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Recollement of Homotopy Categories and Cohen-Macaulay Modules *

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Abstract

We introduce the homotopy category of unbounded complexes with bounded homologies. We study a recollement of its a quotient by the homotopy category of bounded complexes. This leads to the existence of quotient categories which are equivalent to a homotopy category of acyclic comlpexes, that is a stable derived category. In the case of a coherent ring R of self-injective dimension both sides, we show that the above recollement are triangulated equivalent to a recollement of the stable module category of Cohen-Macaulay R-modules.

1 Introduction

We study two types of triangulated categories in this paper. One is the categories of homotopy classes of chain complexes, equipped with triangles induced by chain maps and mapping cones. The other is stable module categories that are module categories mod projective modules. A stable module category is not triangulated in general. Suppose that the module category is Frobenius; it has enough projectives and injectives, and the class of projectives coincide with that of injectives. Then it's projective stabilization is triangulated. This type of triangulated categories are called algebraic triangulated categories. A well-known example is the stable module category of Cohen-Macaulay modules over Gorenstein rings.

Let R be a two-sided noetherian ring. The catogories of right R-modules, of finitely generated right R-modules and of finitely generated projective right R-modules are denoted by Mod$R$ and mod$R$, and *proj* R respectively. Let $\mathsf{K} = \mathsf{K}(\mathrm{proj}R)$ be the category of homotopy classes of complexes of finitely generated R-projective complexes. The following triangulated subcategories of K are of our concern.

$$\mathsf{K}^{\infty,b} = \{C \in \mathsf{K} \mid \mathrm{H}^i(C) = 0 \text{ (except for finite } i\text{'s)}\}$$

*This is joint work with Osamu Iyama and Jun-ichi Miyachi.

$$\mathsf{K}^{-,b} = \{C \in \mathsf{K}^{\infty,b} \mid C^i = 0 \text{ (for sufficiently large } i)\}$$
$$\mathsf{K}^{\infty,\emptyset} = \{C \in \mathsf{K}^{\infty,b} \mid \mathrm{H}^i(C) = 0 \ (i \in \mathbf{Z})\}$$
$$\mathsf{K}^{b} = \{C \in \mathsf{K} \mid C^i = 0 \text{ (except for finite } i\text{'s)}\}$$

Those triangulated categories are all thick, so we have quotient categories which are triangulated.

Definition 1.1 ([Iw]) *A two-sided noetherian ring is called Iwanaga-Gorenstein if* $\mathrm{id}_R R < \infty$ *and* $\mathrm{id}_{R^{op}} R < \infty$.

If R is an Iwanaga-Gorenstein ring, we define a subcategory $\mathrm{CM}(R)$ of $\mathrm{mod}R$ as $\mathrm{CM}(R) = \{X \in \mathrm{mod}R \mid \mathrm{Ext}^i_R(X, R) = 0 \quad (i > 0)\}$.

Theorem 1.2 (Buchweitz [Bu]) *Assume R is Iwanaga-Gorenstein. The quotient category $\mathsf{K}^{-,b}/\mathsf{K}^b$ is triangle equivalent to the stable module category $\underline{\mathrm{CM}}(R)$.*

On the other hand, we observe the following.

Lemma 1.3 *If R is Iwanaga-Gorenstein, the category $\mathsf{K}^{\infty,b}/\mathsf{K}^{-,b}$ is equivalent to the stable module category $\underline{\mathrm{CM}}(R)$.*

Naturally, the question arises: What is $\mathsf{K}^{\infty,b}/\mathsf{K}^b$? Is it realizable as a stable module category? As an answer, we get the following Buchweitz-type theorem.

Theorem 4.7 Assume R is Iwanaga-Gorenstein. The quotient category $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ is triangle equivalent to the stable module category $\underline{\mathrm{CM}}(T_2(R))$ where $T_2(R)$ is the ring of 2×2 upper triangular matrices over R.

2 The category $\mathsf{K}^{\infty,b}$

For an object A of $\mathsf{K}^{\infty,b}$, define objects X_A and T_A of $\mathsf{K}^{\infty,\emptyset}$ as follows.

Let l be the smallest integer such that $\mathrm{H}_l(A^*) \neq 0$. Then $Cok\, d_A^{l-1}$ is a maximal Cohen-Macaulay module. Define $X_A \in \mathsf{K}^{\infty,\emptyset}$ as

$$\tau_{\leq l} X_A = \tau_{\leq l} A$$

and

$$\cdots \to X_A^{l+1*} \to X_A^{l+2*} \to (Cok\, d_A^{l-1})^* \to 0$$

is exact. Then X_A is totally acyclic and $\mathrm{id}_{Cok\, d_A^{l-1*}}$ induces a canonical chain map $\xi_A : X_A \to A$ as $\xi_A^i = \mathrm{id} \quad (i \leq l)$.

Similarly, let r be the largest integer such that $\mathrm{H}^r(A) \neq 0$. Then $Ker\, d_A^r$ is a maximal Cohen-Macaulay module. Define $T_A \in \mathsf{K}^{\infty,\emptyset}$ as

$$\tau_{\geq r} X_A = \tau_{\geq r} A$$

and

$$\cdots \to T_A^{r-1} \to T_A^r \to (Ker\, d_A^r) \to 0$$

is exact. Then T_A is totally acyclic and $\mathrm{id}_{Ker\, d_A^r}$ induces a canonical chain map $\zeta_A : A \to T_A$ as $\zeta_A^i = \mathrm{id} \quad (i \geq r)$.

Set a chain maps $l_A : L_A \to A$ and $r_{L_A} : L_A \to R_{L_A}$ as follows:

$$\tau_{\leq 0} L_A = \tau_{\leq 0} X_A, \tau_{\geq 1} L_A = \tau_{\geq 1} A,$$

$$\tau_{\leq 0} l_A = \tau_{\leq 0} \xi_A, \tau_{\geq 1} l_A = \tau_{\geq 1} \mathrm{id}_A,$$

$$\tau_{\leq 0} R_{L_A} = \tau_{\leq 0} L_A, \tau_{\geq 1} R_{L_A} = \tau_{\geq 1} T_{L_A},$$

$$\tau_{\leq 0} r_{L_A} = \tau_{\leq 0} \mathrm{id}_{L_A}, \tau_{\geq 1} r_{L_A} = \tau_{\geq 1} \zeta_A$$

Obviously $C(l_A)$ and $C(r_{L_A})$ belongs to K^b, hence as an object of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$, A is isomorphic to the complex

$$R_{L_A} : \cdots \to {X_A}^{-1} \to {X_A}^{0} \to {T_A}^{1} \to {T_A}^{2} \to \cdots$$

We may assume $\lambda_A = \mathrm{H}^0(\tau_{\leq 0} \xi_A \zeta_A) : Cok\, d_{X_A}^{-1} \to Ker\, d_{T_A}^1$ to be surjective by adding some split exact sequence of projective modules if necessary.

3 A functor to the category of morphisms

We define category $\mathrm{Mor}(R)$ as follows: objects of $\mathrm{Mor}(R)$ are the morphisms $\alpha : X_\alpha \to T_\alpha$ of $\mathrm{Mod}(R)$. For $\alpha, \beta \in \mathrm{Mor}(R)$, we define the set of morphisms from α t0 β as

$$\{(f_X, f_T) \in \mathrm{Hom}_R(X_\alpha, X_\beta) \times \mathrm{Hom}_R(T_\alpha, T_\beta) \mid f_T \alpha = \beta f_X\}.$$

And the subcategory $\mathrm{mor}_s^{CM}(R)$ of $\mathrm{Mor}(R)$ consists of the objects $\alpha : X_\alpha \to T_\alpha$ of $\mathrm{CM}(R)$ that are surjective. The structure of $\mathrm{mor}_s^{CM}(R)$ is obtained by the next lemma.

Lemma 3.1 *Let $T_2(R)$ be the category of 2×2 upper triangular matrices with entries in R. Then $\mathrm{Mod}(T_2(R))$ is equivalent to $\mathrm{Mor}(R)$. And $\mathrm{mor}_s^{CM}(R)$ is equivalent to the category $\mathrm{CM}(T_2(R))$.*

proof. An object $f : X_f \to T_f$ of $\mathrm{Mor}(R)$ corresponds to an $T_2(R)$-module $M_f = X_f \times T_f$ where $(x\ t) \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = (xa\ f(x)b + tc)$. This correspondence gives an equivalence between $\mathrm{CM}(T_2(R))$ and $\mathrm{mor}_i^{CM}(R)$ consisting of injective maps $\alpha : X_\alpha \to T_\alpha$ with $X_\alpha, T_\alpha, \mathrm{Cok} f \in \mathrm{CM}(R)$. Obviously $\mathrm{mor}_i^{CM}(R)$ is equivalent to $\mathrm{mor}_s^{CM}(R)$. (q.e.d.)

Thus $\mathrm{mor}_s^{CM}(R)$ is a Frobenius category together with projective-injective objects consisting of $p \in \mathrm{mor}_s^{CM}(R)$ that X_p and T_p are projective modules. Hence the stable category $\underline{\mathrm{mor}}_s^{CM}(R)$ is triangulated.

We shall construct a functor between $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ and $\underline{\mathrm{mor}}_s^{CM}(R)$. Let $\alpha : X_\alpha \to T_\alpha$ be an object of $\mathrm{mor}_s^{CM}(R)$ and let F_{X_α} and F_{T_α} be acyclic projective complexes such that $\mathrm{H}^0(\tau_{\leq 0}F_{X_\alpha}) = X_\alpha$ and $\mathrm{H}^0(\tau_{\leq 0}F_{T_\alpha}) = T_\alpha$. Set natural maps $\rho : F^0_{X_\alpha} \to X_\alpha$ and $\epsilon : T_\alpha \to F_{T_\alpha}$. Make a projective complex F_α as

$$\tau_{\leq 0}F_\alpha = \tau_{\leq 0}F_{X_\alpha},\ \tau_{\geq 1}F_\alpha = \tau_{\geq 1}F_{T_\alpha},\ d_{F_\alpha} = \epsilon\alpha\rho.$$

Lemma 3.2 *1) A morphism $f \in \mathrm{mor}_s^{CM}(R)(\alpha,\beta)$ induces a chain map $F_f : F_\alpha \to F_\beta$.*
2) For morphisms $f \in \mathrm{mor}_s^{CM}(R)(\alpha,\beta)$ and $g \in \mathrm{mor}_s^{CM}(R)(\beta,\gamma)$, $F_{gf} = F_gF_f$.
3) An exact sequence $0 \to \alpha \xrightarrow{f} \beta \xrightarrow{g} \gamma \to 0$ in $\mathrm{mor}_s^{CM}(R)$ induces an exact sequence $0 \to F_\alpha \xrightarrow{F_f} F_\beta \xrightarrow{F_g} F_\gamma \to 0$ in $\mathsf{C}^{\infty,b}$.
4) An object p of $\mathrm{mor}_s^{CM}(R)$ is projective if and only if F_p is a bounded complex.

Lemma 3.3 *The operation F gives a functor $\mathrm{mor}_s^{CM}(R) \to \mathsf{K}^{\infty,b}$. And F induces a functor $\underline{F} : \underline{\mathrm{mor}}_s^{CM}(R) \to \mathsf{K}^{\infty,b}/\mathsf{K}^b$.*

Proposition 3.4 *The functor $\underline{F} : \underline{\mathrm{mor}}_s^{CM}(R) \to \mathsf{K}^{\infty,b}/\mathsf{K}^b$ is triangulated.*

We state that $\underline{F}$ is a triangle equivalence. It leads us to the answer to our question together with Lemma 3.1.

Theorem 3.5 *The category $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ is triangle equivalent to $\underline{\mathrm{mor}}_s^{CM}(R)$.*

To prove the equivlence of $\underline{F}$, we have already seen that $\underline{F}$ is dense from the previous section. For the proof of fully faithfulness, we use some torsion structure in our category.

4 Stable t-structures

Definition 4.1 ([Mil]) *For full subcategories $\mathcal{U}$ and $\mathcal{V}$ of a triangulate category $\mathcal{C}$, $(\mathcal{U},\mathcal{V})$ is called a stable t-structure in $\mathcal{C}$ provided that*

- *$\mathcal{U}$ and $\mathcal{V}$ are stable for translations.*
- $\mathrm{Hom}_{\mathcal{C}}(\mathcal{U},\mathcal{V}) = 0$.
- *For every $X \in \mathcal{C}$, there exists a triangle $U \to X \to V \to \Sigma U$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.*

Proposition 4.2 ([BBD], [Mi3]) *Let $\mathcal{C}$ be a triangulated category. The following hold.*

1. *Let $(\mathcal{U}, \mathcal{V})$ be a stable t-structure in $\mathcal{C}$, $i_* : \mathcal{U} \to \mathcal{C}$ and $j_* : \mathcal{V} \to \mathcal{C}$ the canonical embeddings. Then there are a right adjoint $i^! : \mathcal{C} \to \mathcal{U}$ of i_* and a left adjoint $j^* : \mathcal{C} \to \mathcal{V}$ of j_* which satisfy the following.*

   (a) $j^* i_* = 0$, $i^! j_* = 0$.

   (b) *The adjunction arrows $i_* i^! \to \mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{C}} \to j_* j^*$ imply a triangle $i_* i^! X \to X \to j_* j^* X \to \Sigma i_* i^! X$ for any $X \in \mathcal{C}$.*

   In this case, j^* (resp., $i^!$) implies the triangulated equivalence $\mathcal{C}/\mathcal{U} \simeq \mathcal{V}$ (resp., $\mathcal{C}/\mathcal{V} \simeq \mathcal{U}$).

2. *If $\{\mathcal{C}, \mathcal{C}''; j^*, j_*\}$ (resp., $\{\mathcal{C}, \mathcal{C}''; j_!, j^*\}$) is a localization (resp., a colocalization) of $\mathcal{C}$, that is, j_* (resp., i_*) is a fully faithful right (resp., left) adjoint of $i^!$, then $(\mathrm{Ker} j^*, \mathrm{Im} j_*)$ (resp., $(\mathrm{Im} j_!, \mathrm{Ker} j^*)$) is a stable t-structure. In this case, the adjunction arrow $\mathbf{1}_{\mathcal{C}} \to j_* j^*$ (resp., $j_! j^* \to \mathbf{1}_{\mathcal{C}}$) implies triangles*

$$U \to X \to j_* j^* X \to \Sigma U$$
$$(\text{resp., } j_! j^* X \to X \to V \to \Sigma j_! j^* X)$$

   with $U \in \mathrm{Ker} j^*$, $j_* j^* X \in \mathrm{Im} j_*$ (resp., $j_! j^* X \in \mathrm{Im} j_!$, $V \in \mathrm{Ker} j^*$) for all $X \in \mathcal{C}$.

Definition 4.3 *Let $\mathcal{D}_1$, $\mathcal{D}_2$ be triangulated categories. Let $(\mathcal{U}_i, \mathcal{V}_i)$ be stable t-structures in $\mathcal{D}_i$ $(i = 1, 2)$. A triangule functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ sends a stable t-structure $(\mathcal{U}_1, \mathcal{V}_1)$ to a stable t-structure $(\mathcal{U}_2, \mathcal{V}_2)$ if $F(\mathcal{U}_1) \subset \mathcal{U}_2$ and $F(\mathcal{V}_1) \subset \mathcal{V}_2$.*

Lemma 4.4 *If a triangle functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ sends a stable t-structure $(\mathcal{U}_1, \mathcal{V}_1)$ in $\mathcal{D}_1$ to a stable t-structure $(\mathcal{U}_2, \mathcal{V}_2)$ of $\mathcal{D}_1$. Then we have the following:*

1. *If $F \mid_{\mathcal{U}_1}$ is full (faithful), then $\mathrm{Hom}_{\mathcal{D}_1}(U, X) \to \mathrm{Hom}_{\mathcal{D}_2}(FU, FX)$ is surjective(injective) for $U \in \mathcal{U}_1$ and $X \in \mathcal{D}_1$.*

2. *If $F \mid_{\mathcal{V}_1}$ is full (faithful), then $\mathrm{Hom}_{\mathcal{D}_1}(X, V) \to \mathrm{Hom}_{\mathcal{D}_2}(FX, FV)$ is surjective(injective) for $X \in \mathcal{D}_1$ and $V \in \mathcal{V}_1$.*

3. *If F is fully faithful and $F \mid_{\mathcal{U}_1}$ and $F \mid_{\mathcal{V}_1}$ are equivalences, then F is an equivalence.*

Corollary 4.5 *Let $\mathcal{D}_1$, $\mathcal{D}_2$ be triangulated categories. Let $(\mathcal{U}_n, \mathcal{V}_n)$ and $(\mathcal{V}_n, \mathcal{W}_n)$ be stable t-structures in $\mathcal{D}_n$ $(n = 1, 2)$. Assume a triangle functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ sends $(\mathcal{U}_1, \mathcal{V}_1)$ and $(\mathcal{V}_1, \mathcal{W}_1)$ to $(\mathcal{U}_2, \mathcal{V}_2)$ and $(\mathcal{V}_2, \mathcal{W}_2)$ respectively. If $F \mid_{\mathcal{U}_1}$, $F \mid_{\mathcal{V}_1}$ and $F \mid_{\mathcal{W}_1}$ are fully faithful (equivalent), so is F.*

proof For given $X, Y \in \mathcal{D}_1$, there are triangles

$$U_X \to X \to V^X \to \Sigma U_X$$
$$U_Y \to Y \to V^Y \to \Sigma U_Y$$

where $U_X, U_Y \in \mathcal{U}_1$ and $V^X, V^Y \in \mathcal{V}_1$. These triangles induce a diagram of abelian groups with exact rows and columns:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{D}_1}(V^X, U_Y) & \to & \mathrm{Hom}_{\mathcal{D}_1}(X, U_Y) & \to & \mathrm{Hom}_{\mathcal{D}_1}(U_X, U_Y) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{D}_1}(V^X, Y) & \to & \mathrm{Hom}_{\mathcal{D}_1}(X, Y) & \to & \mathrm{Hom}_{\mathcal{D}_1}(U_X, Y) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{D}_1}(V^X, V^Y) & \to & \mathrm{Hom}_{\mathcal{D}_1}(X, V^Y) & \to & \mathrm{Hom}_{\mathcal{D}_1}(U_X, V^Y)
\end{array}$$

Since $\mathrm{Hom}_{\mathcal{D}_1}(U, Z) \to \mathrm{Hom}_{\mathcal{D}_2}(FU, FZ)$, $\mathrm{Hom}_{\mathcal{D}_1}(Z, V) \to \mathrm{Hom}_{\mathcal{D}_2}(FZ, FV)$, and $\mathrm{Hom}_{\mathcal{D}_1}(V, Z) \to \mathrm{Hom}_{\mathcal{D}_2}(FV, FZ)$ are bijective by Lemma 4.4 1) 2), so is
$\mathrm{Hom}_{\mathcal{D}_1}(X, Y) \to \mathrm{Hom}_{\mathcal{D}_2}(FX, FY)$ from five-lemma. In the case that $F|_{\mathcal{U}_1}$, $F|_{\mathcal{V}_1}$ and $F|_{\mathcal{W}_1}$ are equivalences, F is dense by Lemma 4.4 3). (q.e.d.)

Proposition 4.6 *Let $\mathcal{D}_1$, $\mathcal{D}_2$ be triangulated categories. Let $(\mathcal{U}_i, \mathcal{V}_i)$, $(\mathcal{V}_i, \mathcal{W}_i)$ and $(\mathcal{W}_i, \mathcal{U}_i)$ be sable t-structures in $\mathcal{D}_i$ $(i = 1, 2)$. Assume a triangle functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ sends stable t-structures $(\mathcal{U}_1, \mathcal{V}_1)$, $(\mathcal{V}_1, \mathcal{W}_1)$ and $(\mathcal{W}_1, \mathcal{U}_1)$ to $(\mathcal{U}_2, \mathcal{V}_2)$, $(\mathcal{V}_2, \mathcal{W}_2)$ and $(\mathcal{W}_2, \mathcal{U}_2)$ respectively. If $F|_{\mathcal{U}}$ is fully faithful (equivalent) k, so is F.*

Proposition 4.7 *Let R be a coherent ring. Then we have the following.*

- *$(\mathsf{K}^{-,b}, \mathsf{K}^{\infty,\emptyset})$ is a stable t-structure of $\mathsf{K}^{\infty,b}$. Hence $(\mathsf{K}^{-,b}/\mathsf{K}^b, \mathsf{K}^{\infty,\emptyset})$ is a stable t-structure of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.*
- *$(\mathsf{K}^{+,b}/\mathsf{K}^b, \mathsf{K}^{-,b}/\mathsf{K}^b)$ is a stable t-structure of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.*
- *If R is Iwanaga-Gorenstein, then $(\mathsf{K}^{\infty,\emptyset}/\mathsf{K}^b, \mathsf{K}^{+,b}/\mathsf{K}^b)$ is a stable t-structure of $\mathsf{K}^{\infty,b}/\mathsf{K}^b$.*

Let R be an Iwanaga-Gorenstein ring. Let $\underline{\mathcal{CM}}_0$ (resp., $\underline{\mathcal{CM}}_1$, $\underline{\mathcal{CM}}_p$) be the full subcategory of $\underline{\mathrm{mor}}_s^{CM}(R)$ consisting of objects of the form $X \to 0$ (resp., $S \xrightarrow{=} S$, $P \to T$, with P being projective).

Proposition 4.8 *The following are stable t-structures of $\underline{\mathrm{mor}}_s^{CM}(R)$.*

$$(\underline{\mathcal{CM}}_0, \underline{\mathcal{CM}}_1),\ (\underline{\mathcal{CM}}_p, \underline{\mathcal{CM}}_0),\ (\underline{\mathcal{CM}}_1, \underline{\mathcal{CM}}_p).$$

Proposition 4.9 *The triangulated functor F induces equivalences*

$$\underline{F}|_{\underline{\mathcal{CM}}_0} : \underline{\mathcal{CM}}_0 \to \mathsf{K}^{-,b}/\mathsf{K}^b,$$
$$\underline{F}|_{\underline{\mathcal{CM}}_1} : \underline{\mathcal{CM}}_1 \to \mathsf{K}^{\infty,\emptyset},$$
$$\text{and } \underline{F}|_{\underline{\mathcal{CM}}_p} : \underline{\mathcal{CM}}_p \to \mathsf{K}^{+,b}/\mathsf{K}^b.$$

proof. It is well-known that $\underline{\mathrm{CM}}(\mathrm{R})$ is equivalent to $\mathsf{K}^{\infty,\emptyset}$, by the correspondence $P : X \mapsto P_X^\bullet$ where $P_X^\bullet$ is the complete resolution of X; $P_X^{-1} \to P_X^0 \to X \to 0$ and $0 \to X \to P_X^1 \to P_X^2$ are exact. The functor $\underline{F} \mid_{\mathcal{CM}_1}$ is the composite of P and the obvious equivalence $\underline{\mathcal{CM}_1}) \to \underline{\mathcal{CM}}(R)) : (X \xrightarrow{=} X) \mapsto X$, hence is an equivalence. (q.e.d.)

The proof of Theorem 3.5. We easily see $\underline{F}(\underline{\mathcal{CM}_0}) \subset \mathsf{K}^{-,b}/\mathsf{K}^b$, $\underline{F}(\underline{\mathcal{CM}_1}) \subset \mathsf{K}^{\infty,\emptyset}$, and $\underline{F}(\underline{\mathcal{CM}_p}) \subset \mathsf{K}^{+,b}/\mathsf{K}^b$. Propositions 4.6 and 4.9 imply the equivlence of $\underline{F}$.

Together with lemma 3.1, we obtain a Buchweitz-type theorem:

Theorem 4.10 *If R is Iwanaga-Gorenstein, then $\mathsf{K}^{\infty,b}/\mathsf{K}^b$ is triangle equivalent to $\underline{\mathrm{CM}}(T_2(R))$.*

5 Polygon of Recollements

Definition 5.1 *Let $\mathcal{C}$ be a triangulated categories and let $U_1, U_2, \cdots,$ U_n be full subcategories of $\mathcal{C}$. $(U_1, U_2, \cdots, U_n)$ is an n-gon of recollements in $\mathcal{C}$ if $(U_1, U_2), (U_1, U_2), \cdots, (U_{n-1}, U_n), (U_n, U_1)$ are stable t-structures in $\mathcal{C}$.*

The n-gon of recollements is not bizarre. The property of the category $\mathcal{C}$ often naturally induces an n-gon of recollements. For instance, let k be a field, and let $\mathcal{C}$ be a m/n-Calabi-Yau category. That is, $\mathcal{C}$ is a k-linear triangulated category with a Serre functor $S : \mathcal{C} \to \mathcal{C}$, which satisfies $\mathrm{Hom}_{\mathcal{C}}(X,Y) \cong \mathrm{D}\,\mathrm{Hom}_{\mathcal{C}}(Y, SX)$. and $\Sigma^m \cong S^n$. Then we have the following.

Proposition 5.2 *For any functorially finite thick subcategory $\mathcal{U}$ of $\mathcal{C}$, $(\mathcal{U}, \mathcal{U}^\perp, S\mathcal{U}, (S\mathcal{U})^\perp, \cdots, S^{n-1}\mathcal{U}, (S^{n-1}\mathcal{U})^\perp)$ is a $2n$-gon of recollements in $\mathcal{C}$.*

Since $\mathsf{D}^b(\mod T_n(k))$ is an $n-1/n+1$-Calabi-Yau category, Applying the previous lemma, we have the following.

Lemma 5.3 *The bounded derived category $\mathsf{D}^b(\mod T_n(k))$ of $\mod T_n(k)$ has a 2(n+1)-gon of recollements @ $(\mathcal{U}, \mathcal{U}^\perp, S\mathcal{U}, (S\mathcal{U})^\perp,$ $\cdots, S^n\mathcal{U}, (S^n\mathcal{U})^\perp)$ where $\mathcal{U}$ is a functorially finite thick subcategory.*

In the case $n = 2n'$ is even, there is an (n+1)-gon of recollements $(\mathcal{U}, S^{n'+1}\mathcal{U}, S\mathcal{U}, S^{n'+2}\mathcal{U}, \cdots, S^{n'-2}\mathcal{U}, S^{n-1}\mathcal{U}, S^{n'-1}\mathcal{U})$ by a suitable choice of $\mathcal{U}$.

This induces the following, which includes our case.

Corollary 5.4 *Let R be an Iwanaga-gorenstein ring. Then, $\underline{\mathrm{CM}}(\mathrm{T_n}(\mathrm{R}))$ has a $2(n+1)$-gon of recollements. If n is even, then $\underline{\mathrm{CM}}(\mathrm{T_n}(\mathrm{R}))$ has an $(n+1)$-gon of recollements.*

Finally, we give a definition of "recollement" appeared in the title.

Definition 5.5 ([BBD]) *A nine-tuple* $\{\mathcal{C}', \mathcal{C}, \mathcal{C}''; j^*, j_*, j^!, s_!, s^*, s_*\}$ *consisting of triangulated categories and functors*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "j_*"] & \mathcal{C} \arrow[l, bend right, "j^*"'] \arrow[l, bend left, "j^!"] \arrow[r, "s^*"] & \mathcal{C}'' \arrow[l, bend right, "s_!"'] \arrow[l, bend left, "s_*"] \end{tikzcd}$$

is called a recollement if it satisfies the following:

- j_*, $s_!$, *and* s_* *are fully faithful.*
- (j^*, j_*), $(j_*, j^!)$, $(s_!, s^*)$, *and* (s^*, s_*) *are adjoint pairs.*
- $j^*s_! = 0$, $s^*j_* = 0$, *and* $j^!s_* = 0$.
- *For each object* C *of* $\mathcal{C}$ *has triangles*

$$j_*j^!C \to C \to s_!s^*C \to \Sigma j_*j^!C,$$
$$s_*s^*C \to C \to j_*j^*C \to \Sigma s_*s^*C.$$

As a recollement implies two consecutive stable t-structures as a localization does one stable t-structure.

Proposition 5.6 ([BBD], [Mil]) *1) If* $(\mathcal{U}, \mathcal{V})$ *and* $(\mathcal{V}, \mathcal{W})$ *are stable t-structures of* $\mathcal{C}$, *then the canonical embedding* $j_* : \mathcal{V} \to \mathcal{C}$ *produces a recollement*

$$\begin{tikzcd} \mathcal{V} \arrow[r, "j_*"] & \mathcal{C} \arrow[l, bend right, "j^*"'] \arrow[l, bend left, "j^!"] \arrow[r, "s^*"] & \mathcal{C}/\mathcal{V} \arrow[l, bend right, "s_!"'] \arrow[l, bend left, "s_*"] \end{tikzcd}$$

2) If $\{\mathcal{C}', \mathcal{C}, \mathcal{C}''; j^*, j_*, j^!, s_!, s^*, s_*\}$ *is a recollement, then* $(\mathrm{Im} j_*, \mathrm{Im} s_*)$ *and* $(\mathrm{Im} s_!, \mathrm{Im} j_*)$ *are stable t-structures.*

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