数理解析研究所講究録
第 1710 巻 2010 年 154-165

COMPLETE ASYMPTOTIC EXPANSIONS FOR THE PRODUCT AVERAGES OF HIGHER DERIVETIVES OF LERCH ZETA-FUNCTIONS

MASANORI KATSURADA
DEPARTMENT OF MATHEMATICS, HIYOSHI CAMPUS, KEIO UNIVERSITY
(慶應大学・経済学部・桂田 昌紀)

ABSTRACT. This is a preannouncement version of the forthcoming paper [Ka11].

Let $\phi(s,x,\lambda)$ be the Lerch zeta-function defined by (1.1) below, and $I_{m_1,m_2}(s_1,s_2;a,\lambda)$ the product average of higher derivatives of $\phi(s,x,\lambda)$, given in the form (1.2). The present investigation proceeds with our previous study [Ka2][Ka9] to establish a general explicit formula for (1.2) (Theorem 1); this further leads us to show that a complete asymptotic expansion exists for (1.2) when $s_1=\sigma+it$ and $s_2=\sigma_2-it$ in the descending order of t as $t\to\pm\infty$ (Theorem 2). The existence of such an asymptotic expansion of (1.2) has been shown in particular when $m_1=m_2=0$ and $a=1$ by the author [Ka2]; however, it is rather remarkable that a similar asymptotic series still exists in the most general setting into this direction. Our main formula (2.13) with (2.14) and (2.15) is reduced, for e.g., to an improvement upon the previous result (1.6) on the critical line $\sigma=1/2$ (see Corollary 2.3), and to similar asymptotic expansions of (1.2) in more extended regions (Corollaries 2.1 and 2.2), in particular including the line $\sigma=1$ (Corollary 2.4).

1. INTRODUCTION

Throughout the following, $s=\sigma+it$ denotes a complex variable, x and λ complex parameters with $x>0$, and the notation $e(\lambda)=e^{2\pi i\lambda}$ is frequently used. The Lerch zeta-function $\phi(s,x,\lambda)$ is defined by

$$\phi(s,x,\lambda)=\sum_{l=0}^{\infty}e(\lambda l)(l+x)^{-s}\qquad(\operatorname{Re}s=\sigma>1),\tag{1.1}$$

and its meromorphic continuation over the whole s-plane; it is an entire function for $\lambda\in\mathbb{R}\setminus\mathbb{Z}$, while if $\lambda\in\mathbb{Z}$ it is reduced to the Hurwitz zeta-function $\zeta(s,x)$, and further to the Riemann zeta-function $\zeta(s)=\zeta(s,1)$.

We write $\phi^{(m)}(s,x,\lambda)=(\partial/\partial s)^m\phi(s,x,\lambda)$ $(m=0,1,\dots)$ in the sequel. The present paper proceeds further with our previous study [Ka2][Ka9] of the mean squares of Lerch zeta-functions. We shall first prove a general explicit formula for the product average of

---

2000 *Mathematics Subject Classification.* Primary 11M35; Secondary 11M06.

Key words and phrases. Lerch zeta function, asymptotic expansion, Mellin-Barnes integral, mean square.

A portion of the present research was made during the first author's academic stay at Mathematisches Institut, Westfälische Wilhelms-Universität Münster. He would like to express his sincere gratitude to Professor Christopher Deninger and the institution for their warm hospitality and constant support. The author is also indebted to Grant-in-Aid for Scientific Research (No. 19540049), The Ministry of Education, Science, Sports, Culture of Japan.

$\phi^{(m)}(s,x,\lambda)$, in the form

$$(1.2)\qquad I_{m_1,m_2}(s_1,s_2;a,\lambda)=\int_0^1 \phi^{(m_1)}(s_1,a+x,\lambda)\phi^{(m_2)}(s_2,a+x,-\lambda)dx$$

for any nonnegative integers m_1 and m_2, where s_1 and s_2 are independent complex variables, and $a>0$ and λ fixed real numbers (Theorem 1); this leads us to show that a complete asymptotic expansion exists for (1.2) when $s_1=\sigma_1+it$ and $s_2=\sigma_2-it$ in the descending order of t as $t\to\pm\infty$ (Theorem 2), the case $\sigma_1=\sigma_2$ and $m_1=m_2$ of which in particular yields complete asymptotic expansions of the mean square

$$(1.3)\qquad \int_0^1 |\phi^{(m)}(s,a+x,\lambda)|^2dx \qquad (m=0,1,2,\dots)$$

as $\operatorname{Im} s\to\pm\infty$ (Corollaries 2.3–2.5). When $m=0$ and $a=1$, the existence of complete asymptotic expansions of (1.3) were shown in [Ka1]; however, it is rather remarkable that similar asymptotic series still exist for more general product averages such as (1.2).

We give here a brief overview of the history of research related to the integrals of the type (1.2). Let $\Gamma(s)$ denote the gamma function. Then Mikolás [Mi1] in 1956 proved the formula

$$(1.4)\qquad \int_0^1 \zeta(s_1,x)\zeta(s_2,x)dx=2(2\pi)^{s_1+s_2-2}\Gamma(1-s_1)\Gamma(1-s_2)\times\cos\left\{\frac{\pi}{2}(s_1-s_2)\right\}\zeta(2-s_1-s_2)$$

if $\max(\operatorname{Re}s_1,\operatorname{Re}s_2,\operatorname{Re}(s_1+s_2))<1$; otherwise the integral diverges since $\zeta(s,x)$ has a singularity at $x=0$ (see also [Mi2] for variants of (1.4)). It is hence natural to consider the function $\zeta(s,x)-x^{-s}=\zeta(s,1+x)$ (by (1.1)), for which the singularity in x is removed. The mean square $I_0(s)=\int_0^1|\zeta(s,1+x)|^2dx$ was already studied in 1952 by Koksma-Leckerkerker [KL], who proved that $I_0(1/2+it)=O(\log t)$ for $t\geq 2$. Improvements upon this result were due to various authors; we refer the reader, for e.g., to [KM3] or [Ka9].

As for asymptotic aspects of Lerch zeta-functions, hybrid type mean value theorems for the weighted mean square $\int_0^\infty|\phi(\sigma+it,a,\lambda)|^2e^{-\delta t}dt$ as $\delta\to+0$ were proved by Klusch [Kl1], while an asymptotic formula for the mean square $I_0(s;\lambda)=\int_0^1|\phi(s,1+x,\lambda)|^2dx$, where $\phi(s,1+x,\lambda)=e(-\lambda)\{\phi(s,x,\lambda)-x^{-s}\}$ (by (1.1)) as $\operatorname{Im}s=t\to+\infty$ with the error term $O(t^{-1})$ was derived by Zhang [Z1]. The author [Ka2] established a complete asymptotic expansion of $I(s;\lambda)$ in the descending order of $\operatorname{Im}s$ as $\operatorname{Im}s\to\pm\infty$, where Atkinson's [At] dissection method was applied upon combined with Mellin-Barnes type integrals. This type of integrals were extensively applied by Motohashi to investigate higher power moments and spectral theory of zeta and allied functions (see, for e.g., [Mo1]–[Mo3]). It is worth-while noting that the integrals have advantage over heuristic treatments in studying certain asymptotic aspects and transformation properties of zeta and theta functions (see also [Ka3]–[Ka8][Ka10][KN]). Egami-Matsumoto [EM] applied this type of integrals to investigate a discrete analogue of higher power moments of $\zeta(s,x)$.

Furthermore, a multiple mean square of $\phi(s,x,\lambda)$, in the form

$$\int_0^1\cdots\int_0^1 |\phi(s,a+x_1+\cdots+x_m,\lambda)|^2dx_1\cdots dx_m$$

for any integer $m\geq 1$, was recently studied by the author [Ka9], who established its complete asymptotic expansion in the descending order of $\operatorname{Im}s$ as $\operatorname{Im}s\to\pm\infty$; crucial rôles here were played by various properties of hypergeometric functions, which were again manipulated with Mellin-Barnes type integrals.

The mean square of the derivative of $\zeta(s,x)$, on the other hand, were first treated by Zhang [Z1], who proved an asymptotic formula for $I_1(s)=\int_0^1|\zeta'(s,x)|^2dx$ on the critical line $\sigma=1/2$ as $t\to+\infty$ with the error term $O(t^{-1/6}\log t)$. Guo [G1][G2] showed the same formula for $I_1(1/2+it)$ upon making its coefficients more explicit, together with the improved error term $O(t^{-1}\log^2 t)$. Let $\gamma_j(x)$ $(j=0,1,\dots)$ denote the coefficients of the Taylor series expansion

$$\zeta(s,x)=(s-1)^{-1}+\sum_{j=0}^{\infty}\gamma_j(x)(s-1)^j \tag{1.5}$$

at $s=1$ (cf. [Iv]), where $\gamma_j(1)=\gamma_j$ $(j=0,1,\dots)$ are the ordinary Euler-Stieltjes constants. Then a more general mean square

$$I_m(s)=\int_0^1|\zeta^{(m)}(s,1+x)|^2dx\qquad(m=1,2,\dots)$$

was investigated on the lines $\sigma=1/2$ and $\sigma=1$ by Katsurada-Matsumoto [KM5], who in particular showed the asymptotic formula

$$\begin{aligned}I_m\Big(\frac12+it\Big)&=\frac{1}{2m+1}\log^{2m+1}\Big(\frac{t}{2\pi}\Big)+\sum_{j=0}^{2m}\frac{(2m)!\gamma_j}{(2m-j)!}\log^{2m-j}\Big(\frac{t}{2\pi}\Big)\\&\quad+\frac{1}{t^2}\mathcal{P}_m\Big(\log t,\frac1t\Big)-2\,\mathrm{Re}\left\{\frac{m!\zeta^{(m)}(\frac12+it)}{(\frac12+it)^{m+1}}\right\}+O(t^{-m-1})\end{aligned} \tag{1.6}$$

for $t\ge 2$, where $\mathcal{P}_m(\log t,1/t)$ denotes some polynomial in $\log t$ and $1/t$, and the implied O-constant depends only on m.

It seems quite difficult to determine the exact form of $\mathcal{P}_m(\log t,1/t)$ and to sharpen the error term $O(t^{-m-1})$ above by elaborating the method developed in [KM5]; considerable computational complexity arises along with the increase of the multiplicity of differentiation, where the profound difficulty here lies in the asymptotic analysis of the (successively differentiated) product of the zeta-function and the quotient of gamma functions (see (2.2) and (2.3) below). We can in fact pass through this crucial step by introducing a certain auxiliary zeta-function, which allows us to establish (complete) Stirling's type formula for the quotient of gamma functions, together with its explicit remainder term whose representation is uniformly valid throughout the whole sector $|\arg z|<\pi$; this uniformity of the representation is most appropriate for the analysis of (2.3) after successive differentiations.

2. Statement of results

Let $\Gamma(s)$ denote the gamma function, and $(s)_k=\Gamma(s+k)/\Gamma(s)$ for any $k\in\mathbb{Z}$ Pochhammer's symbol. Note in particular that $(s)_{-h}=1/(s-1)\cdots(s-h)$ for any $h\ge1$. We write

$$f^{(m,n)}(u_0,v_0)=\frac{\partial^{m+n}f}{\partial u^m\partial v^n}\bigg|_{(u,v)=(u_0,v_0)}\qquad(m,n=0,1,\dots)$$

for a function $f(u,v)$ holomorphic at $(u,v)=(u_0,v_0)$, where the index (m,n) indicates (in this order) the multiplicities of each differentiation in terms of the first or the second variable.

The proofs of Theorems 1 and 2 will in fact be initiated from the case $m=1$ of [Ka9, Theorem 2] yielding Formula (3.3) with (3.4) below, one of the merits of which is that it contains the independent complex variables s_1 and s_2. We can therefore differentiate both sides of (3.3) successively to obtain the following Theorem 1. Let $L(s,\chi)$ denote the Dirichlet L-function attached to a Dirichlet character χ modulo q. Then the same principle was first applied by the author [Ka1] to study the discrete mean square $\sum_{\chi(\mathrm{mod}\,q)}|L^{(m)}(s,\chi)|^2$ for any integer $m\geq 1$, where the summation is taken over all Dirichlet characters χ modulo q.

Our first mein result asserts

Theorem 1. *Let $I_{m_1,m_2}(s_1,s_2;a,\lambda)$ be defined by (1.2) with any nonnegative integers m_1 and m_2, where s_1 and s_2 are independent complex variables, and $a>0$ and λ are any real numbers. Define the set $\widetilde{E}\subset\mathbb{C}^2$ by*

$$\widetilde{E}=\{(s_1,s_2);\ s_1+s_2\in\mathbb{Z},\ s_1+s_2\leq 2\}\cup\{(s_1,s_2);\ s_1\in\mathbb{Z}\ \text{or}\ s_2\in\mathbb{Z}\}. \tag{2.1}$$

Then for any integer $N\geq 1$ in the region $1-N<\operatorname{Re}s_j=\sigma_j<1+N$ $(j=1,2)$ except the points at $\widetilde{E}$ the formula

$$\begin{aligned}I_{m_1,m_2}(s_1,s_2;a,\lambda)&=-a^{1-s_1-s_2}\sum_{j=0}^{m_1+m_2}\frac{(m_1+m_2)!}{(m_1+m_2-j)!}\frac{(-\log a)^{m_1+m_2-j}}{(1-s_1-s_2)^{j+1}}\\&\quad+R^{(m_1,m_2)}(s_1,s_2;\lambda)+R^{(m_2,m_1)}(s_2,s_1;-\lambda)\\&\quad-S_N^{(m_1,m_2)}(s_1,s_2;a,\lambda)-S_N^{(m_2,m_1)}(s_2,s_1;a,-\lambda)\\&\quad-T_N^{(m_1,m_2)}(s_1,s_2;a,\lambda)-T_N^{(m_2,m_1)}(s_2,s_1;a,-\lambda)\end{aligned} \tag{2.2}$$

holds, where R, S_N and T_N are defined by

$$R(s_1,s_2;\lambda)=\zeta_\lambda(s_1+s_2-1)\Gamma(s_1+s_2-1)\frac{\Gamma(1-s_2)}{\Gamma(s_1)}, \tag{2.3}$$

$$S_N(s_1,s_2;a,\lambda)=\sum_{n=0}^{N-1}\frac{(s_1)_n}{(1-s_2)_{n+1}}a^{1-s_2+n}e(\lambda)\phi(s_1+n,a+1,\lambda), \tag{2.4}$$

$$T_N(s_1,s_2;a,\lambda)=\frac{(s_1)_N}{(1-s_2)_N}a^{1-s_2+N}\sum_{l=1}^{\infty}\frac{e(l\lambda)}{l^{s_1+s_2-1}}\int_l^\infty\frac{y^{s_1+s_2-2}}{(a+y)^{s_1+N}}dy. \tag{2.5}$$

Furthermore, for any integer $K\geq 0$ the expression

$$T_N^{(m_1,m_2)}(s_1,s_2;a,\lambda)=\sum_{k=1}^{K}U_{N,k}^{(m_1,m_2)}(s_1,s_2;a,\lambda)+V_{N,K}^{(m_1,m_2)}(s_1,s_2;a,\lambda) \tag{2.6}$$

follows in the same region of (s_1,s_2) above, where $U_{N,k}$ and $V_{N,K}$ are given by

$$\begin{aligned}U_{N,k}(s_1,s_2;a,\lambda)&=\frac{(-1)^{k-1}(2-s_1-s_2)_{k-1}(s_1)_{N-k}}{(1-s_2)_N}a^{1-s_2+N}\\&\quad\times\sum_{l=1}^{\infty}\frac{e(l\lambda)}{l^k(a+l)^{s_1+N-k}},\end{aligned} \tag{2.7}$$

$$V_{N,K}(s_1,s_2;a,\lambda)=\frac{(-1)^K(2-s_1-s_2)_K(s_1)_{N-K}}{(1-s_2)_N}a^{1-s_2+N}\times\sum_{l=1}^{\infty}\frac{e(l\lambda)}{l^{s_1+s_2-1}}\int_l^{\infty}\frac{y^{s_1+s_2-K-2}}{(a+y)^{s_1+N-K}}dy. \tag{2.8}$$

Here the empty sums are to be regarded as null.

Remark. The exceptional set $\widetilde{E}$ in (2.1) is defined by collecting all singular points of the factors on the right side of (2.2); formulae similar to (2.2) for the exceptional points $(s_1,s_2)\in\widetilde{E}$ can be deduced as the limiting cases of Theorem 1 (see, for e.g., Corollaries 2.1, 2.3 and 2.4).

Remark. On the right sides of (2.7) and (2.8) (which is reduced to (2.5) if $K=0$), both the infinite series converge in the region $\operatorname{Re}s_1>1-N$, since the integral in each term is of order $O(l^{-\operatorname{Re}s_2-N-1})$ as $l\to+\infty$; the expressions on the right sides are hence valid for $\operatorname{Re}s_1>1-N$ and $\operatorname{Re}s_2<1+N$.

Let α and ν be any complex parameters. In order to describe our second main result, we introduce Nörlund's generalized Bernoulli polynomials $B_h^{(\nu)}(\alpha)$ $(h=0,1,\dots)$ defined by the Taylor series expansion

$$\left(\frac{z}{e^z-1}\right)^{\nu}e^{\alpha z}=\sum_{h=0}^{\infty}\frac{B_h^{(\nu)}(\alpha)}{h!}z^h \tag{2.9}$$

for $|z|<2\pi$, where $\{z/(e^z-1)\}^{\nu}=\exp[\nu\log\{z/(e^z-1)\}]$ and the $\log\{\cdot\}$ here takes the principal branch of logarithms. Note that $B_h^{(1)}(\alpha)=B_h(\alpha)$ $(h=0,1,\dots)$ are the usual Bernoulli polynomials. We write $\operatorname{sgn}t=t/|t|$ for $t\neq0$, and use the convention that $\zeta(s,0)=\zeta(s)$ throughout the following. Theorem 1 particularly yields complete asymptotic expansions of (1.2) when $s_1=\sigma_1+it$ and $s_2=\sigma_2-it$ in the descending order of t as $t\to\pm\infty$.

Our second main result asserts

Theorem 2. *Let m_1, m_2, a, λ, I_{m_1,m_2}, R, S_N, T_N, $U_{N,k}$ and $V_{N,K}$ be as in Theorem 1, and define the set $E\subset\mathbb{R}^2$ by*

$$E=\{(\sigma_1,\sigma_2);\ \sigma_1+\sigma_2\in\mathbb{Z},\ \sigma_1+\sigma_2\leq2\}.$$

Let further $P_m(\sigma,\tau;\log(|t|/2\pi))$ and $Q_h^{m_1,m_2}(\sigma_1,\sigma_2,\tau;\log(|t|/2\pi))$ be the polynomials in $\log(|t|/2\pi)$ defined by

$$P_m\Bigl(\sigma,\tau;\log\Bigl(\frac{|t|}{2\pi}\Bigr)\Bigr)=(-1)^m\sum_{j=0}^{m}\binom{m}{j}\zeta^{(m-j)}(2-\sigma,\tau)\log^{m-j}\Bigl(\frac{|t|}{2\pi}\Bigr) \tag{2.10}$$

for $\sigma\neq1$ and $m=0,1,\dots$, and

$$Q_h^{m_1,m_2}\Bigl(\sigma_1,\sigma_2,\tau;\log\Bigl(\frac{|t|}{2\pi}\Bigr)\Bigr)=(-1)^{m_1+m_2}\sum_{j=0}^{m_1+m_2}A_{h,j}^{m_1,m_2}(\sigma_1,\sigma_2,\tau)\log^{m_1+m_2-j}\Bigl(\frac{|t|}{2\pi}\Bigr) \tag{2.11}$$

for $h=1,2,\ldots,$ *where*

$$A_{h,j}^{m_1,m_2}(\sigma_1,\sigma_2,\tau)=(-1)^j\sum_{\substack{j_1+j_2=j\\0\le j_1\le m_1\\0\le j_2\le m_2}}\binom{m_1}{j_1}\binom{m_2}{j_2}\frac{\partial^j}{\partial\sigma_1^{j_1}\partial\sigma_2^{j_2}}\{B_h^{(2-\sigma_1-\sigma_2)}(1-\sigma_2)\times(\sigma_1+\sigma_2-1)_h\zeta(2-\sigma_1-\sigma_2,\tau)\}\tag{2.12}$$

for any real $\tau\ge 0$. *Then for any integer* $N\ge 1$, *in the region* $1-N<\sigma_j<1+N$ $(j=1,2)$ *except the cases of* $(\sigma_1,\sigma_2)\in E$ *the formula*

$$\begin{aligned}&I_{m_1,m_2}(\sigma_1+it,\sigma_2-it;a,\lambda)\\&=-a^{1-\sigma_1-\sigma_2}\sum_{j=0}^{m_1+m_2}\frac{(m_1+m_2)!}{(m_1+m_2-j)!}\frac{(-\log a)^{m_1+m_2-j}}{(1-\sigma_1-\sigma_2)^{j+1}}\\&\quad+R^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;\lambda)+R^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;-\lambda)\\&\quad-S_N^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;a,\lambda)-S_N^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;a,\lambda)\\&\quad-T_N^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;a,\lambda)-T_N^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;a,-\lambda)\end{aligned}\tag{2.13}$$

holds for any $t\in\mathbb{R}\setminus\{0\}$. *Furthermore, for any integer* $H\ge 0$ *the expression*

$$\begin{aligned}&R^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;\lambda)+R^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;-\lambda)\\&=\left(\frac{|t|}{2\pi}\right)^{1-\sigma_1-\sigma_2}P_{m_1+m_2}\left(\sigma_1+\sigma_2,\{\lambda\operatorname{sgn}t\};\log\left(\frac{|t|}{2\pi}\right)\right)\\&\quad+\sum_{h=1}^{H}\frac{(-1)^h(it)^{-h}}{h!}\left(\frac{|t|}{2\pi}\right)^{1-\sigma_1-\sigma_2}Q_h^{m_1,m_2}\left(\sigma_1,\sigma_2,\{\lambda\operatorname{sgn}t\};\log\left(\frac{|t|}{2\pi}\right)\right)\\&\quad+R_H^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;\lambda)+R_H^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;-\lambda)\end{aligned}\tag{2.14}$$

follows, where $R_H^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;\lambda)+R_H^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;-\lambda)$ *is the remainder term represented by a certain Mellin-Barnes type integral, and also for any integer* $K\ge 0$ *the expression*

$$T_N^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;a,\lambda)=\sum_{k=1}^{K}U_{N,k}^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;a,\lambda)+V_{N,K}^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;a,\lambda)\tag{2.15}$$

together with that of $T_N^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;a,-\lambda)$ *follows, both in the same region of* $(\sigma_1+it,\sigma_2-it)$ *above; Formula* (2.13) *with* (2.14) *and* (2.15) *gives a complete asymptotic expansion in the descending order of* t *as* $t\to\pm\infty$, *where each term of the asymptotic series is estimated as*

$$\frac{(-1)^h(it)^{-h}}{h!}\left(\frac{|t|}{2\pi}\right)^{1-\sigma_1-\sigma_2}Q_h^{m_1,m_2}\left(\sigma_1,\sigma_2,\{\lambda\operatorname{sgn}t\};\log\left(\frac{|t|}{2\pi}\right)\right)=O(|t|^{1-h-\sigma_1-\sigma_2}\log^{m_1+m_2}|t|),\tag{2.16}$$

$$R_H^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;\lambda)+R_H^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;-\lambda)=O(|t|^{-H-\sigma_1-\sigma_2}\log^{m_1+m_2}|t|),\tag{2.17}$$

$$\begin{aligned} U_{N,k}^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;a,\lambda) &= O(|t|^{-k}), \\ U_{N,k}^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;a,-\lambda) &= O(|t|^{-k}) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} V_{N,K}^{(m_1,m_2)}(\sigma_1+it,\sigma_2-it;a,\lambda) &= O(|t|^{-K-1}), \\ V_{N,K}^{(m_2,m_1)}(\sigma_2-it,\sigma_1+it;a,-\lambda) &= O(|t|^{-K-1}) \end{aligned} \tag{2.19}$$

for any $H \geq h \geq 1$ and $K \geq k \geq 1$, and for any σ_j and t with $1-N < \sigma_j < 1+N$ $(j=1,2)$ and $|t| \geq 2$. Here the implied O-constants depend at most on H, K, a, m_j and σ_j $(j=1,2)$.

One can observe that the first term on the right side of (2.13) has a singularity at each point (σ_1, σ_2) with $\sigma_1+\sigma_2=1$; this in fact cancels out with that included in the first term on the right side of (2.14). We use hereafter the convention that $\gamma_j = \gamma_j(0)$ for $j=0,1,\dots$ (see (1.5)). The limiting case $(\sigma_1,\sigma_2) \to (\sigma, 1-\sigma)$ of Theorem 2 then asserts

Corollary 2.1. *Let m_1, m_2, a, λ, I_{m_1,m_2}, R, S_N, T_N, $U_{N,k}$ and $V_{N,K}$ be as in Theorem 1, $\widehat{P}_m(\tau;\log(|t|/2\pi))$ the polynomial in $\log(|t|/2\pi)$ defined by*

$$\widehat{P}_m\Big(\tau;\log\Big(\frac{|t|}{2\pi}\Big)\Big) = (-1)^m\Big\{\frac{1}{m+1}\log^{m+1}\Big(\frac{|t|}{2\pi}\Big) + \sum_{j=0}^{m}\frac{m!\gamma_j(\tau)}{(m-j)!}\log^{m-j}\Big(\frac{|t|}{2\pi}\Big)\Big\} \tag{2.20}$$

with $\tau \geq 0$ and $m=0,1,\dots$, and $Q_h^{m_1,m_2}$ by (2.11). Then for any integer $N \geq 1$, in the region $1-N < \sigma < N$ the formula

$$\begin{aligned} I_{m_1,m_2}(\sigma+it,1-\sigma-it;a,\lambda) &= \frac{\partial^{m_1+m_2}}{\partial\sigma_1^{m_1}\partial\sigma_2^{m_2}}\Big\{-\frac{a^{1-\sigma_1-\sigma_2}}{1-\sigma_1-\sigma_2} \\ &\quad + R(\sigma_1+it,\sigma_2-it;\lambda) + R(\sigma_2-it,\sigma_1+it;-\lambda)\Big\}\Big|_{\substack{\sigma_1=\sigma \\ \sigma_2=1-\sigma}} \\ &\quad - S_N^{(m_1,m_2)}(\sigma+it,1-\sigma-it;a,\lambda) - S_N^{(m_2,m_1)}(1-\sigma-it,\sigma+it;a,-\lambda) \\ &\quad - T_N^{(m_1,m_2)}(\sigma+it,1-\sigma-it;a,\lambda) - T_N^{(m_2,m_1)}(1-\sigma-it,\sigma+it;a,-\lambda) \end{aligned} \tag{2.21}$$

holds. Furthermore, for any integer $H \geq 0$ the expression

$$\begin{aligned} &\frac{\partial^{m_1+m_2}}{\partial\sigma_1^{m_1}\partial\sigma_2^{m_2}}\Big\{-\frac{a^{1-\sigma_1-\sigma_2}}{1-\sigma_1-\sigma_2} \\ &\quad + R(\sigma_1+it,\sigma_2-it;\lambda) + R(\sigma_2-it,\sigma_1+it;-\lambda)\Big\}\Big|_{\substack{\sigma_1=\sigma \\ \sigma_2=1-\sigma}} \\ &= \frac{(-\log a)^{m_1+m_2+1}}{m_1+m_2+1} + \widehat{P}_{m_1+m_2}\Big(\{\lambda\operatorname{sgn} t\};\log\Big(\frac{|t|}{2\pi}\Big)\Big) \\ &\quad + \sum_{h=1}^{H}\frac{(-1)^h(it)^{-h}}{h!}Q_h^{m_1,m_2}\Big(\sigma,1-\sigma,\{\lambda\operatorname{sgn} t\};\log\Big(\frac{|t|}{2\pi}\Big)\Big) \\ &\quad + R_H^{(m_1,m_2)}(\sigma+it,1-\sigma-it;\lambda) + R_H^{(m_2,m_1)}(1-\sigma-it,\sigma+it;-\lambda) \end{aligned} \tag{2.22}$$

follows, and also the expression (2.15) follows in particular for $T_N^{(m_1,m_2)}(\sigma+it,1-\sigma-it;a,\lambda)$ and for $T_N^{(m_2,m_1)}(1-\sigma-it,\sigma+it;a,-\lambda)$, both in the same region of $\sigma+it$ above; Formula (2.21) with (2.22) and (2.15) gives a complete asymptotic expansion in the *descending order of* t *as* $t \to \pm\infty$, *where each term of the asymptotic series is estimated as* (2.16)–(2.19).

The case $m_1 = m_2$ and $(\sigma_1, \sigma_2) = (\sigma, \sigma)$ of Theorem 2 is reduced to

Corollary 2.2. *Let* $m \geq 0$ *be an arbitrarily fixed integer,* a, λ, P_l, $Q_h^{m,m}$, R, S_N, T_N, $U_{N,k}$, *and* $V_{N,K}$ *as in Theorem 2. Then for any integer* $N \geq 1$, *in the region* $1 - N < \sigma < 1 + N$ *except on the line* $\sigma = n/2$ $(n = 2, 1, 0, -1, \ldots)$, *the formula*

$$
\begin{aligned}
(2.23)\qquad \int_0^1 |\phi^{(m)}(\sigma + it, a + x, \lambda)|^2 dx &= -a^{1-2\sigma} \sum_{j=0}^{m_1+m_2} \frac{(m_1+m_2)!}{(m_1+m_2-j)!} \frac{(-1)^j \log^{2m-j} a}{(1-2\sigma)^{j+1}} \\
&\quad + 2\operatorname{Re} R^{(m,m)}(\sigma + it, \sigma - it; \lambda) - 2\operatorname{Re} S_N^{(m,m)}(\sigma + it, \sigma - it; a, \lambda) \\
&\quad - 2\operatorname{Re} T_N^{(m,m)}(\sigma + it, \sigma - it; a, \lambda)
\end{aligned}
$$

holds. Furthermore, for any integer $H \geq 0$ *the expression*

$$
\begin{aligned}
(2.24)\qquad 2\operatorname{Re} R^{(m,m)}(\sigma + it, \sigma - it; \lambda) &= \left(\frac{|t|}{2\pi}\right)^{1-2\sigma} P_{2m}\left(2\sigma, \{\lambda \operatorname{sgn} t\}; \log\left(\frac{|t|}{2\pi}\right)\right) \\
&\quad + \sum_{h=1}^{[H/2]} \frac{(-1)^h t^{-2h}}{(2h)!} \left(\frac{|t|}{2\pi}\right)^{1-2\sigma} Q_{2h}^{m,m}\left(\sigma, \sigma, \{\lambda \operatorname{sgn} t\}; \log\left(\frac{|t|}{2\pi}\right)\right) \\
&\quad + 2\operatorname{Re} R_H^{(m,m)}(\sigma + it, \sigma - it; \lambda)
\end{aligned}
$$

follows, and also for any integer $K \geq 0$ *the expression* (2.15) *follows in particular for* $T_N^{(m,m)}(\sigma + it, \sigma - it; a, \lambda)$, *both in the same region of* $\sigma + it$ *above; Formula* (2.23) *with* (2.24) *and* (2.15) *gives a complete asymptotic expansion in the descending order of* t *as* $t \to \pm\infty$, *where each term of the asymptotic series is estimated as* (2.16)–(2.19).

We next supplement two exceptional (but important) cases of Theorem 2. One can observe that the region with $N = 1$ in Corollary 2.1 or 2.2 includes the lines $\sigma = 1/2$ and $\sigma = 1$. When $N = 1$ either the case $\sigma = 1/2$ of Corollary 2.1 or the limiting case $\sigma \to 1/2$ of Corollary 2.2 gives

Corollary 2.3. *Let* $m \geq 0$ *be an arbitrarily fixed integer, and* a, λ, R, T_1, $U_{1,k}$ *and* $V_{1,K}$ *be as in Theorem 1, and* $\widehat{P}_m$ *and* $Q_h^{m,m}$ *defined by* (2.20) *and* (2.11) *respectively. Then the formula*

$$
\begin{aligned}
(2.25)\qquad \int_0^1 \left|\phi^{(m)}\left(\frac{1}{2} + it, a + x, \lambda\right)\right|^2 dx &= \frac{\partial^{2m}}{\partial\sigma_1^m \partial\sigma_2^m} \left\{ -\frac{a^{1-\sigma_1-\sigma_2}}{1-\sigma_1-\sigma_2} \right. \\
&\quad \left. + 2\operatorname{Re} R(\sigma_1 + it, \sigma_2 - it; \lambda) \right\}\Bigg|_{\substack{\sigma_1=1/2 \\ \sigma_2=1/2}} \\
&\quad - 2\operatorname{Re}\left\{ e(\lambda)\phi^{(m)}\left(\frac{1}{2} + it, a + 1, \lambda\right) a^{1/2+it} \sum_{j=0}^{m} \frac{m!}{(m-j)!} \frac{(-\log a)^{m-j}}{\left(\frac{1}{2} + it\right)^{j+1}} \right\} \\
&\quad - 2\operatorname{Re} T_1^{(m,m)}\left(\frac{1}{2} + it, \frac{1}{2} - it; a + 1, \lambda\right)
\end{aligned}
$$

holds for any $t \in \mathbb{R}$. *Furthermore, for any integer* $H \geq 0$ *the expression*

$$\frac{\partial^{2m}}{\partial\sigma_1^m \partial\sigma_2^m}\left\{-\frac{a^{1-\sigma_1-\sigma_2}}{1-\sigma_1-\sigma_2} + 2\operatorname{Re} R(\sigma_1+it, \sigma_2-it; \lambda)\right\}\Bigg|_{\substack{\sigma_1=1/2\\ \sigma_2=1/2}} \tag{2.26}$$

$$= -\frac{\log^{2m+1} a}{2m+1} + \widehat{P}_{2m}\Big(\{\lambda \operatorname{sgn} t\}; \log\Big(\frac{|t|}{2\pi}\Big)\Big)$$

$$+ \sum_{h=1}^{[H/2]} \frac{(-1)^h t^{-2h}}{(2h)!} Q_{2h}^{m,m}\Big(\frac{1}{2}, \frac{1}{2}; \{\lambda \operatorname{sgn} t\}; \log\Big(\frac{|t|}{2\pi}\Big)\Big)$$

$$+ 2\operatorname{Re} R_H^{(m,m)}(\sigma+it, \sigma-it; \lambda)$$

follows, and also the expression (2.15) *follows in particular for* $T_1^{(m,m)}(1/2+it, 1/2-it; a, \lambda)$, *both on the lines* $t \in \mathbb{R} \setminus \{0\}$; *Formula* (2.25) *with* (2.26) *and* (2.15) *gives a complete asymptotic expansion in the descending order of* t *as* $t \to \pm\infty$, *where each term of the asymptotic series is estimated as* (2.16)–(2.19).

One can further observe that the case $N = 1$ of Corollary 2.2 implies the formula on the line $\sigma = 1$; this asserts

Corollary 2.4. *Let* $m \geq 0$ *be an arbitrarily fixed integer,* a, λ, R, T_1, $U_{1,k}$ *and* $V_{1,K}$ *as in Theorem 1, and* P_m *and* $Q_h^{m_1,m_2}$ *defined by* (2.10) *and* (2.11) *respectively. Then the formula*

$$\int_0^1 |\phi^{(m)}(1+it, a+x, \lambda)|^2 dx = a^{-1}\sum_{j=0}^{2m} \frac{(2m)!}{(2m-j)!}\log^{2m-j} a \tag{2.27}$$

$$+ 2\operatorname{Re} R^{(m,m)}(1+it, 1-it; \lambda)$$

$$- 2\operatorname{Re}\left\{e(\lambda)\phi^{(m)}(1+it, a+1, \lambda)a^{it}\sum_{j=0}^{m}\frac{m!}{(m-j)!}\frac{(-\log a)^{m-j}}{(it)^{j+1}}\right\}$$

$$- 2\operatorname{Re} T_1^{(m,m)}(1+it, 1-it; a, \lambda)$$

holds for any $t \in \mathbb{R} \setminus \{0\}$. *Furthermore, for any integer* $H \geq 0$ *the expression*

$$2\operatorname{Re} R^{(m,m)}(1+it, 1-it; \lambda) = \Big(\frac{|t|}{2\pi}\Big)^{-1} P_{2m}\Big(2, \{\lambda \operatorname{sgn} t\}; \log\Big(\frac{|t|}{2\pi}\Big)\Big) \tag{2.28}$$

$$+ \sum_{h=1}^{[H/2]} \frac{(-1)^h t^{-2h}}{(2h)!}\Big(\frac{|t|}{2\pi}\Big)^{-1} Q_{2h}^{m,m}\Big(1, 1, \{\lambda \operatorname{sgn} t\}; \log\Big(\frac{|t|}{2\pi}\Big)\Big)$$

$$+ 2\operatorname{Re} R_H^{(m,m)}(1+it, 1-it; \lambda)$$

follows, and also the expression (2.15) *follows in particular for* $T_1^{(m,m)}(1+it, 1-it; a, \lambda)$, *both on the lines* $t \in \mathbb{R} \setminus \{0\}$; *Formula* (2.27) *with* (2.28) *and* (2.15) *gives a complete asymptotic expansion in the descending order of* t *as* $t \to \pm\infty$, *where each term of the asymptotic series is estimated as* (2.16)–(2.19).

3. A FUNDAMENTAL FORMULA

The detailed proofs of Theorems 1 and 2, together with their corollaries, will be given in the forthcoming paper [Ka11], so we content ourselves here by describing a formula which is fundamental in proving Theorems 1 and 2.

Atkinson [At] first developed the dissection device to treat the product $\zeta(s_1)\zeta(s_2)$ in two independent complex variables; this method was further applied, upon enhanced by a Mellin-Barnes type integral technique, to study the product $\phi(s_1,x,\lambda)\phi(s_2,x,-\lambda)$ by the author [Ka2][Ka9], in which an initial rôle was played by the dissection formula

$$\begin{aligned}(3.1)\qquad \phi(s_1,x,\lambda)\phi(s_2,x,-\lambda) &= \zeta(s_1+s_2,x)+R(s_1,s_2;\lambda)+R(s_2,s_1;-\lambda)\\ &\quad + g(s_1,s_2;x,\lambda)+g(s_2,s_1;x,-\lambda),\end{aligned}$$

where R is defined by (2.3), and g by the Mellin-Barnes type integral

$$(3.2)\qquad g(s_1,s_2;x,\lambda)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Gamma(s_1+w)\Gamma(-w)}{\Gamma(s_1)}\zeta(s_1+s_2+w,x)\zeta_\lambda(-w)dw.$$

Here $\mathcal{C}$ denotes the vertical path, directed upward, which is suitably indented to separate the (possible) poles of $\Gamma(s_1+w)\zeta(s_1+s_2+w,x)$ at $w=1-s_1-s_2$ and $w=-s_1-n$ $(n=0,1,\dots)$ from those of $\Gamma(-w)\zeta_\lambda(-w)$ at $w=-1-n$ $(n=0,1,\dots)$. The formula which is fundamental in proving Theorem 1 is obtained (in principle) by integrating both sides of (3.2); the case $m=1$ of our previous result [Ka9, Theorem 1] asserts

Proposition 1. *Let $\widetilde{E}\subset\mathbb{C}^2$ be the set defined by* (2.1). *Then for any integer $N\geq 0$ in the region $1-N<\sigma_j<1+N$ $(j=1,2)$ except the points of $\widetilde{E}$, the formula*

$$\begin{aligned}(3.3)\qquad &\int_0^1 \phi(s_1,a+x,\lambda)\phi(s_2,a+x,-\lambda)dx\\ &=-\frac{a^{1-s_1-s_2}}{1-s_1-s_2}+R(s_1,s_2;\lambda)+R(s_2,s_1;-\lambda)\\ &\quad -S_N(s_1,s_2;a,\lambda)-S_N(s_2,s_1;a,-\lambda)\\ &\quad -T_N(s_1,s_2;a,\lambda)-T_N(s_2,s_1;a,-\lambda)\end{aligned}$$

holds, where R, S_N and T_N are given in (2.3)–(2.5). *Furthermore, for any integer $K\geq 0$ the expression*

$$(3.4)\qquad T_N(s_1,s_2;a,\lambda)=\sum_{k=1}^{K}U_{N,k}(s_1,s_2;a,\lambda)+V_{N,K}(s_1,s_2;a,\lambda),$$

together with that of $T_N(s_2,s_1;a,-\lambda)$, follows in the same region of (s_1,s_2) above, where $U_{N,k}$ and $V_{N,K}$ are given by (2.7) *and* (2.8) *respectively.*

Remark. The particular case $a=1$ of (3.3) was first established by the author [Ka1], and it has recently been rederived by Balasubramanian-Kanemitsu-Tsukada [BKT] in a different manner.

References

[An] J. Andersson, *Mean value properties of the Hurwitz zeta-function*, Math. Scand. **71** (1992), 295–300.

[At] F. V. Atkinson, *The mean-value of the Riemann zeta function*, Acta Math. **81** (1949), 353–376.

[Ba] R. Balasubramanian, *A note on Hurwitz's zeta-function*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1979), 41–44.

[BKT] R. Balasubramanian, S. Kanemitsu and H. Tsukada, *Contributions to the theory of Lerch zeta-function*, The Riemann zeta function and related themes: papers in honour of Professor K. Ramachandra, pp. 29–38, Ramanujan Math. Soc. Lect. Notes Ser., 2, Ramanujan Math. Soc., Mysore, 2006.

[EM] S. Egami and K. Matsumoto, *Asymptotic expansions of multiple zeta functions and power mean values of Hurwitz zeta functions*, J. London Math. Soc. (2) **66** (2002), 41–60.

[Er] A. Erdélyi (ed.), W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I (1953), McGraw-Hill, New York

[EM1] O. Espinosa and V. H. Moll, *On some integrals involving the Hurwitz zeta function: Part 1*, Ramanujan J. **6** (2002), 159-188.

[EM2] ———, *On some integrals involving the Hurwitz zeta function: Part 2*, Ramanujan J. **6** (2002), 449-468.

[G1] Guo Jinbao, *On the mean value formula of the derivative of Hurwitz zeta-function* (in Chinese), J. Yanan Univ. **13** (1994), 45-51, 65.

[G2] ———, *A class of new mean value formulas for the derivative of the Hurwitz zeta-function* (in Chinese), J. Math. Res. Expos. **16** (1996), 549-553.

[Iv] A. Ivić, *The Riemann Zeta-Function*, 1985, John Wiley & Sons, New York

[Ka1] M. Katsurada, *Asymptotic expansions of the mean values of Dirichlet L-functions III*, Manuscripta Math. **83** (1994), 425-442.

[Ka2] ———, *An application of Mellin-Barnes' type integrals to the mean square of Lerch zeta-functions*, Collect. Math. **48** (1997), 137-153.

[Ka3] ———, *On Mellin-Barnes type of integrals and sums associated with the Riemann zeta-function*, Publ. Inst. Math. (Beograd) (N.S.) **62(76)** (1997), 13-25.

[Ka4] ———, *An application of Mellin-Barnes type of integrals to the mean square of L-functions*, Liet. Mat. Rink. **38** (1998), 98-112.

[Ka5] ———, *Power series and asymptotic series associated with the Lerch zeta-function*, Proc. Japan Acad. Ser. A **74** (1998), 167-170.

[Ka6] ———, *Rapidly convergent series representations for* $\zeta(2n+1)$ *and their* χ*-analogue*, Acta Arith. **90** (1999), 79-89.

[Ka7] ———, *On an asymptotic formula of Ramanujan for a certain theta-type series*, Acta Arith. **97** (2001), 157-172.

[Ka8] ———, *Asymptotic expansions of certain q-series nd a formula of Ramanujan for specific values of the Riemann zeta-function*, Acta Arith. **107** (2003), 269-298.

[Ka9] ———, *An application of Mellin-Barnes type integrals to the mean square of Lerch zeta-function II*, Collect. Math. **56** (2005), 57-83.

[Ka10] ———, *Complete asymptotic expansions associated with Epstein zeta-functions*, Ramanujan J. **14** (2007), 249-275.

[Ka11] ———, *An application of Mellin-Barnes type integrals to the mean square of Lerch zeta-functions III*, (preprint).

[KL] J. F. Koksma and C. G. Lekkerkerker, *A mean value theorem for* $\zeta(s,w)$, Indag. Math. **14** (1952), 446-452.

[Kl1] D. Klusch, *Asymptotic equalities for the Lipschitz-Lerch zeta-function*, Arch. Math. (Basel) **49** (1987), 38-43.

[Kl2] ———, *A hybrid version of a theorem of Atkinson*, Rev. Roumaine Math. Pures Appl. **34** (1989), 721-728.

[KM1] M. Katsurada and K. Matsumoto, *Discrete mean values of Hurwitz zeta-functions*, Proc. Japan Acad. Ser. A **69** (1993), 164-169.

[KM2] ———, *Explicit formulas and asymptotic expansions for certain mean square of Hurwitz zeta-functions*, Proc. Japan Acad. Ser. A **69** (1993), 303-307.

[KM3] ———, *Explicit formulas and asymptotic expansions for certain mean square of Hurwitz zeta-functions I*, Math. Scand. **78** (1996), 161-177.

[KM4] ———, *Explicit formulas and asymptotic expansions for certain mean square of Hurwitz zeta-functions II*, in "New Trends in Probability and Statistics, Vol. 4" A. Laurinčikas, E. Manstavičius and V. Stakėnas (Eds.) VSP(Utrecht)/TEV(Vilnius), 1997, pp. 119-134.

[KM5] ———, *Explicit formulas and asymptotic expansions for certain mean square of Hurwitz zeta-functions III*, Compositio Math. **131** (2002), 239-266.

[KN] M. Katsurada and T. Noda, *Differential actions on the asymptotic expansions of non-holomorphic Eisenstein series*, Int. J. Number Theory **5** (2009), 1061-1088.

[L] M. Lerch, *Note sur la fonction* $K(w,x,s)=\sum_{n\geq 0}\exp\{2\pi inx\}(n+w)^{-s}$, Acta Math. **11** (1887), 19-24.

[Mi1] M. Mikolás, *Mellinsche Transformation und Orthogonalität bei* $\zeta(s,u)$*. Verallgemeinerung der Riemannschen Functionalgleichung von* $\zeta(s)$, Acta Sci. Math. Szeged **17** (1956), 143-164.

[Mi2] ———, *Integral formulae of arithmetical characteristics relating to the zeta-function of Hurwitz*, Publ. Math. Debrecen **5** (1957), 44–53.

[Mo1] Y. Motohashi, *Spectral mean values of Maass waveform of L-functions*, J. Number Theory **42** (1992), 258–284.

[Mo2] ———, *An explicit formula for the fourth power mean of the Riemann zeta-function*, Acta Math. **170** (1993), 181-220.

[Mo3] ———, *Spectral Theory of the Riemann-Zeta Function*, Cambridge University Press, Cambridge, 1997.

[R] V. V. Rane, *On Hurwitz zeta-function*, Math. Ann. **264**, (1983), 147–151.

[WW] [WW]E. T. Whittaker and G. N. Watson, *A course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1927.

[Z1] W. Zhang, *The Hurwitz zeta-function* (in Chinese), Acta Math. Sinica **33** (1990), 160–171.

[Z2] ———, *On the mean square value formula of Lerch zeta-function*, Adv. Math. (China) **22** (1993), 367–369.

[Z3] ———, *On the mean square value of Hurwitz zeta-function*, Illinois J. Math. **38** (1994), 71–78.

DEPARTMENT OF MATHEMATICS, HIYOSHI CAMPUS, KEIO UNIVERSITY, 4–1–1 HIYOSHI, KOUHOKU-KU, YOKOHAMA 223–8521, JAPAN

E-mail address: katsurad@hc.cc.keio.ac.jp; katsurad@z3.keio.jp