorem 1.1.** *The complexity of the spin module $D^{(m+1,m)}$ of $\Sigma_{2m+1}$ is equal to $[m/2]$.*

Uno's conjecture on the spin module $D^{(m+1,m)}$ proposed in [14] concerns with the cohomological variety of it and we also discussed the conjecture at 2007 RIMS meeting [11]. Little progress has been made with our study, but in this lecture, we shall explain some idea to attack the conjecture.

Let $G$ be a finite group $G$ and $k$ be an algebraically closed field of characteristic $p > 0$. The variety, the maximal ideal spectrum, of the cohomology algebra $H^*(G, k)$ is denoted by $V_G(k)$. For a finitely generated $kG$-module $M$, the annihilator ideal of $H^*(G, k)$-module $\mathrm{Ext}^*_{kG}(M, M)$ is denoted by $I_G(M)$ and the subvariety determined by $I_G(M)$ is denoted by $V_G(M)$. The complexity $c_G(M)$ of $M$ is equal to the dimension of $V_G(M)$. For a homogeneous element $\zeta \in H^*(G, k)$, denote by $V_G(\zeta) \subset V_G(k)$ the set of maximal ideals of $H^*(G, k)$ containing $\zeta$. Thanks to theorems of Quillen [12] and Avrunin-Scott [2], the variety of $V_G(M)$ is determined by knowing the varieties $V_E(M\downarrow_E)$ for all the elementary abelian $p$-subgroups $E$ of $G$.

For our spin module, the most important case is the case that $m = 2^{s-1}$ for some positive integer $s$ and $E \subset \Sigma_{2m} \subset \Sigma_{2m+1}$ is a maximal regular elementary abelian 2-sugroup (of order $2^s$) in $\Sigma_{2m}$. The cohomology algebra $H^*(E, k)$ of an elementary abelian 2-group $E$ of rank $s$ is a polynomial ring of $s$ variables (of degree 1) on which $GL(s,2)$ acts in an obvious way. The invariant subalgebra $H^*(E,k)^{GL(s,2)}$ is generated by so called Dickson invariants $c_i(E)$ $(0 \leqq i \leqq s-1)$ of degree $2^s - 2^i$. For precise definition and properties of Dickson invariants, see [1], [4]. Uno's conjecture is described as follows.

**Conjecture 1.2.** *Let $m = 2^{s-1}$ and $E$ be a maximal regular elementary abelian 2-sugroup of $\Sigma_{2m}$. Then the variety $V_E(D^{(m+1,m)}\downarrow_E)$ is $V_E(c_{s-1}(E))$.*

In Section 2, we give a lemma concerning the varieties of Carlson modules which will be used to discuss varieties of the spin modules. To investigate the variety of the spin module, it may help to know properties of the cohomology algebra of the symmetric group. In Section 3, we construct some cohomology elements of the symmetric group $\Sigma_{2^s}$ which restrict to Dickson invariants $c_i(E)$, $0 \leqq i \leqq s-1$ in the cohomology algebra $H^*(E,k)$ of a maximal regular elementary abelian 2-subgroup $E$. In Section 4, using a presentation of the spin module $D = D^{(m+1,m-1)}$ of the symmetric group $G = \Sigma_{2m}$ due to Nagai-Uno [10], we describe a cohomology element $\rho \in \mathrm{Ext}^1_{k\Sigma_{2m}}(D,D)$. In Section 5, we propose a problem on relations between $\mathrm{res}_{G,E}(\rho)$ and $c_{s-1}(E) \otimes id_D$ where $m = 2^{s-1}$ and $E$ is a maximal regular elementary abelian 2-subgroup of $\Sigma_{2m}$.

## 2. A Lemma From Cohomology Theory of Finite Groups

We refer to a book of Benson [3] for the cohomology theory of finite groups. Here we give a lemma concerning Carlson module $L_\zeta$ of a homogeneous element $\zeta \in H^n(G,k)$. Let $\hat{\zeta} : \Omega^n(k) \to k$ be the corresponding cocycle and set $L_\zeta = \mathrm{Ker}\,\hat{\zeta}$ if $\zeta \neq 0$. If $\zeta = 0$, then set $L_\zeta = \Omega^n(k) \oplus \Omega(k)$. Then it holds that $V_G(L_\zeta) = V_G(\zeta)$ is the set of maximal ideals that contain $\zeta$. Following to the Carlson's argument in a proof of this fact [6], we have the similar result for an arbitraly $kG$-module $M$ and $\rho \in \mathrm{Ext}^n_{kG}(M,M)$.

Take any cocycle $\hat{\rho}' : \Omega^n(M) \to M$ representing $\rho$ and then take a projective $kG$-module $P$, $\pi \in \mathrm{Hom}_{kG}(P,M)$ so that $\hat{\rho} = (\hat{\rho}', \pi) : \Omega^n(M) \oplus P \to M \to 0$ becomes exact. Then we obtain a short exact sequence of $kG$-modules,

$$0 \to \mathrm{Ker}\,\hat{\rho} \to \Omega^n(M) \oplus P \xrightarrow{\hat{\rho}} M \to 0$$

The isomorphism classes $\hat{\rho}$ and $\mathrm{Ker}\,\hat{\rho}$ in the stable category of $kG$-modules are uniquely determined by $\rho$. We denote $\mathrm{Ker}\,\hat{\rho}$ by $L_{\rho,M}$.

**Lemma 2.1.** *In the notations above, the following statements hold.*

(1) $V_G(L_{\rho,M}) = V_G(L_{\rho^s,M})$ *for any* $s > 0$.
(2) *If* $\rho^s = \zeta \otimes id_M$ *for some homogeneous element* $\zeta \in H^{sn}(G,k)$ *and some positive integer* $s$*, Then* $V_G(L_{\rho,M}) = V_G(\zeta) \cap V_G(M)$.

## 3. Some Cohomology Elements in $H^*(\Sigma_n, k)$, $n = 2^s$

For the cohomology of the symmetric group, we refer to a book of Adem and Milgram [1]. Our aim in this section is to construct some cohomology elements of $\Sigma_n$ to be able to understand the restrictions of them to Young subgroups easier.

### 3.1. The Complex of Rickard of a Finite Coxeter Group.

Let $W$ be a finite Coxeter group with a generating set $S$. Let $\ell : W \to \mathbb{N}$ be the length function with respect to $S$. For $I \subset S$, we set $W_I =< I >$, a parabolic subgroup of $W$ and $w_I \in W_I$ is the unique element of maximal length in $W_I$.

We fix a total order on $S$ in the following discussion. For $s \in S \setminus I$, set $n(I,s) = |\{ s' \in I \ ; \ s' < s \}|$.

The $W$-poset, called the Coxeter poset, consists of all right cosets

$$\{ W_I w \ ; \ w \in W, \ I \subset S \}$$

The complex of right $kW$-modules associated to the Coxeter poset is described as follows (the complex can be defined for any commutative ring). For $I \subset S$, let $[I] = \sum_{w \in W_I} w \in kW_I$. Then for $I, J \subset S$, there exists a unique $kW$- homomorphism

$$\pi_{IJ} \ : \ [I]kW \to [J]kW, \quad \text{sending } [I] \text{ to } [J]$$

Whenever $I \subset J \subset K \subset S$, we have that $\pi_{JK} \circ \pi_{IJ} = \pi_{IK}$.

**Definition 3.1.** *Let* $\mathbf{A} = \mathbf{A}(W,S)$ *be the cochain complex of right* $kW$*-modules concentrated in degree* 0 *to* $|S|$ *defined as follows ;*

$$(\mathbf{A})^t = A^t = \oplus \sum_{I \subset S, |I|=t} [I]kW$$

*for each* $t$ $(0 \leqq t \leqq |S|)$ *and the differentials* $d^t : A^t \to A^{t+1}$ $(0 \leqq t \leqq |S|-1)$ *on* $[I]kW \subset A^t$ *are given by*

$$d^t = \sum_{s \in S \setminus I} (-1)^{n(I,s)} \pi_{I I \cup \{s\}}$$

It is easily checked that for $s', s \in S \setminus I$ with $s' < s$, $n(I,s') = n(I \cup \{s\}, s')$, $n(I,s) = n(I \cup \{s'\}, s) - 1$ and that $d^{t+1} \circ d^t = 0$.

In these notations, Rickard proved in [13] that the complex $\mathbf{A}$ of $kW$-modules satisfies deep interesting properties. Among his results, we need the following facts.

**Theorem 3.2.** *The following statements hold.*

1. $H^t(\mathbf{A}(W,S)) = 0$ *for* $0 < t \leqq |S|$ *and* $H^0(\mathbf{A}(W,S))$ *is isomorphic to the sign representation of* $W$.
2. *Let* $K \subset S$. *Then in the homotopy category* $K^b(\text{mod-}kW_K)$ *of right* $kW_K$*-modules,* $\mathbf{A}(W,S) \downarrow_{W_K}$ *is isomorphic to* $\mathbf{A}(W_K, K)$.
3. *For each* $t \leqq |S| - 1$, *the short exact sequence* $0 \to \operatorname{Ker} d^t \to A^t \to \operatorname{Im} d^t \to 0$ *is* $W_K$*-split for all* $K \subset S$ *with* $|K| = t$.
4. *Assume that we have a decomposition* $S = K \cup L$ *such that* $st = ts$ *for all* $s \in K$ *and* $t \in L$. *Then* $W = W_K \times W_L$, $kW \cong kW_K \otimes kW_L$ *and* $\mathbf{A}(W,S) \cong \mathbf{A}(W_K,K) \otimes \mathbf{A}(W_L,L)$ *in the homotopy category* $K^b(\text{mod-}kW))$ *of right* $kW$*-modules.*

$A^{|S|} = [S]kW \cong k_W$. And if the characteristic of $k$ is 2, then $H^0(\mathbf{A}(W,S)) \cong k_W$ and we have the $|S|$-fold extension of $k_W$ by itsef of the following form as the reduced complex $\tilde{\mathbf{A}}(W,S)$ of $\mathbf{A}(W,S)$

$$0 \to k_W \xrightarrow{\iota} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{|S|-2}} A^{|S|-1} \xrightarrow{d^{|S|-1}} k_W \to 0$$

We shall use these facts due to Rickard to construct several cohomology elements of the symmetric group over a field of characteristic 2.

In the rest of the section, assume that our field $k$ has characteristic 2.

Let $M$ be a $kW$-module and suppose that we are given a $kW$-homomorphism $f : M \to \operatorname{Im} d^{t-1} = \operatorname{Ker} d^t$ for $0 \leqq t \leqq |S|$. Then by taking a pull back diagram

$$\begin{tikzcd}
0 \arrow[r] & k_W \arrow[r] \arrow[d, equal] & A^0 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & A^{t-2} \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & k_W \arrow[r] & A^0 \arrow[r] & \cdots \arrow[r] & A^{t-2} \arrow[r] & A^{t-1} \arrow[r, "d^{t-1}"] & \operatorname{Im} d^{t-1} \arrow[r] & 0
\end{tikzcd}$$

we obtain a cohomology element in $\operatorname{Ext}^t_{kW}(M,k)$. If $M = k_W$, then we denote this cohomology elements by $co(f) = co(W,f) \in H^t(W,k)$. The complex $\tilde{\mathbf{A}}(W)$ itself gives a cohomology element of degree $|S|$ which is $co(id_W)$, where $id_W : k_W \to k_W = A^{|S|}$ is the identity map. The following lemma follows from the statement (3) in Theorem 3.2.

**Lemma 3.3.** $\operatorname{res}_{W,W_K}(co(f)) = 0$ *for all* $K \subset S$ *with* $|K| < t$. *In particular,* $\operatorname{res}_{W,W_K}(co(id_W)) = 0$ *for all proper parabolic subgroup* $W_K$.

Let $K$ be a subset of $S$ and denote by $d_K^{(t)}$ be the differentials for the complex $\mathbf{A}(W_K,K)$. Then by the statement (2) in Theorem 3.2, we have a $kW_K$-map $\pi(K)^t : \mathbf{A}(W,S)^t \downarrow_{W_K} \to \mathbf{A}(W_K,K)^t$ which induces the map from $\operatorname{Ker} d^{(t)}$ to $\operatorname{Ker} d_K^{(t)}$ for each $t$. For $f \in \operatorname{Hom}_{kW}(k_W, \operatorname{Ker} d^{(t)})$, $\pi(K)^t \circ f \in \operatorname{Hom}_{kW_K}(k_{W_K}, \operatorname{Ker} d_K^{(t)})$.

**Lemma 3.4.** *In the notations above,* $\operatorname{res}_{W,W_K}(co(f)) = co(\pi(K)^t \circ f)$.

Assume that we have a decomposition $S = K \cup L$ such that $st = ts$ for all $s \in K$ and $t \in L$. Let us denote differentials for $\mathbf{A}(W_K,K)$ and $\mathbf{A}(W_L,L)$ by $d_1^{(i)}$ and $d_2^{(j)}$, respectively. Let $g \in \operatorname{Hom}_{kW_K}(k_{W_K}, \operatorname{Ker} d_1^{(i)})$ and $h \in \operatorname{Hom}_{kW_L}(k_{W_L}, \operatorname{Ker} d_2^{(j)})$. Then $\operatorname{Ker} d_1^{(i)} \otimes \operatorname{Ker} d_2^{(j)} \subset \operatorname{Ker} d^{(i+j)}$ under the isomorphism given in the statement (4) in Theorem 3.2. Thus we obtain $g \otimes h \in \operatorname{Hom}_{kW}(k_W, \operatorname{Ker} d_1^{(i)} \otimes \operatorname{Ker} d_2^{(j)}) \subset \operatorname{Hom}_{kW}(k_W, \operatorname{Ker} d^{(i+j)})$.

**Lemma 3.5.** *In the notations above, we have*

$$co(g \otimes h) = co(g) \otimes co(h)$$

*under the isomorphism* $H^*(W,k) \cong H^*(W_K,k) \otimes H^*(W_L,k)$.

3.2. **Some Cohomology Elements in $H^*(\Sigma_n, k)$.**

The symmetric group $\Sigma_n$ of degree $n$ is a finite Coxeter group with generating set $S = \{ \sigma_i = (i\ i+1),\ 1 \leqq i \leqq n-1 \}$ of $A_{n-1}$-type. The parabolic subgroups are what are called Young subgroups of $\Sigma_n$. We set $W = \Sigma_n$ in this subsection. For notaional simplicity, we sometimes write $S = \{1, 2, \cdots, n-1\}$ and for $I \subset S$, $I = \{ i\ ;\ \sigma_i \in I\}$. From the complex $\mathbf{A}(\Sigma_n) = \mathbf{A}(W)$, we saw that we obtain an $(n-1)$-fold extension of $k_W$ by itself. We denote the corresponding cohomology element $co(id_W) \in H^{n-1}(W, k)$ by $c_0(W) = c_0(\Sigma_n)$. The following proposition follows by Lemma 3.3.

**Proposition 3.6.** *In the notations above, the following statements hold.*

(1) $\mathrm{res}_{W,W_K}(c_0(W)) = 0$ *for any proper parabolic subgroup* $W_K$ *of* $W$. *In particular,* $c_0(W) = 0$ *if* $n$ *is not a power of* 2.

(2) *Assume that* $n = 2^s$ *and let* $E$ *be a maximal regular elementary abelian* 2*-subgroup (of order* $2^s$*) of* $W = \Sigma_{2^s}$. *Then*

 (a) $\mathrm{res}_{W,E}(c_0(W)) \neq 0$.

 (b) $\mathrm{res}_{W,F}(c_0(W)) = 0$ *for all proper subgroup* $F$ *of* $E$.

In the rest, assume that $n = 2^s$ for some $s$. Set $W = \Sigma_{2^s}$ and $E$ be a maximal regular elementary abelian 2-subgroup of $\Sigma_{2^s}$. By the distinguished properties of Dickson invariant $c_0(E)$, we can see that $\mathrm{res}_{W,E}(c_0(W)) = c_0(E)$ by Proposition 3.6.

For each $t$ with $0 \leqq t \leqq s-1$, we shall construct a cohomology element $c_t = c_t(\Sigma_{2^s}) = c_t(W)$ such that $\mathrm{res}_{W,E}(c_t) = c_t(E)$. Let fix such $t$ in the following disucussion. Set, for $1 \leqq i \leqq 2^t$,

$$K_i = K_i^{(t)} = \{ (i-1)2^{s-t} + 1, (i-1)2^{s-t} + 2, \cdots, (i-1)2^{s-t} + 2^{s-t} - 1\}$$

$$K = K^{(t)} = \cup_{i=1}^{2^t} K_i^{(t)} = S \setminus \{ i2^{s-t}\ ;\ 1 \leqq i \leqq 2^t - 1\}$$

Then $W_{K_i} \cong \Sigma_{2^{s-t}}$ and $W_K = W_{K_1} \times \cdots \times W_{K_{2^t}}$. If $t = 0$, then $K = S$ and $W_K = W$. If $t = s-1$, then $K = \{ 1, 3, \cdots, 2^s - 1 \}$ and $W_K \cong \mathbb{Z}_2{}^{2^{s-1}}$.

Set $k = |K| = 2^t(2^{s-t} - 1)$ and consider $[K]kW \subset A^k$. We can see that a $W$-invariant element $u(K) = u(K^{(t)}) = \sum_{w \in W} w = [K] \sum_{x \in W_K \backslash W} x \in [K]kW \subset A^k$ is contained in $\mathrm{Ker}\, d^k = \mathrm{Im}\, d^{k-1}$.

Let $f_{K^{(t)}} : k_W \to \mathrm{Im}\, d^{k-1}$ be a $kW$-homomorphim sending 1 to $u(K^{(t)})$. Then we obtain a cohomology element $co(f_{K^{(t)}}) \in H^k(W, k) = H^{2^t(2^{s-t}-1)}(W, k)$. The cohomology algebra $H^*(W_K, k)$ is isomorophic to $H^*(W_{K_1}, k) \otimes \cdots \otimes H^*(W_{K_{2^t}}, k)$.

Denote $co(f_{K^{(t)}})$ by $c_t(W) = c_t(\Sigma_{2^s})$. The statements (1) and (2) in the following proposition will be obtained as an easy consequence of Lemmas 3.4 and 3.5. The statement (3) follows by the distinguished properties of Dickson invariant $c_t(E)$.

**Proposition 3.7.** *Assume that* $n = 2^s$. *Then the following statements hold.*

(1) $\mathrm{res}_{W,W_K}(c_t(W)) = c_0(W_{K_1}) \otimes \cdots \otimes c_0(W_{K_m})$.

(2) *For* $I \subset S$, *if* $\mathrm{res}_{W,W_K}(c_t(W)) \neq 0$, *then* $I \supseteq K$.

(3) $\mathrm{res}_{W,E}(c_t(W)) = c_t(E)$.

## 4. The Spin Module $D^{(m+1,m-1)}$ of $\Sigma_{2m}$

Set $\Omega = \{1,2,\cdots,2m\} \subset \Omega_0 = \{0,1,2,\cdots,2m,2m+1\}$ and $\Sigma_{2m} = \Sigma_\Omega \subset \Sigma_{2m+2} = \Sigma_{\Omega_0}$. The spin module $D^{(m+2,m)}$ is of dimension $2^m$. Using a presentation of the spin module $D^{(m+2,m)}$ given by Nagai and Uno [14], we have the following lemma.

**Lemma 4.1.** *Set $\tau_i = (i\ 2m+1)$, $0 \leqq i \leqq 2m$. Then for $\sigma \in \Sigma_{2m}$, $\sigma\tau_i\sigma^{-1} = \tau_{\sigma(i)}$ and the following statements hold.*

(1) *For a subset $\Lambda \subset \Omega$, $\sum_{i\in\Lambda}\tau_i$ is centralized by $\Sigma_\Lambda \times \Sigma_{\Omega\setminus\Lambda}$.*
(2) *The following elements in $k\Sigma_{2m+2}$ annihilate $D^{(m+2,m)}$.*
  (a) *For $i \neq j$, $\tau_i + \tau_j + (i\ j)$.*
  (b) *For $i \neq j$, $\tau_i\tau_j + \tau_i\tau_j + 1$.*
  (c) *For three distinct $i,j,k$, $(\tau_i+\tau_j)\tau_k + \tau_k(\tau_i+\tau_j)$.*
  (d) $(m+1)\cdot 1 + \tau_0 + \sum_{i=1}^{m}\sigma_{2i-1}$.
(3) *The following elements in $k\Sigma_{2m+2}$ annihilate $D^{(m+2,m)}(\tau_0+1)$.*
  (a) $\tau_0 + 1$.
  (b) *For $i \neq 0$, $1 + \tau_i(\tau_0+1)$*
  (c) *For a subset $\Lambda \subset \Omega$, $|\Lambda|\cdot 1 + (\sum_{i\in\Lambda}\tau_i)(\tau_0+1)$.*
(4) *Let $\Sigma_3 = \langle\sigma_1,\sigma_2\rangle$ and $\Sigma_2 = \langle\sigma_1\rangle$. Then*
  (a) *$\sigma_1$ is $\Sigma_2$-invariant and $\mathrm{Tr}_{\Sigma_2,\Sigma_3}(\sigma_1)$ annihilates $D^{(m+2,m)}$.*
  (b) *$\sigma_1\tau_3$ is $\Sigma_2$-invariant and $1 + \mathrm{Tr}_{\Sigma_2,\Sigma_3}(\sigma_1\tau_3)$ annihilates $D^{(m+2,m)}$.*
(5) *Let $\Sigma_4 = \langle\sigma_1,\sigma_2,\sigma_3\rangle$ and $\Sigma_2\times\Sigma_2 = \langle\sigma_1\rangle\times\langle\sigma_3\rangle$. Then $\sigma_1\sigma_3$ is $(\Sigma_2\times\Sigma_2)$-invariant and $\mathrm{Tr}_{\Sigma_2\times\Sigma_2,\Sigma_4}(\sigma_1\sigma_3)$ annihilates $D^{(m+2,m)}$.*

### 4.1. The Restriction of $D^{(m+2,m)}$ to $\Sigma_{2m}$.

The restriction of the spin module $D^{(m+2,m)}$ of $\Sigma_{2m+2}$ to $\Sigma_{2m}$ is a self extension of the spin module $D^{(m+1,m-1)}$ of $\Sigma_{2m}$. This is isomorphic to the restriction of the spin module $D^{(m+1,m)}$ of $\Sigma_{2m+1}$ to $\Sigma_{2m}$ and described as follows.

Set $M = M^{(m+1,m-1)} = D^{(m+2,m)}\downarrow_{\Sigma_{2m}}$ and $D = M(\tau_0+1) \subset M$ where $\tau_0 = (0\ 2m+1)$. $\tau_0$ centralizes $\Sigma_{2m}$ and therefore $D$ is a $k\Sigma_{2m}$-submodule of $M$. We can see that $D \cong D^{(m+1,m-1)}$ and $D = \{u \in M\ ;\ u(\tau_0+1) = 0\}$ and we have a short exact sequence of $k\Sigma_{2m}$-modules,

$$0 \to D \xrightarrow{g} M \xrightarrow{f} D \to 0 \tag{1}$$

where $f$ is the multiplication map by the element $\tau_0 + 1$ and $g$ is the inclusion map.

Let $k \in \Omega$ and consider $\Sigma_{\Omega\setminus\{k\}} \cong \Sigma_{2m-1}$. Set $\tau_k = (k\ 2m+1)$. Then $\langle\tau_0,\ \tau_k\rangle \cong \Sigma_3$ centralizes $\Sigma_{\Omega\setminus\{k\}}$. By Lemma 4.1, we have the following equalities on the effects on $M$ by multiplying the following elements,

$$\cdot(\tau_0+1)\tau_k(\tau_0+1) = \cdot(\tau_0+1)(\tau_0\tau_k\tau_0+\tau_k) = \cdot(\tau_0+1)\tau_0 = \cdot(\tau_0+1)$$

Thus if we define a map $h_k : M \to M$ by $h_k(u) = u\cdot\tau_k$, $u \in M$, then $h_k$ is a $k\Sigma_{\Omega\setminus\{k\}}$-homomorphism and

$$f \circ h_k \circ g = id_D$$

Thus the short exact sequence (1) splits as a sequence of $k\Sigma_{\Omega\setminus\{k\}}$-modules.

Let $\rho \in \mathrm{Ext}^1_{k\Sigma_{2m}}(D,D) \subset \mathrm{Ext}^*_{k\Sigma_{2m}}(D,D)$ be the cohomology element determined by the exact sequence (1).

We shall use notations from the notion of Coxeter groups introduced in Section 3. We shall investigate the cohomology element $\rho$.

Set $S = \{\sigma_k \ ; \ 1 \leqq k \leqq 2m-1\}$ and $\Sigma_{2m} = W = (W,S)$. We use the letter $\mathbf{X}$, $X^t$ to describe the complex $\mathbf{A}(W,S)$.

$$\tilde{\mathbf{X}} \qquad 0 \to k \xrightarrow{\iota} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \to X^{2m-3} \xrightarrow{d^{2m-3}} X^{2m-2} \xrightarrow{d^{2m-2}} k \to 0$$

If $|I| \leqq m-1$, then $W_I \subset \Sigma_{\Omega\setminus\{k\}}$ for some $k \in \Omega$. We have observed that the sequence $0 \to D \xrightarrow{g} M \xrightarrow{f} D \to 0$ is $\Sigma_{\Omega\setminus\{k\}}$-split. Thus we can lift the identity map $id_D : D \to D$ to have the following commutative diagram of $kW$-modules,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{g} & M & \xrightarrow{g\circ f} & M & \xrightarrow{g\circ f} & \cdots \\
 & & \| & & \downarrow{\scriptstyle \lambda^0} & & \downarrow{\scriptstyle \lambda^1} & & \\
0 & \longrightarrow & D & \xrightarrow{id_D\otimes\iota} & D\otimes X^0 & \xrightarrow{id_D\otimes d^0} & D\otimes X^1 & \xrightarrow{id_D\otimes d^1} & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M & \xrightarrow{g\circ f} & M & \xrightarrow{f} & D & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda^{m-2}} & & \downarrow{\scriptstyle \lambda^{m-1}} & & \downarrow{\scriptstyle \mu^m} & & \\
\cdots & \longrightarrow & D\otimes X^{m-2} & \xrightarrow{id_D\otimes d^{m-2}} & D\otimes X^{m-1} & \xrightarrow{id_D\otimes d^{m-1}} & D\otimes \mathrm{Im}\, d^{m-1} & \longrightarrow & 0
\end{array}$$

$\mu^m$ is uniquely determined modulo maps which through the map $id_D \otimes d^{m-1}$.

In the following discussion, we shall give maps $\lambda^t$ $(0 \leqq t \leqq m-1)$ and the resulting map $\mu^m$ in degree $m$-term explicitly.

We have an isomorphism $\mathrm{Hom}_{kW}(M, D\otimes[I]kW) \cong \mathrm{Hom}_{kW_I}(M\downarrow_{W_I}, D)$ given by $\mathrm{Hom}_{kW}(M, D\otimes[I]kW) \ni \lambda \leftrightarrow \hat{\lambda} \ni \mathrm{Hom}_{kW_I}(M\downarrow_{W_I}, D)$, where

$$\lambda(v) = \sum_{x\in W_I\setminus W} \hat{\lambda}(vx^{-1})x \otimes [I]x, \ v \in M$$

If $I \subset K$, then $(id_D\otimes\pi_{I\ K})\circ\lambda \in \mathrm{Hom}_{kW}(M, D\otimes[K]kW)$ corredponds to $\mathrm{Tr}_{W_I,W_K}(\hat{\lambda}) \in \mathrm{Hom}_{kW_K}(M\downarrow_{W_K}, D)$.

The maps we shall define are multiplication maps of elements in $k\Sigma_{\Omega_0}$. Let $\tau_i = (i\ 2m+1)$ $0 \leqq i \leqq 2m$ and set

$$\tau(0) = \tau_1, \quad \tau(i) = \sigma_1\sigma_3\cdots\sigma_{2i-1}\cdot\{\sigma_1+\sigma_3+\cdots+\sigma_{2i-1}+\tau_{2i+1}\} \ (1 \leqq i \leqq m-1)$$

And let $I_i$ be an $i$-points subset of $S$ given by

$$I_i = \{1,3,\cdots,2i-1\}, \qquad 1 \leqq i \leqq m$$

Set $\sigma = \sigma_1\sigma_3\cdots\sigma_{2m-1}$, $\rho_i = (2i-1\ 2i+1)(2i\ 2i+2)$, $1 \leqq i \leqq m-1$ and $F = \langle \rho_1, \cdots, \rho_{m-1} \rangle$. Then

$$C_W(\sigma) = W_{I_m} \rtimes F, \quad F \cong \Sigma_m$$

Define $\lambda^0 : M \to D \otimes X^0$ as follows. For $v \in M$,

$$\lambda^0(v) = \sum_{x \in W_\varnothing \backslash W} vx^{-1} \cdot \{\tau(0)(\tau_0 + 1)\} x \otimes [\varnothing] x$$

$$= \sum_{x \in W_\varnothing \backslash W} vx^{-1} \cdot \{\tau_1(\tau_0 + 1)\} x \otimes [\varnothing] x \in D \otimes [\varnothing] kW = D \otimes X^0$$

For each $i$ with $1 \leqq i \leqq m-1$, define $\lambda^i : M \to D \otimes X^i$ as follows. For $v \in M$,

$$\lambda^i(v) = \sum_{x \in W_{I_i} \backslash W} vx^{-1} \cdot \{\tau(i)(\tau_0 + 1)\} x \otimes [I_i] x \in D \otimes [I_i] kW \subset D \otimes X^i$$

Define $\mu^m : D \to D \otimes X^m$ as follows. For $u \in D$,

$$\mu^m(u) = \sum_{x \in W_{I_m} \backslash W} ux^{-1} \cdot \{\sigma_1 \sigma_3 \cdots \sigma_{2m-1}\} x \otimes [I_m] x$$

$$= \sum_{x \in C_W(\sigma) \backslash W} ux^{-1} \sigma x \otimes [C_W(\sigma)] x \in D \otimes [I_m] kW \subset D \otimes X^m$$

where $[C_W(\sigma)] = \sum_{h \in C_W(\sigma)} h = [I_m] \sum_{y \in F} y$.

**Lemma 4.2.** *The maps $\lambda^i$ $(0 \leqq i \leqq m-1)$ and $\mu^m$ defined above make the diagram given above to be commutative.*

A proof uses Lemma 4.1 and will be done direct calculations.

4.2. **The Restriction of $D^{(m+1,m-1)}$ to $\Sigma_m \times \Sigma_m$, $\Sigma_m \wr \Sigma_2$, $m = 2\ell$.**

In this subsection let $m = 2\ell$. We shall investigate the restriction of $D^{(m+1,m-1)}$ to $\Sigma_m \times \Sigma_m$. Set

$$S_1 = \{\sigma_1, \sigma_2, \ldots, \sigma_{2\ell-1}\}, \quad S_2 = \{\sigma_{2\ell+1}, \sigma_{2\ell+3}, \cdots, \sigma_{2m-1}\}, \quad T = S_1 \cup S_2 = S \backslash \{\sigma_{2\ell}\}$$

$$\Omega_1 = \{1, 2, \cdots, 2\ell\}, \quad \Omega_2 = \{2\ell+1, 2\ell+2, \cdots, 2m\}$$

$$W_{S_1} = \Sigma_{\Omega_1} \cong \Sigma_{2\ell}, \quad W_{S_2} = \Sigma_{\Omega_2} \cong \Sigma_{2\ell}, \quad W_T = W_1 \times W_2$$

$W_T$ is a Coxeter group with generating set $T$ of type $A_{2\ell-1} \times A_{2\ell-1}$. Set

$$\mu = (1\ 2\ell+1)(2\ 2\ell+2) \cdots (i\ 2\ell+i) \cdots (2\ell\ 2m)$$

Then $W_{S_1}^{\mu} = W_{S_2}$ and we can consider a semidiect product

$$W_T \subset (W_{S_1} \times W_{S_2}) \rtimes \langle \mu \rangle \cong \Sigma_m \wr \Sigma_2$$

In fact, $(W_{S_1} \times W_{S_2}) \rtimes \langle \mu \rangle = N_W(W_T)$. We denote this subgroup by $W_0$.

For $1 \leqq i \leqq 2\ell - 1$,

$$\sigma_i^{\mu} = \sigma_{2\ell+i}$$

so that $\mu$ acts on $T$. Consider the complex $\mathbf{Y} = \mathbf{A}(W_T, T)$ associated to $W_T$. Each term $Y^t = \oplus \sum_{I \subset T,\ |I|=t} [I] kW_T$ of $\mathbf{Y}$ becomes $kW_0$-module by defining the action of $\mu$ as follows. For $[I]\gamma \in [I] kW_T \subset Y^t$, define $[I]\gamma \cdot \mu$ to be

$$[I]\gamma \cdot \mu = [I^{\mu}]\gamma^{\mu} \in [I^{\mu}] kW_T \subset Y^t$$

It is not hard to see that the differentials $d^t$ commute with this action of $\mu$. Thus $\mathbf{Y}$ becomes a complex of $kW_0$-modules.

By Lemma 4.1, the action of the following two elements

$$\ell \cdot 1 + \sigma_1 + \sigma_3 + \cdots + \sigma_{2\ell-1}, \quad \ell \cdot 1 + \sigma_{2\ell+1} + \sigma_{2\ell+3} + \cdots + \sigma_{2m-1}$$

on $D^{(m+1,m-1)}$ coincide. Thus the action of $\tau$ on $D^{(m+1,m-1)}$ commutes with those of elements in $W_T = W_{S_1} \times W_{S_2}$ and of $\mu$. In this subsection, set

$$D = D^{(m+1,m-1)} \downarrow_{W_0}, \quad L = D \cdot \tau \subset D$$

$L$ is a $kW_0$-submodule of $D$ and $L = \{u \in D_0 \ ; \ u\tau = 0\}$ as $\tau^2 = 0$. So we have a short exact sequence of $kW_0$-modules

$$0 \longrightarrow L \xrightarrow{g_0} D \xrightarrow{f_0} L \longrightarrow 0 \tag{2}$$

where $f_0$ is a multiplication map by $\tau$ and $g_0$ is an inclusion map.

If we denote the spin module $D^{(\ell+1,\ell-1)}$ of $\Sigma_m = \Sigma_{2\ell}$ by $D_1$, then we can see that

$$L \cong D_1^{\otimes \Sigma_m \wr \Sigma_2}, \quad L \downarrow_{W_{S_1} \times W_{S_2}} \cong D_1 \otimes D_1$$

where $D_1^{\otimes \Sigma_m \wr \Sigma_2}$ is a tensor induction.

Let $\eta \in \mathrm{Ext}^1_{k[\Sigma_m \wr \Sigma_2]}(L, L) \subset \mathrm{Ext}^*_{k[\Sigma_m \wr \Sigma_2]}(L, L)$ be the cohomology element determined by the exact sequence (2).

We shall construct maps $\gamma^i : D \longrightarrow L \otimes Y^i$ and $\delta^{i+1} = (id_L \otimes d^{i+1}) \circ \gamma^i$ which give the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{g_0} & D & \xrightarrow{g_0 \circ f_0} & D & \xrightarrow{g_0 \circ f_0} & \cdots \\
 & & \Vert & & \downarrow{\scriptstyle \gamma^0} & & \downarrow{\scriptstyle \gamma^1} & & \\
0 & \longrightarrow & L & \xrightarrow{id_L \otimes \iota} & L \otimes Y^0 & \xrightarrow{id_L \otimes d^0} & L \otimes Y^1 & \xrightarrow{id_L \otimes d^1} & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{g_0 \circ f_0} & D & \xrightarrow{g_0 \circ f_0} & D & \xrightarrow{f_0} & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma^{\ell-2}} & & \downarrow{\scriptstyle \gamma^{\ell-1}} & & \downarrow{\scriptstyle \delta^{\ell}} & & \\
\cdots & \xrightarrow{id_L \otimes d^{\ell-3}} & L \otimes Y^{\ell-2} & \xrightarrow{id_L \otimes d^{\ell-2}} & L \otimes Y^{\ell-1} & \xrightarrow{id_L \otimes d^{\ell-1}} & L \otimes \mathrm{Im}\, d^{\ell-1} & \longrightarrow & 0
\end{array}$$

Set

$$\sigma(0,0) = (1 \ 2\ell+1), \quad \sigma(0,0)_0 = 1, \quad \sigma(1,1)_0 = \sigma_1, \quad \sigma(0,1)_0 = \sigma_{2\ell+1}$$

For $1 \leqq i \leqq \ell$, set

$$\sigma(i,i) = (\sigma_1 \sigma_3 \cdot \sigma_{2i-1}) \cdot \{(\sigma_1 + \sigma_3 + \cdots + \sigma_{2i-1}) + (2i+1 \ 2\ell+1)\}$$

$$\sigma(0,i) = (\sigma_{2\ell+1}\sigma_{2\ell+3} \cdots \sigma_{2\ell+2i-1}) \cdot \{(\sigma_{2\ell+1} + \cdots + \sigma_{2\ell+2i-1}) + (1 \ 2\ell+2i+1)\}$$

For $1 \leqq j < i \leqq \ell$, set

$$\begin{aligned}
&\sigma(j,i) \\
&= (\sigma_1 \cdots \sigma_{2j-1})(\sigma_{2\ell+1} \cdots \sigma_{2\ell+2(i-j)-1}) \\
&\quad \cdot \{(\sigma_1 + \cdots + \sigma_{2j-1}) + (\sigma_{2\ell+1} + \cdots + \sigma_{2\ell+2(i-j)-1}) + (2j+1 \ 2\ell+2(i-j)+1)\}
\end{aligned}$$

$$\sigma(i,i)_0 = \sigma_1 \sigma_3 \cdot \sigma_{2i-1}, \quad \sigma(0,i)_0 = \sigma_{2\ell+1}\sigma_{2\ell+3} \cdots \sigma_{2\ell+2i-1}$$

$$\sigma(j,i)_0 = (\sigma_1 \cdots \sigma_{2j-1})(\sigma_{2\ell+1} \cdots \sigma_{2\ell+2(i-j)-1})$$

We have the following equalities.

$$\sigma(j,i)^{\mu} = \sigma(i-j,i), \qquad \sigma(j,i)_0^{\mu} = \sigma(i-j,i)_0$$

By Lemma 4.1, for $0 \leqq j \leqq \ell - 1$, we have the following equality on the action on $D$

$$\begin{aligned}&\tau \cdot (2j+1\ 2\ell+2(i-j)+1) + (2j+1\ 2\ell+2(i-j)+1) \cdot \tau \\ &\quad = \sigma_{2j+1} \cdot (2j+1\ 2\ell+2(i-j)+1) + (2j+1\ 2\ell+2(i-j)+1) \cdot \sigma_{2j+1} \equiv 1\end{aligned}$$

Thus the following lemma holds.

**Lemma 4.3.** *For $u \in L$, $u \cdot \sigma(j,i)\tau = u \cdot \sigma(j,i)_0$ for each $0 \leqq j \leqq i \leqq \ell - 1$.*

For each $1 \leqq i \leqq \ell$, we define the certain $i$-elements subsets of $\{1, 3, \cdots, 2\ell - 1, 2\ell+1, \cdots, 2m-1\}$ as follows,

$$\begin{aligned}&I(i,i) = \{1, 3, \cdots, 2i-1\} \\ &I(j,i) = \{1, 3, \cdots, 2j-1, 2\ell+1, \cdots, 2\ell+2(i-j)-1\},\ 1 \leqq j \leqq i-1 \\ &I(0,i) = \{2\ell+1, 2\ell+3, \cdots, 2\ell+2i-1\}\end{aligned}$$

Then

$$I(j,i)^{\mu} = I(i-j,i)$$

And we set $W_{I(j,i)} = W(j,i) \subset W$.

**Lemma 4.4.** *$\sigma(j,i)$ and $\sigma(j,i)_0$ are $W(j,i)$-invariant. Let $k \notin I(j,i)$ and set $I = I(j,i) \cup \{k\}$. Then the following statements hold.*

(1) *If $k \neq 2j+1,\ 2\ell+2(i-j)+1$, then $\mathrm{Tr}_{W(j,i),W_I}(\sigma(j,i)) \equiv 0$.*
(2) *If $k = 2j+1$, then $I = I(j+1,i+1)$ and $\mathrm{Tr}_{W(j,i),W(j+1,i+1)}(\sigma(j,i)) \equiv \sigma(j+1,i+1)_0$.*
(3) *If $k = 2\ell+2(i-j)+1$, then $I = I(j,i+1)$ and $\mathrm{Tr}_{W(j,i),W(j,i+1)}(\sigma(j,i)) \equiv \sigma(j,i+1)_0$.*

Let

$$i = a_0 + a_1 \cdot 2 + \cdots + a_s \cdot 2^s, \quad j = b_0 + b_1 \cdot 2 + \cdots + b_s \cdot 2^s, \quad 0 \leqq a_u, b_v \leqq 1$$

be the 2-adic expansions of $i$ and $j$. We say that $i$ contains $j$, or $j$ is contained in $i$ provided that $b_0 \leqq a_u$ for all $u$. We write $i \supseteq j$ or $j \subseteq i$ if $i$ contains $j$. Notice that the definition here is slightly different from Definition 24.12 [9]. 0 and $i$ are always contained in $i$.

We have an isomorphism $\mathrm{Hom}_{kW_T}(D, L \otimes [I]kW_M) \cong \mathrm{Hom}_{kW_I}(D\downarrow_{W_I}, L)$ given by $\mathrm{Hom}_{kW_T}(D, L \otimes [I]kW_T) \ni \gamma \leftrightarrow \hat{\gamma} \in \mathrm{Hom}_{kW_I}(D\downarrow_{W_I}, L)$, where

$$\gamma(v) = \sum_{x \in W_I \backslash W} \hat{\gamma}(vx^{-1})x \otimes [I]x,\ v \in D$$

If $I \subset K$, then $(id_L \otimes \pi_{I\ K}) \circ \gamma \in \mathrm{Hom}_{kW_T}(D, L \otimes [K]kW_T)$ corredponds to $\mathrm{Tr}_{W_I,W_K}(\hat{\gamma}) \in \mathrm{Hom}_{kW_K}(D\downarrow_{W_K}, L)$.

Define $\gamma^0 : D \to L \otimes Y^0$ as follows. For $u \in D$,

$$\gamma^0(u) = \sum_{x \in W_{\varnothing} \backslash W_T} u \cdot x^{-1}\sigma(0,0)\tau x \otimes [\varnothing]x) \in L \otimes [\varnothing]kW_T = L \otimes Y^0$$

For $1 \leqq i \leqq \ell - 1$, define $\gamma^i : D \to L \otimes Y^i$ as follows. For $u \in D$,

$$\gamma^i(u) = \sum_{j \subseteq i} \left\{ \sum_{x \in W(j,i) \backslash W_T} u \cdot x^{-1} \sigma(j,i) \tau x \otimes [I(j,i)]x \right\} \in \oplus \sum_{j \subseteq i} L \otimes [I(j,i)] kW_T \subset L \otimes Y^i$$

We have observed that $\sigma(j,i)$ is $W(j,i)$-invariant. So the maps $\gamma^i$ are $kW_T$-homomorphsms. And we can see easily that $\gamma^i$ commute with action of $\mu$ so that $\gamma^i$ are $kW_0$-homomorphisms.

For $1 \leqq i \leqq \ell$, define $\delta^i : L \to L \otimes Y^i$ as follows. For $u \in L$,

$$\delta^i(u) = \sum_{j \subseteq i} \left\{ \sum_{x \in W(j,i) \backslash W_T} u \cdot x^{-1} \sigma(j,i)_0 x \otimes [I(j,i)]x \right\} \in \oplus \sum_{j \subseteq i} L \otimes [I(j,i)] kW_T \subset L \otimes Y^i$$

We can see that $\delta^i$ is also a $kW_0$-homomorphism.

**Lemma 4.5.** *The maps $\gamma^i$ $(0 \leqq i \leqq \ell - 1)$ and $\delta^\ell$ defined above make the diagram given above to be commutative.*

Assume that $n = 2m = 2^s$ for some integer $s \geqq 2$ so that $m = 2^{s-1}, \ell = 2^{s-2}$. Then only 0 and $\ell$ are contained in $\ell$ and

$$I(\ell,\ell) = \{1, 3, \cdots, 2\ell - 1\}, \quad \sigma(\ell,\ell)_0 = \sigma_1 \sigma_3 \cdots \sigma_{2\ell-1}$$

$$I(0,\ell) = \{2\ell+1, 2\ell+3, \cdots, 2m-1\}, \quad \sigma(0,\ell)_0 = \sigma_{2\ell+1}\sigma_{2\ell+3} \cdots \sigma_{2m-1}$$

Set

$$I(1) = I(\ell,\ell) = \{1, 3, \cdots, 2\ell - 1\} \subset S_1$$

$$\sigma(1) = \sigma(\ell,\ell)_0 = \sigma_1 \sigma_3 \cdots \sigma_{2\ell-1} \in W_{I(1)}$$

$$I(2) = I(0,\ell) = \{2\ell+1, 2\ell+3, \cdots, 2m-1\} \subset S_2$$

$$\sigma(2) = \sigma(0,\ell)_0 = \sigma_{2\ell+1}\sigma_{2\ell+3} \cdots \sigma_{2m-1} \in W_{I(2)}$$

Then

$$\begin{aligned}
\delta^\ell(u) &= \sum_{x \in W_{I(1)} \backslash W_T} u \cdot x^{-1}\sigma(1)x \otimes [I(1)]x + \sum_{x \in W_{I(2)} \backslash W_T} u \cdot x^{-1}\sigma(2)x \otimes [I(2)]x \\
&= \sum_{x \in C_{W_{S_1}}(\sigma(1)) \backslash W_{S_1}} u \cdot x^{-1}\sigma(1)x \otimes [C_{W_{S_1}}(\sigma(1))]x \cdot [S_2] \\
&\qquad + \sum_{x \in C_{W_{S_2}}(\sigma(2)) \backslash W_{S_2}} u \cdot x^{-1}\sigma(2)x \otimes [S_1] \cdot [C_{W_{S_2}}(\sigma(2))]x
\end{aligned}$$

## 5. The Varieties $V_E(D^{(m+1,m-1)})$, $V_E(M^{(m+1,m-1)})$ for $\Sigma_{2m}$ with $m = 2^{s-1}$

Assume that $n = 2m = 2^s$ for some integer $s \geqq 2$ and let $E$ be a maximal regular elementary abelian 2-subgroup of $\Sigma_{2^s}$. Let $D$ and $M$ be $k\Sigma_n$-modules and $L$ be a $k\Sigma_m \wr \Sigma_2$-module defined in Section 4. Let $\mu \in \Sigma_m \wr \Sigma_2$ be a regular involution and set $\Gamma = C_{\Sigma_m \wr \Sigma_2}(\mu)$. Then $\Gamma = \Delta(\Sigma_m) \times \langle \mu \rangle$ where $\Delta(\Sigma_m)$ is the diagonal subgroup in $\Sigma_m \times \Sigma_m \subset \Sigma_m \wr \Sigma_2$. We shall propose a problem concerning the cohomology elements $\rho \in \mathrm{Ext}^1_{k\Sigma_n}(D, D)$ and $\eta \in \mathrm{Ext}^1_{k\Sigma_m \wr \Sigma_2}(L, L)$. If $n = 2^2$, then we can see that

$$\mathrm{res}_{\Sigma_n, E}(\rho^2) = c_1(E) \otimes id_D, \qquad \mathrm{res}_{\Sigma_m \wr \Sigma_2,\ \Gamma}(\eta) = 0$$

**Problem 5.1.** *Assume that $n = 2^s$. Then are the following statements true ?*

(1) *There exists a positive integer $k$, $t$ such that* $\mathrm{res}_{\Sigma_n, E}(\rho^k) = c_{s-1}(E)^t \otimes id_D$.
(2) *There exists a positive integer $u$ such that* $\mathrm{res}_{\Sigma_m \wr \Sigma_2,\ \Gamma}(\eta^u) = 0$.

By an inductive argument and Lemma 2.1, we have the following remark.

**Remark 5.2.** *If the problems have affirmative answers, then the following statements hold.*

(1) $V_E(D^{(m+1,m)} \downarrow_E) = V_E(c_{s-1}(E))$.
(2) $V_E(D^{(m+1,m-1)} \downarrow_E) = V_E(k)$.

*In particular, Conjecture 1.2 is true.*

## References


[1] A.Adem and R.J.Milgram, Cohomology of Finite Groups, Grundlehren der mathematischen Wissenschaften 309, Springer-Verlag, 1995

[2] G.S.Avrunin and L.L.Scott, *Quillen stratification for modules*, Invent. Math. 66(1982), 277-286

[3] D.J.Benson, Representations and Cohomology I,II, Cambridge Studies in Advanced Mathematics, 30,31, Cambridge University Press, 1991

[4] D.J.Benson, Polynomial Invariants of Finite Groups , London Math. Soc. Lecture Note Series, 190, Cambridge University Press, 1993

[5] D.J.Benson, *Spin modules for symmetric groups*, J. London Math. Soc.(2) 38(1988), 250–262

[6] J.F.Carlson, *The variety of an indecomposable module is connected*, Invent. Math., 77(1984), 291–299

[7] S.Danz and B.Külshammer, *On the vertices of the basic spin module for the symmetric group in characteristic* 2 Preprint, 2008

[8] R.Gow and A.Kleshchev, *Connections between the representations of the symmetric group and the symplectic group in characteritic* 2, J

[9] G.D.James, The Representation Theory of the Symmetric Groups, J. of Algebra, 221, 60–89, 221, 1999

[10] T.Nagai, *The rank variety of the spin module of a symmetric groups over a field of characteritic* 2, In Japanese, Master's thesis , Osaka University, 2002

[11] T.Okuyama, *On a certain simple module and cohomology of the symmetric group over* $GF(2)$, RIMS Kokyuroku, 1581, Cohomology Theory of Finite Groups and Related Topics (in Japanese「有限群のコホモロジー論とその周辺」), 58-63, 2008

[12] D.Quillen, *The spectrum of an equiivariant cohomology ring, I,II*, Ann. of Math. 94(1971), 549–572, 573–602

[13] J.Rickard, *Splendid equivalences: Derived categories and permutation modules*, Proc. London Math. Soc.(3), 212, 512–523, 1995

[14] K. Uno, *The rank variety of a simple module of a symmetric group*, In Japanese, RIMS Kokyuroku, 1251, Cohomology Theory of Finite Groups and Related Topics (in Japanese「有限群のコホモロジー論とその周辺」), 8–15, 2002