S-UNIT EQUATIONS IN NUMBER FIELDS: EFFECTIVE RESULTS, GENERALIZATIONS, ABC-CONJECTURE

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INTRODUCTION

There is a very extensive literature on S-unit equations and their applications. The purpose of this paper is to give a short overview of our recent effective results concerning S-unit equations in two unknowns and their generalizations. For earlier, more detailed surveys on the subject we refer to Győry (1980, 1992a, 1996), Shorey and Tijdeman (1986), Evertse, Győry, Stewart and Tijdeman (1988) and Sprindžuk (1993).

In Section 1, the best explicit bounds to date are presented for the solutions of S-unit equations in two unknowns. As a consequence, in Sections 2 and 3 new effective upper bounds are formulated in connection with the abc-conjecture in number fields. Section 4 is devoted to a common generalization of S-unit equations over $\mathbb{Q}$ and binomial Thue equations with unknown exponents. Finally, in Section 5 some generalizations of S-unit equations are considered e.g. for polynomial equations in two variables where the unknowns are taken from the division group of a multiplicative subgroup of finite rank of $\overline{\mathbb{Q}}^*$.

1. S-UNIT EQUATIONS IN TWO UNKNOWNS

Let K be an algebraic number field with ring of integers O_K and unit group O_K^*. Let M_K denote the set of places on K, S a finite subset of M_K containing the set S_∞ of infinite places, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ the prime ideals of O_K corresponding to the finite places in S. An element $\alpha \in K$ is called S-integer if $\text{ord}_{\mathfrak{p}}(\alpha) \geq 0$ for all prime ideals $\mathfrak{p}$ of O_K different from $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. The S-integers form a subring of K which is denoted by O_S. It contains O_K as a subring, and for $t = 0$ it is just O_K. The unit group O_S^* of O_S is called the group of S-units. Let s denote the cardinality of S.

Many diophantine problems can be reduced to S-unit equations of the form

$$(1.1) \qquad \alpha x + \beta y = 1 \quad \text{in } x, y \in O_S^*,$$

where α, β are non-zero elements of K. The group O_S^* is of rank $s - 1$. If $\{\varepsilon_1, \ldots, \varepsilon_{s-1}\}$ is a fundamental system of S-units then x and y can be written in the form

$$x = \zeta \varepsilon_1^{a_1} \cdots \varepsilon_{s-1}^{a_{s-1}}, \quad y = \rho \varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}},$$

where ζ, ρ are unknown roots of unity in K and a_i, b_i are unknown rational integers for $i = 1, \ldots, s - 1$. Hence equation (1.1) can be regarded as an exponential diophantine equation.

It was implicitly proved by Siegel (1921) for ordinary units and by Mahler (1933) for S-units that equation (1.1) has only finitely many solutions. The first explicit proof is due to Lang (1960). These proofs are ineffective, i.e. they do not provide any algorithm for determining the solutions of (1.1).

1.1. **Effective results.** The first explicit bounds for the heights of the solutions of (1.1) were given by Győry (1974, 1979). Further bounds were obtained by Kotov and Trelina (1979), Schmidt (1992), Sprindžuk (1993), Bugeaud and Győry (1996), Haristoy (2003), Győry and Yu (2006) and Győry (2008). All these results were proved by using Baker's theory of logarithmic forms.

An alternative effective method was elaborated by Bombieri (1993) and Bombieri and Cohen (1997, 2003) for deriving upper bounds for the heights of the solutions. This method is based on an extension of the Thue-Siegel method, Dyson lemma and some geometry of numbers.

Bugeaud (1998) combined the above-mentioned effective methods to bound the solutions of (1.1).

The effective results concerning (1.1) led to many important applications, among others to Thue equations, decomposable form equations, superelliptic equations, recurrence sequences, polynomials and binary forms of given discriminant, power integral bases, irreducible polynomials, certain arithmetic graphs and the abc-conjecture; cf. Győry (1980, 1992a, 1996, 2008), Shorey and Tijdeman (1986), Evertse, Győry, Stewart and Tijdeman (1988), Győry and Yu (2006) and the references given there.

1.2. **Bounds for the solutions.** Keeping the above notation, let n denote the degree of the number field K over $\mathbb{Q}$, and let $P = \max_{1 \le i \le t} N(\mathfrak{p}_i)$ with the convention that $P = 1$ if $t = 0$. It follows from prime number theory that $t \le 2nP/\log^* P$. We use here the notation

$$\log^* a = \max(\log a, 1) \quad \text{for } a \ge 1.$$

Let R and R_S be the regulator and S-regulator of K, respectively. We have

$$R_S = i_S R \prod_i \log N(\mathfrak{p}_i),$$

where i_S is a positive divisor of h, the class number of K.

For $\gamma \in \overline{\mathbb{Q}}$, we denote by $h(\gamma)$ the absolute logarithmic height of γ. Put

$$H = \max(h(\alpha), h(\beta), 1)$$

where α, β denote the coefficients in (1.1).

Using the theory of logarithmic forms Bugeaud and Győry (1996) proved that all solutions x, y of (1.1) satisfy

$$(1.2) \qquad \max(h(x), h(y)) \le C_1 P R_S (\log^* R_S)^2 H,$$

where $C_1 = (c_1 n s)^{c_2 s}$ with explicitly given positive absolute constants c_1, c_2.

By means of a combination of the methods of Bugeaud and Győry (1996) as well as of Bombieri (1993) and Bombieri and Cohen (1997, 2003), Bugeaud (1998) derived the estimate

$$(1.3) \qquad \max(h(x), h(y)) \le C_2 P(\log^* P) R_S \max(C_3 P(\log^* P) R_S, H)$$

for the solutions x, y of (1.1), where C_2, C_3 are constants of the same form as C_1.

We note that (1.2) and (1.3) are best possible in terms of H. Further, in terms of S, s^s is the dominating factor in (1.2) and (1.3) whenever $t > \log P$.

We now present some recent improvements of (1.2) and (1.3).

Theorem 1 (Győry and Yu, 2006). *All solutions x, y of equation (1.1) satisfy*

$$(1.4) \qquad \max(h(x), h(y)) \leq C_4 P R_S (\log^* R_S) H,$$

where $C_4 = (c_3 n s)^{c_4 s}$ with explicitly given positive absolute constants c_3, c_4.

It should be remarked that the values of the absolute constants c_3, c_4 in C_4 are much smaller than those occurring in C_1, C_2 and C_3.

Theorem 2 (Győry, 2008). *Every solution x, y of (1.1) satisfies*

$$(1.5) \qquad \max(h(x), h(y)) \leq C_5 (P/\log^* P) R_S H,$$

where $C_5 = c_3^s$ with an explicit constant c_3 depending only on n, h and R.

For $t > 0$, (1.5) is a modified, more precise version of Theorem 2 of Győry and Yu (2006). It provides the first bound not having the factor s^s. This fact is important for certain applications, for example for the abc-conjecture. The appearance of s^s in (1.4) is due to the use of Minkowski's theorem on successive minima.

The new ingredients in the proofs of Theorems 1 and 2 are among other things some improved estimates for S-units, a recent theorem of Loher and Masser (2004) on multiplicatively independent algebraic numbers, some refined arguments of Győry (1979) for (1.5) and Bugeaud and Győry (1996) for (1.4), and recent estimates of Matveev (2000) and Yu (2007) on logarithmic forms.

Consider now the special case $K = \mathbb{Q}$. Then (1.1) can be written in the form

$$(1.6) \qquad Aa + Bb + Cc = 0,$$

where A, B, C are relatively prime non-zero integers, and a, b, c are relatively prime unknown integers composed of fixed primes $p_1, \ldots, p_t$. Let P be the greatest of these primes, and suppose that

$$\max(|A|, |B|, |C|) \leq H, \quad |abc| > 1.$$

The explicit version of (1.5) gives the following.

Corollary 1 to Theorem 2. *If (1.6) holds then we have*

$$(1.7) \qquad \log\max(|a|, |b|, |c|) < 2^{10t+22} t^4 (P/\log P) \Big(\prod_{p|abc} \log p\Big) \log^* H.$$

The *radical* of $(a, b, c) \in (\mathbb{Z} \setminus \{0\})^3$ is defined as

$$N = N(a, b, c) = \prod_{p|abc} p.$$

Put $N^* = \max(N, 16)$, and denote by $\log_i$ the i-th iterate of the logarithmic function. Then we have

$$(1.8) \qquad \begin{cases} P \leq N, \quad \prod_{p|abc} \log p \leq (\log N/t)^t \\ \\ t < 1.5 \log N / \log_2 N^*. \end{cases}$$

The first and second inequalities are obvious, while for the third one we refer to Győry (2008).

The inequalities (1.7) and (1.8) give immediately an upper bound for $\max(|a|, |b|, |c|)$ which depends on H and N only. It is a natural **question**: what is the best possible upper bound? In case of $A = B = C = 1$ the answer is provided by the abc-conjecture.

2. ABC-CONJECTURE

Oesterlé and, in a refined form, Masser (1985) proposed the following.

abc-conjecture. *For any given $\varepsilon > 0$, and for any coprime positive integers a, b, c with*

$$(2.1) \qquad a + b = c \quad \text{and radical } N = N(a, b, c),$$

$$(2.2) \qquad c \ll_\varepsilon N^{1+\varepsilon}$$

holds.

It is known that (2.2) is already best possible in terms of ε.

For any positive integer m we shall denote by $P(m)$ and $\omega(m)$ the greatest prime factor and the number of distinct prime factors of m with the convention that $P(1) = 1$.

Baker (1998, 2004) and Granville (1998) formulated such refinements of the abc-conjecture which involve also $\omega(abc)$. The following completely explicit version is due to Baker (2004).

A refined version of the abc-conjecture. *If* (2.1) *holds then*

$$c < \frac{6}{5} N (\log N)^t / t!,$$

where $t = \omega(abc)$.

The abc-conjecture has a very extensive literature. It has many extraordinary consequences. Further, it unifies and motivates a number of results and problems in number theory. For details, we refer the reader to the *abc-conjecture home page* created and maintained by Nitaj, http://www.math.unicaen.fr/~nitaj/abc.html.

Although the conjecture seems completely out of reach, there are some effective results towards its truth. By means of the theory of logarithmic forms Stewart and Tijdeman (1986) and Stewart and Yu (1991, 2001) obtained upper bounds for c as a function of $N = N(a, b, c)$. Stewart and Yu (2001) proved that if

$$p' = \min(P(a), P(b), P(c)),$$

then (2.1) implies that

$$(2.3) \qquad p' N^{c_4 \log_3 N^* / \log_2 N}$$

and

$$(2.4) \qquad c_5 N^{1/3} (\log N)^3$$

are upper bounds for $\log c$, where c_4, c_5 are effectively computable positive absolute constants. Chi (2005) showed that in (2.3) one can take $c_4 = 710$.

Together with (1.8), our Corollary 1 to Theorem 2 gives the following.

Corollary 2 to Theorem 2. *If a, b, c are coprime positive integers with $a + b = c$, $abc > 1$, $P = P(abc)$, $t = \omega(abc)$ and $N = N(a, b, c)$, then $\log c$ is bounded above by the following bounds:*

$$(2.5) \qquad 2^{10t+22} t^4 (P/\log P)\Big(\prod_{p \mid abc} \log p \Big),$$

$$(2.6) \qquad \left(2^{10t+22} / t^{t-4} \right) N (\log N)^t,$$

$$(2.7) \qquad 2^{23} (P/\log P) N^{653 \log_3 N^* / \log_2 N^*},$$

and, if $\varepsilon > 0$ and N is large enough with respect to ε,

$$(2.8) \qquad 2^{23} N^{1+\varepsilon}.$$

Here (2.8) is a result in the direction of the conjecture of Oesterlé and Masser, while (2.6) in the direction of Baker's conjecture.

Comparing Corollary 2 with the results of Stewart and Yu (2001), one can observe that (2.3) and (2.4) are slightly better than (2.7) and (2.8). The reason is that (2.3) and (2.4) were proved in a direct way, using some specific properties of $\mathbb{Z}$; for example, the fact that $a + b = c$ and $b > a$ imply $2b > c > b$. It should, however, be remarked that (2.5) gives in general better upper bound for c than (2.3) with $c_4 = 710$. We illustrate this on the following example which is due to de Weger

$$11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 23.$$

Here, for $c = 2^{21} \cdot 23$ we have $\log c > 2^4$. The bound in (2.3) is then greater than 2^{4950}, while the bound in (2.5) is smaller than 2^{100}.

3. ABC CONJECTURE IN NUMBER FIELDS

Let K be an algebraic number field of degree n, and M_K the set of places on K. Assume that every $v \in M_K$ is normalized in the usual way: if $\alpha \in K^*$ and v is infinite then

$$|\alpha|_v = |\sigma(\alpha)|^{n_v}, \quad \text{where } \sigma : K \mapsto \mathbb{C}, \; n_v = \begin{cases} 1 & \text{if } \sigma(K) \subseteq \mathbb{R}, \\ 2 & \text{otherwise,} \end{cases}$$

while if v is finite and corresponds to the prime ideal $\mathfrak{p}$ then

$$|\alpha|_v = N_{K/\mathbb{Q}}(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(\alpha)}.$$

The *height* of $(a, b, c) \in (K^*)^3$ is defined as

$$H_K(a, b, c) = \prod_{v \in M_K} \max \left(|a|_v, |b|_v, |c|_v \right),$$

and the *radical* of (a, b, c) as

$$(3.1) \qquad N_K(a, b, c) = \prod_v N_{K/\mathbb{Q}}(\mathfrak{p})^{\operatorname{ord}_{\mathfrak{p}}(p)}.$$

Here p is the rational prime lying below $\mathfrak{p}$, and the product is taken over all finite v for which $|a|_v$, $|b|_v$, $|c|_v$ are not all equal. Denote by Δ_K the absolute value of the discriminant of K.

Vojta (1987) proposed a very general conjecture and, as a consequence, suggested the first number field version of the abc-conjecture. Later, Vojta's version was refined by Elkies (1991), Broberg (1999), Granville and Stark (2000), Browkin (2000) and Masser (2002), respectively. The following uniform version is due to Masser (2002).

abc-conjecture in K**.** *For every* $\varepsilon > 0$ *there exists* C_ε, *depending only on* ε, *such that*

$$H_K(a,b,c) < C_\varepsilon^n \left(\Delta_K N_K(a,b,c)\right)^{1+\varepsilon}$$

for all a, b, $c \in K^*$ *with* $a + b + c = 0$.

For $K = \mathbb{Q}$, this reduces to the Oesterlé-Masser conjecture. The upper bound is again best possible in terms of ε.

This general conjecture has also very rich literature, and has many profounds implications; see e.g. the abc-conjecture home page mentioned above.

3.1. Unconditional bounds for $H_K(a,b,c)$. The effective results concerning S-unit equations can be used to obtain weaker but unconditional effective upper bounds for $H_K(a,b,c)$. Let

$$(3.2) \qquad\qquad a + b + c = 0, \quad \text{where } a,b,c \in K^*.$$

Denote by S_∞ the set of infinite places of K, and let

$$S = S_\infty \cup \{\text{finite } v \in M_K \text{ such that } |a|_v, |b|_v, |c|_v \text{ are not all equal}\}.$$

Then $x = -a/c$, $y = -b/c$ is a solution of the S-unit equation

$$(3.3) \qquad\qquad x + y = 1 \text{ in } x,y \in O_S^*.$$

Every bound for $h(x)$, $h(y)$ gives a bound for $H_K(a,b,c)$. Surroca (2007) showed that the bound of Bugeaud and Győry (1996) occurring in (1.2) yields the bound

$$(3.4) \qquad\qquad \left((c_6 n \Delta_K)^{c_7} N_K(a,b,c)^{c_8}\right)^n$$

for $\log H_K(a,b,c)$, where c_6, c_7 and c_8 are effectively computable positive absolute constants.

Our Theorem 2 enabled us to considerably improve the bound (3.4).

Theorem 3 (Győry, 2008). *Let* $\varepsilon > 0$. *Then* (3.2) *implies that* $\log H_K(a,b,c)$ *can be estimated from above by*

$$(3.5) \qquad\qquad c_9(n, \Delta_K, \varepsilon) N^{1+\varepsilon}$$

and, if

$$N = N_K(a,b,c) > \max\left(\exp\exp(\max(\Delta_K, e)), \Delta_K^{2/\varepsilon}\right),$$

by

$$(3.6) \qquad\qquad c_{10}(n, \varepsilon)(\Delta_K N)^{1+\varepsilon},$$

where c_9 *and* c_{10} *are effectively computable constants depending only on the parameters occurring in the parentheses* .

The main steps in the proof are as follows. Putting $P = \max N_{K/\mathbb{Q}}(\mathfrak{p})$ in (3.1), for appropriate choice of S Theorem 2 gives

$$\log H_K(a,b,c) < c_3(n,h,R)^s P R_S,$$

where $c_3(n,h,R)$ denotes a constant given explicitly and depending only on n, h and R. Then estimating h, R, s, P and R_S in terms of Δ_K and N we get (3.6) which implies (3.5).

4. COMMON GENERALIZATION OF S-UNIT EQUATIONS AND BINOMIAL THUE EQUATIONS

4.1. Binomial Thue equations.

Let a, b, c, n be non-zero integers with $n \geq 3$, and consider the equation

$$(4.1) \qquad ax^n - by^n = c \quad \text{in } x, y \in \mathbb{Z}.$$

It follows from a general theorem of Thue (1909) that (4.1) has only finitely many solutions. This was extended by Mahler (1933) to the case when x and y are coprime and c is also unknown with $c \in \mathscr{S}$, where $\mathscr{S}$ denotes the set of integers composed of fixed primes $p_1, \ldots, p_t$. These results were later made effective by Baker (1968) and Coates (1969), respectively.

In the case when n is also *unknown*, Tijdeman (1976) derived an effective upper bound for n depending on a, b and c. This was extended to the case $c \in \mathscr{S}$ by van der Poorten (1977). In terms of a, b, c and $\mathscr{S}$, a completely explicit bound was given for n by Bugeaud and Győry (2004).

4.2. Common generalization of (4.1) and S-unit equations.

Consider now more generally the equation

$$(4.2) \qquad ax^n - by^n = c \quad \text{where } x, y, a, b, c, n \in \mathbb{Z}$$

are all unknowns with $|xy| \geq 1$, a, b, $c \in \mathscr{S}$, $n \geq 3$ and

$$(4.3) \qquad \gcd(ax, by, c) = 1, \quad a, b, c \text{ n-th powerfree.}$$

Denoting by S the set of places on $\mathbb{Q}$ consisting of the infinite place and the finite ones corresponding to $p_1, \ldots, p_t$, the unknowns a, b, c are S-units in $\mathbb{Q}$.

Theorem 4 (Győry and Pintér, 2008). *All solutions of* (4.2) *with* (4.3) *satisfy*

$$(4.4) \qquad \max(|ax^n|, |by^n|, |c|) \leq c_1^{eff}(Q) \leq c_2^{eff}(P),$$

where $Q = p_1 \cdots p_t$, $P = \max_i p_i$. *Further, if* $|xy| > 1$, *then*

$$(4.5) \qquad n \leq c_3^{eff} Q^3,$$

where c_3 *is a positive absolute constant.*

Theorem 4 has been proved in a more general form, in the number field case. The proof involves the theory of logarithmic forms. Choosing $x = y = 1$, Theorem 4 gives an effective result for S-unit equations. Further, when a and b are fixed our Theorem 4 implies the theorems of Tijdemna and van der Poorten concerning equation (4.1).

The estimate (4.5) is not optimal. The abc-conjecture implies that $n < c_4 \log Q$ with an absolute constant c_4.

4.3. Parametric families of S-unit equations.

Let again $\mathscr{S}$ be the set of integers composed of fixed primes $p_1, \ldots, p_t$, and consider the corresponding S-unit equations of the form

$$(4.6) \qquad Aa + Bb + Cc = 0 \quad \text{in } a, b, c \in \mathscr{S},$$

where the coefficients A, B, C are relatively prime integers with $\gcd(ABC, p_1 \cdots p_t) = 1$.

Evertse, Győry, Stewart and Tijdeman (1988) *proved that*

(i) *there are only finitely many equations of the form* (4.6) *with more than two non-proportional solutions;*

(ii) *there are infinitely many equations of the form* (4.6) *with exactly two non-proportional solutions.*

We note that the proof of (i) does not make it possible to determine the exceptional equations. Our Theorem 4 implies the following.

(iii) *There exists infinitely many and effectively determinable equations of the form* (4.6) *which have no solution.*

Indeed, it follows from Theorem 4 that there is an effective constant c_4 depending only on the maximum of $p_1, \ldots, p_t$ such that the 3-parameter family of S-unit equations

$$(4.7) \qquad t^n a - w^n b = c \quad \text{in } a, b, c \in \mathscr{S}$$

with parameters t, w and n has no solution for t, w, n with $n \geq 3$, $|tw| > 1$, $\gcd(tw, p_1 \cdots p_t) = 1$ and $\max(|t|, |w|, n) > c_4$.

We note that the statement (iii), in an ineffective form, follows also from some general results of Corvaja and Zannier (2006) and Levin (2006).

4.4. A generalization of statement (i).

Let K be a field of characteristic 0, Γ a multiplicative subgroup of rank r of K^*, and $a, b \in K^*$. It was proved by Beukers and Schlickewei (1996) that the equation

$$(4.8) \qquad ax + by = 1 \quad \text{in } x, y \in \Gamma$$

has at most $c_1(r)$ solutions, where $c_1(r)$ was given explicitly. This bound has been recently improved by Hirata-Kohno (2008).

Equation (4.8) and equation

$$a'x' + b'y' = 1 \quad \text{in } x', y' \in \Gamma$$

are called *equivalent* if a'/a, $b'/b \in \Gamma$. In this case the two equations have the same number of solutions.

Denote by $\mathcal{N}$ the number of equivalence classes of equations of the form (4.8) which have more than two solutions. As a generalization of statement (i) above Evertse, Győry, Stewart and Tijdeman (1988) showed that $\mathcal{N}$ is finite. Let $\mathcal{N}_n$ denote the number of those solutions of the equation

$$x_1 + x_2 + \cdots + x_n = 1 \quad \text{in } x_1, \ldots, x_n \in \Gamma$$

for which $x_1 + x_2 + \cdots + x_n$ has no vanishing subsum. Evertse, Schlickewei and Schmidt (2002) gave an explicit upper bound $c_2(n, r)$ for $\mathcal{N}_n$. This bound has been improved by Amoroso and Viada (2009). Győry (1992b) proved that

$$\mathcal{N} \leq \mathcal{N}_5 + 12\mathcal{N}_3 + 30\mathcal{N}_2^2.$$

which, together with $\mathcal{N}_n \le c_2(n,r)$ gives

$$\mathcal{N} \le c_3(r)$$

with an explicit constant $c_3(r)$ depending only on r.

5. POLYNOMIAL EQUATIONS IN TWO UNKNOWNS FROM A MULTIPLICATIVE DIVISION GROUP

Let $P(X,Y) \in \overline{\mathbb{Q}}[X,Y]$ be an absolute irreducible polynomial, Γ a finitely generated multiplicative subgroup of $\left(\overline{\mathbb{Q}}^*\right)^2$, and consider the following generalization of equation (1.1):

(5.1) $$P(x,y) = 0 \quad \text{in } (x,y) \in \Gamma.$$

Suppose that P has at least three terms (otherwise (5.1) may have infinitely many trivial solutions). To give a *geometric interpretation* of (5.1), consider the curve $\mathscr{C}: P(x,y) = 0$ in $\left(\overline{\mathbb{Q}}^*\right)^2$, and the set of points

(5.2) $$\mathscr{C} \cap \Gamma.$$

We deal with the solutions/points from the following larger sets. Consider the *division group* $\overline{\Gamma}$ of Γ defined by

$$\overline{\Gamma} := \left\{ \mathbf{x} \in \left(\overline{\mathbb{Q}}^*\right)^2 \mid \exists k \in \mathbb{Z}_{>0} \text{ with } \mathbf{x}^k \in \Gamma \right\}.$$

We have $(\zeta, \rho) \in \overline{\Gamma}$ for any roots of unity ζ, ρ. For $\varepsilon > 0$, let

$$\overline{\Gamma}_\varepsilon := \left\{ \mathbf{x} \in \left(\overline{\mathbb{Q}}^*\right)^2 \mid \exists \, \mathbf{y}, \mathbf{z} \text{ with } \mathbf{y} \in \overline{\Gamma}, \, \mathbf{z} = (z_1, z_2) \in \left(\overline{\mathbb{Q}}^*\right)^2 \text{ such that} \right.$$
$$\left. \mathbf{x} = \mathbf{y} \cdot \mathbf{z} \text{ and } h(\mathbf{z}) = h(z_1) + h(z_2) \right\}$$

and

$$\mathcal{C}\left(\overline{\Gamma}, \varepsilon\right) := \left\{ \mathbf{x} \in \left(\overline{\mathbb{Q}}^*\right)^2 \mid \exists \, \mathbf{y}, \mathbf{z} \text{ with } \mathbf{y} \in \overline{\Gamma}, \, \mathbf{z} \in \left(\overline{\mathbb{Q}}^*\right)^2 \text{ such that} \right.$$
$$\left. \mathbf{x} = \mathbf{y} \cdot \mathbf{z} \text{ and } h(\mathbf{z}) < \varepsilon(1 + h(\mathbf{y})) \right\}.$$

The sets $\overline{\Gamma}_\varepsilon$ and $\mathcal{C}\left(\overline{\Gamma}, \varepsilon\right)$ can be regarded as a "cylinder" and a "truncated cone", respectively, around $\overline{\Gamma}$. The points $\mathbf{x} \in \overline{\Gamma}_\varepsilon$ are close to $\overline{\Gamma}$ if $\varepsilon > 0$ is small.

$\overline{\Gamma}_\varepsilon$ was introduced by Poonen (1999), and $\mathcal{C}\left(\overline{\Gamma}, \varepsilon\right)$ by Evertse (2002) in more general context. Clearly

(5.3) $$\Gamma \subset \overline{\Gamma} \subset \overline{\Gamma}_\varepsilon \subset \mathcal{C}\left(\overline{\Gamma}, \varepsilon\right).$$

It is important to note that $\overline{\Gamma}_\varepsilon$ and $\mathcal{C}\left(\overline{\Gamma}, \varepsilon\right)$ are not groups. Further, in general the coordinates of $\mathbf{x}$ in $\overline{\Gamma}$, $\overline{\Gamma}_\varepsilon$ or $\mathcal{C}\left(\overline{\Gamma}, \varepsilon\right)$ are not contained in a prescribed number field.

5.1. Ineffective results.

5.1. Ineffective results. Liardet (1974) proved that $\mathscr{C} \cap \overline{\Gamma}$ is finite. As a consequence of more general results, Poonen (1999) showed that $\mathscr{C} \cap \overline{\Gamma}_\varepsilon$ is also finite for small $\varepsilon > 0$, Evertse (2002) gave a bound for the cardinality of $\mathscr{C} \cap \overline{\Gamma}_\varepsilon$, and proved that even $\mathscr{C} \cap \mathcal{C}\left(\overline{\Gamma}, \varepsilon\right)$ is finite if $\varepsilon > 0$ is small, and finally Rémond (2002) derived a bound for the cardinality of $\mathscr{C} \cap \mathcal{C}\left(\overline{\Gamma}, \varepsilon\right)$. We should still mention Pontreau (201?) who improved Rémond's bound in case of curves $\mathscr{C}$.

Various multivariate /higher dimensional generalizations, and description of the (infinite) set of solutions/ points were established by Laurent, Győry, Bombieri, Masser, Zannier, Faltings, Vojta, McQuillan, Zhang, Szpiro, Ullmo, Poonen, David, Philippon, Chambert-Loir, Evertse, Schlickewei, Schmidt, Rémond and others.

5.2. Effective results.

5.2. Effective results. Consider the equation

$$(5.4) \qquad P(x,y) = 0 \quad \text{in } (x,y) \in \Gamma, \overline{\Gamma}, \overline{\Gamma}_\varepsilon, \quad \text{resp. } \mathcal{C}\left(\overline{\Gamma}, \varepsilon\right),$$

where $P(X,Y) \in \overline{\mathbb{Q}}[X,Y]$ is an absolute irreducible polynomial, not of the form

$$\alpha X^m + \beta Y^n \quad \text{or} \quad \gamma X^m Y^n + \delta.$$

Then the corresponding curve $\mathscr{C}$ is not a translate of a proper algebraic subgroup of $\left(\overline{\mathbb{Q}}^*\right)^2$. Solving equation (5.4) is equivalent to finding the points of the sets

$$(5.5) \qquad \mathscr{C} \cap \Gamma, \quad \mathscr{C} \cap \overline{\Gamma}, \quad \mathscr{C} \cap \overline{\Gamma}_\varepsilon \quad \text{resp. } \mathscr{C} \cap \mathcal{C}\left(\overline{\Gamma}, \varepsilon\right).$$

In the case when $P(X,Y)$ is linear and $\Gamma = \left(O_S^*\right)^2$ in a number field, the corresponding equation (5.3) is an S-unit equation. Then, as was seen in Section 1, equation (5.4) has only finitely many solutions which can be determined. For linear $P(X,Y)$, Bombieri (1993) and Bombieri and Cohen (1997, 2003) proved in an effective form the finiteness of $\mathscr{C} \cap \Gamma$. This was extended by Bombieri and Grubler (2006) to polynomials $P(X,Y)$ considered above.

5.3. New effective and quantitative results.

5.3. New effective and quantitative results. To obtain effective results for $\mathscr{C} \cap \overline{\Gamma}$, $\mathscr{C} \cap \overline{\Gamma}_\varepsilon$ and $\mathscr{C} \cap \mathcal{C}\left(\overline{\Gamma}, \varepsilon\right)$, one has to give effective bounds not only for the heights but also for the degrees of the points of these sets. In case of linear $P(X,Y)$ Bérczes, Evertse and Győry (2009), while in the general case Bérczes, Evertse, Győry and Pontreau (2009) derived such effective bounds. For these special classes of varieties, the effective results mentioned provide effective versions of some more general but ineffective theorems of Laurent, Poonen, Evertse and Rémond, respectively.

To present our effective results in quantitative form, we have to introduce some notation. Let Γ be a finitely generated subgroup of rank r of $\left(\overline{\mathbb{Q}}^*\right)^2$, $\{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ a basis of $\Gamma/\Gamma_{\text{tors}}$, and

$$h_0 = \max(h(\mathbf{w}_1), \ldots, h(\mathbf{w}_r), 1).$$

Let K be a number field such that $\Gamma \subset \left(K^*\right)^2$, n the degree of K over $\mathbb{Q}$, and S a finite subset of M_K, the set of places on K, such that $S \supseteq S_\infty$ and $\Gamma \subset \left(O_S^*\right)^2$. Put $s = |S|$, and

$$N = \max(\max_{v \in S \backslash S_\infty} N(\mathfrak{p}_v), 2),$$

where $\mathfrak{p}_v$ denotes the prime ideal corresponding to $v \in S \setminus S_\infty$. Let

$$\delta = \deg P, \quad H = \max(h(P), 1)$$

and

$$A = \left(e^{13}\delta^7 n^3 r\right)^{r+3} \cdot s \cdot \frac{N^{2\delta^2}}{\log N} h_0^r \log(\max(\delta nsN, \delta h_0)).$$

where

$$h(P) = \sum_{v \in M_K} \log \max_i |a_i|_v,$$

the a_i being the coefficients of P. Finally, let L be the extension of K generated by the coefficients of P.

Theorem 5 (Bérczes, Evertse, Győry and Pontreau, 2009). *Let*

$$\varepsilon = \left(2^{48}\delta(\log \delta)^5\right)^{-1}.$$

Then for every $\mathbf{x} = (x_1, x_2) \in \mathscr{C} \cap \overline{\Gamma}_\varepsilon$ *we have*

(5.6) $$h(\mathbf{x}) \le rh_0\delta A + AH$$

and

(5.7) $$[L(\mathbf{x}) : L] \le 2^{50}\delta(\log \delta)^6.$$

Here for $\mathbf{x} = (x_1, x_2)$, we use the notation

$$h(\mathbf{x}) = h(x_1) + h(x_2) \quad \text{and} \quad L(\mathbf{x}) = L(x_1, x_2).$$

It is interesting to observe the independence of the bounds on L, and the good dependence on H, h_0, n, s and N

It is clear that, in view of (5.3), (5.6) and (5.7) hold for each point $\mathbf{x}$ of $\mathscr{C} \cap \overline{\Gamma}$ as well. Almost the same holds for the points of $\mathscr{C} \cap \mathcal{C}\left(\overline{\Gamma}, \varepsilon\right)$ with smaller ε.

Theorem 6 (Bérczes, Evertse, Győry and Pontreau, 2009). *Let*

$$\varepsilon = \left(2^{50}\delta(\log \delta)^5\right)^{-1} \cdot \left(rh_0\delta A + AH\right)^{-1}.$$

Then for every $\mathbf{x} \in \mathscr{C} \cap \mathcal{C}\left(\overline{\Gamma}, \varepsilon\right)$

$$h(\mathbf{x}) \le 2rh_0\delta A + 2AH$$

and (5.7) *hold.*

We note that for $P(X, Y) = \alpha X + \beta Y - 1$ much better bounds were established in an earlier paper of Bérczes, Evertse and Győry (2009). For example, in this case one can take $0 < \varepsilon < 0.0025$ in Theorem 5, and one obtains $[L(\mathbf{x}) : L] \le 2$ as well as smaller bound for $h(\mathbf{x})$ in place of (5.7) and (5.6).

The *main tools* in the proofs of Theorems 5 and 6 are as follows. The proofs of the bounds concerning $h(\mathbf{x})$ are based on a recent effective approximation theorem of Bérczes, Evertse and Győry (2009), giving an explicit lower bound for $|\alpha - \xi|_v$ $(v \in M_K)$ in terms of $h(\xi)$, where $\alpha \in K^*$ and ξ is an element of a finitely generated multiplicative subgroup of K^*. This approximation theorem was proved by means of a combination of logarithmic form estimates and some geometry of numbers. To derive a bound for $[L(\mathbf{x}) : L]$, we used some estimates on the number of points of small height.

Theorems 5 and 6 have many applications, especially in the case $P(X,Y) = \alpha X + \beta Y - 1$ and Γ, $\overline{\Gamma}$. Using Γ in place of $(O_S^*)^2$, in many cases one can get much better quantitative results concerning, for example, purely exponential diophantine equations, discriminant and related equations, decomposable form equations and linear recurrences.

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