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On (α, β, γ) triple systems [1]

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Abstract

We introduce a notion of (α, β, γ) triple system which generalizes the familiar generalized Jordan triple system. We then discuss its realization by some bilinear algebras and vice versa. We also give a characterization of the structurable algebra of Allison in terms of $(-1,1)$ Freudenthal-Kantor triple system by imposing some additional triple product constraints.

1 Introduction

A triple product system V is a vector space over a field F with a tri-linear map $V \otimes V \otimes V \to V$.

We denote the tri-linear product by juxtaposition $xyz \in V$ hereafter for x, y, $z \in V$. A well-studied example is the (ϵ, δ) Freudenthal-Kantor triple system (hereafter referred to as (ϵ, δ) FKTS) with ϵ and δ being either $+1$ or -1 ([15]). See also ([8],[9]) for many earlier references on the subject. The (ϵ, δ) FKTS is defined by relations

$$\text{(i)} \qquad uv(xyz) = (uvx)yz + \epsilon x(vuy)z + xy(uvz)\,, \tag{1}$$

and

$$\text{(ii)} \qquad K(x,y) \in \text{ End } V \text{ defined by}\,,$$

$$K(x,y)z := xzy - \delta yzx\,, \tag{2}$$

satisfies

$$K(uvx, y) + K(x, uvy) + \delta K(u, K(x,y)v) = 0\,, \tag{3}$$

[1] This paper is a survey note, the details are described in other article.

for u, v, x, y, $z \in V$.

The special case of $\epsilon = -1$ for Eq. (1) without assuming Eq. (3) defines a generalized Jordan triple system. Also, a (ϵ, δ) FKTS is called balanced, if there exists a non-zero bilinear form $(.|.) : V \otimes V \to F$ such that

$$K(x, y) = (x|y)1_V\,, \tag{4}$$

for any x, $y \in V$ with 1_V being the identity map in End V.

One interesting problem for triple systems is its classification and realizations. If the underlying field F is of characteristic 0, then such a classification has been found in ([2]) for finite dimensional simple balanced $(-1, -1)$ FKTS. For the case of F being an algebraically closed field of zero characteristic, Meyberg ([11]) has earlier classified another triple system which is essentially equivalent to simple balanced $(1, 1)$ FKTS by a simple transformation.

However, the realization of the triple system in terms of some bilinear algebras becomes quite easy for some classes of triple systems when we assume in addition the existence of a privileged element $e \in V$ which behaves as an analogue of the identity element for bilinear algebras. As a matter of fact, essentially only one realization can often exist for such systems. To illustrate it, let us consider the case of Jordan triple system J where we have

$$\text{(i)} \qquad uv(xyz) = (uvx)yz - x(vuy)z + uv(xyz) \tag{a}$$

$$\text{(ii)} \qquad xyz = zyx\,, \tag{b}$$

which is a special case of $\epsilon = -1$ and $\delta = +1$ with $K(x, y) = 0$ in Eqs. (1) and (2). Such a system, of course, possesses many different realizations. However, suppose that we impose an extra ansatz of

(iii) There exists a privileged element $e \in J$ satisfying

$$exe = x\,, \tag{c}$$

for any $x \in J$.

Then, Loos ([10]) has proved:

Theorem 1.1

Let J be a Jordan triple system over a field F of characteristic $\neq 2$, $\neq 3$, which satisfies the extra ansatz of Eq. (c). Then, the homotope algebra $J^{(e)}(= A)$ with the bilinear product defined by

$$x \cdot y := xey\,, \tag{$*$}$$

is a unital Jordan algebra with e being the unit element of A. Moreover, for any $x, y\,, z \in J$, we have

$$xyz := x \cdot (y \cdot z) - y \cdot (x \cdot z) + (y \cdot x) \cdot z \,. \tag{**}$$

Conversely, if A is a unital Jordan algebra with the unit element e, then the triple product xyz given by Eq. (**) defines a Jordan triple system satisfying the extra condition Eq. (c).

In ([3], Theorem 3.1 with Remark 3.16), we have also proved the following theorem which essentially generalizes the previous one.

Theorem 1.2

Let J be a generalized Jordan triple system over a field F of characteristic $\neq 2,\ 3$, possessing a privileged element $e \in J$ satisfying

$$eex = xee = x\,, \tag{5}$$

for any $x \in J$. Then, the resulting homotope algebra $A(\equiv J^{(e)})$ defined in the vector space J with the multiplication given by $x \cdot y = xey$ and with the linear map $x \to \overline{x} = exe$ is a unital involutive non-commutative Jordan algebra (i.e., flexible Jordan-admissible algebra, see [14]), satisfying the following additional property: $D_{x,y} \in \mathrm{End}\, A$ defined by

$$D_{x,y} := (x, y, \cdot) - (y, x, \cdot) = (\cdot, x, y) - (\cdot, y, x)\,, \tag{6}$$

is a derivation of A. Moreover, the original triple product is expressed as

$$xyz = x \cdot (\overline{y} \cdot z) - \overline{y} \cdot (x \cdot z) + (\overline{y} \cdot x) \cdot z\,, \tag{7}$$

in terms of the bilinear products of A for any $x\,, y,\ z \in A$. Conversely, let $(A, \cdot, -)$ be a unital involutive non-commutative Jordan algebra over a field F of characteristic not 2, satisfying the condition that $D_{x,y}$ defined by Eq. (6) is a derivation of A. Then, the triple product xyz given by Eq. (7) defines a generalized Jordan triple system satisfying the extra relation Eq. (5).

In the subsequent section 2, we will first discuss triple systems which we call (α, β, γ) triple system ($\alpha,\ \beta,\ \gamma \in F$), defined by

$$uv(xyz) = \alpha(uvx)yz + \beta x(vuy)z + \gamma xy(uvz)\,, \tag{8}$$

which possesses the privileged element $e \in V$ satisfying Eq. (5). We call then V to be a unital (α, β, γ) TS.

In ending this section, we remark that the (α, β, γ) TS is intimately related to a Lie algebra for the case of $\gamma = 1$ but to a Jordan algebra for $\gamma = -1$ by the following reason. Introducing a multiplication operator $L(x, y)$ by

$$L(x, y)z := xyz\,, \tag{9}$$

Eq. (8) is rewritten as

$$L(u,v)L(x,y) - \gamma L(x,y)L(u,v) = \alpha L(uvx,y) + \beta L(x,vuy)\,, \tag{10}$$

so that it defines a Lie algebra for $\gamma = 1$ and a Jordan algebra for $\gamma = -1$. Some cases of $\gamma = -1$ satisfying $xyz = yxz$ and its super generalization have been discussed in ([7],[12]) for construction of some Jordan superalgebras. Letting $x \leftrightarrow u$ and $y \leftrightarrow v$ in Eq. (10), it also gives

$$L(x,y)L(u,v) - \gamma L(u,v)L(x,y) = \alpha L(xyu,v) + \beta L(u,yxv)\,.$$

Eliminating $L(x,y)L(u,v)$ from both relations, we obtain

$$\begin{aligned}&(1-\gamma^2)L(u,v)L(x,y)\\ &\quad = \alpha L(uvx,y) + \beta L(x,vuy) + \alpha\gamma L(xyu,v) + \beta\gamma L(u,yxv)\,.\end{aligned} \tag{11}$$

2 (α, β, γ) Triple Systems

In this section, we will study some properties of unital (α,β,γ) TS so that the triple product satisfies

$$\text{(i)} \qquad uv(xyz) = \alpha(uvx)yz + \beta x(vuy)z + \gamma xy(uvz)\,, \tag{12}$$

$$\text{(ii)} \qquad eex = xee = x\,. \tag{13}$$

We introduce a bilinear product $x \cdot y$ and a linear map $x \to \overline{x}$ again by

$$x \cdot y := xey\,, \tag{14}$$

$$\overline{x} := exe\,. \tag{15}$$

We will then show the following theorem first.

<u>**Theorem 2.1**</u>

Let V be a non-zero (i.e., $V \neq 0$) unital (α,β,γ) TS with $\beta \neq 0$. We then have

(i)

$$\alpha + \beta + \gamma = 1\,. \tag{16}$$

(ii) $A(\equiv V^{(e)})$ is unital with the unit element e, i.e.,

$$e \cdot x = x \cdot e = x\,. \tag{17}$$

(iii)
$$\overline{\overline{x}} = x \quad \text{with} \quad \overline{e} = e\,. \tag{18}$$

(iv) The original triple product in V can be expressed as

$$xyz = \frac{1}{\beta}\{\overline{y}\cdot(x\cdot z) - \alpha(\overline{y}\cdot x)\cdot z - \gamma x\cdot(\overline{y}\cdot z)\}\,, \tag{19}$$

in terms of the bilinear products in A.

(v) If the constants α, β, γ satisfy condition

$$(1-\alpha)(1-\gamma) = 0\,, \tag{20}$$

then $x \to \overline{x}$ is a involution of A, i.e., we have

$$\overline{x\cdot y} = \overline{y}\cdot\overline{x}\,. \tag{21}$$

(vi) If the constants α, β, γ satisfy

$$\alpha^2 + \beta^2 + \gamma^2 = 1\,, \tag{20$'$}$$

then $x \to \overline{x}$ is an automorphism of A, i.e.,

$$\overline{x\cdot y} = \overline{x}\cdot\overline{y}\,. \tag{22}$$

(vii) Suppose that $\gamma \neq 1$. Then, the associator defined by

$$(x, y, z) := (x\cdot y)\cdot z - x\cdot(y\cdot z)\,, \tag{23}$$

satisfies

$$(1-\alpha)(x, y, z) = (1-\beta)(y, x, z)\,. \tag{24}$$

Theorem 2.2

Let V be a unital (α, β, γ) triple system such that $\alpha \neq 1$, $\gamma^2 \neq 1$, and $\beta + 2\gamma \neq 0$. Then, the associated homotope algebra A is a unital, involutive, commutative, and associative algebra. Moreover, the triple product is given by

$$xyz = (x\cdot\overline{y})\cdot z = x\cdot(\overline{y}\cdot z)\,. \tag{25}$$

Conversely, if A is a involutive commutative associative algebra, then the triple product given by Eq. (25) defines a (α, β, γ) triple system with $\alpha + \beta + \gamma = 1$ for any α, β, γ $\in$ F.

<u>**Theorem 2.3**</u>

Let V be a unital $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ TS over the field F of characteristic $\neq 2, \neq 3$. Then, the associated homotope algebra A is a unital associative algebra satisfying $[A, [A, A]] = 0$ with an automorphism $x \to \overline{x}$ of order 2. Moreover, the triple product is determined to be

$$xyz = \frac{1}{2}(x \cdot \overline{y} + \overline{y} \cdot x) \cdot z\,. \tag{26}$$

Conversely, let A be an associative algebra satisfying $[A, [A, A]] = 0$ with a order two automorphism $x \to \overline{x}$ over a field F of characteristic $\neq 2, \neq 3$. Then, the triple product given by Eq. (26) determines a $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ triple system. Further if A is unital in addition, then the triple system is also unital.

3 (1,1,-1) Triple System

Here, we will first show the following theorem.

<u>**Theorem 3.1**</u>

Let V be the unital $(1, 1, -1)$ triple system over the field F of characteristics not 2. Then, the homotope algebra A (see Theorem 2.1) is a unital, involutive, alternative algebra. Moreover, the original triple product is expressed as

$$xyz = (x \cdot \overline{y}) \cdot z\,, \tag{27}$$

in terms of bilinear products of A. Conversely, if A is a unital involutive alternative algebra, then the triple product given by Eq. (27) defines a unital $(1, 1, -1)$ triple system.

<u>**Proposition 3.2**</u>

Let A be a involutive alternative algebra over a field F with the bilinear product $x \cdot y$ and with the involutive map $x \to \overline{x}$. Then, the triple product given by Eq. (27) defines a $(1, 1, -1)$ TS. Moreover, if A is unital with the unit element e, then the triple system is also unital, i.e., it satisfies Eqs. (13).

<u>**Corollary 3.3**</u>

Let V be a $(1, 1, -1)$ triple system over a field F of characteristic not 2 with a symmetric bilinear non-degenerate form $< .|. >$ satisfying

$$xxy = yxx = < x|x > y\,. \tag{28}$$

Then for any element $e \in V$ satisfying $< e|e > = 1$, V becomes a unital $(1, 1, -1)$ triple system. Moreover, the associated homotope algebra A is a Hurwitz algebra (i.e., unital composition algebra) satisfying

$$< xy|xy > = < x|x >< y|y > .$$

Conversely, if A is a Hurwitz algebra with the unit element e, then the triple product xyz given by Eq. (23) defines a unital $(1, 1, -1)$ triple system satisfying Eq. (28).

4 (-1,1,1) Triple System

We will describe the following theorem in this section.

Theorem 4.1

Let V be a unital $(-1, 1, 1)$ TS over the field F of characteristic not 2. Then, its homotope algebra A is a unital involutive associative algebra. Moreover it satisfies an additional constraint of

$$[A, [A, A]] = 0 . \tag{29}$$

Finally, the original triple product is now expressed as

$$xyz = (2\overline{y} \cdot x - x \cdot \overline{y}) \cdot z , \tag{30}$$

in terms of the bilinear products of A. Conversely, if A is a unital, involutive associative algebra satisfying Eq. (29), then the triple product xyz given by Eq. (30) defines a unitary $(-1, 1, 1)$ TS.

Proposition 4.2

Let A be a involutive associative algebra. Then, a triple product xyz given by Eq. (30) satisfies

$$uv(xyz) + (uvx)yz - x(vuy)z - xy(uvz) = 2[[x, \overline{y}], w] \cdot z , \tag{31}$$

with

$$w := 2\overline{v} \cdot u - u \cdot \overline{v} . \tag{32}$$

Especially, if we have $[[A, A], A] = 0$. Then, xyz defines a $(-1, 1, 1)$ triple system.

5 (1,-1,1) Triple System and Structurable Algebra

The case of $\alpha = \gamma = 1$ and $\beta = -1$ is perhaps the most interesting case, since Eqs. $(1 - \alpha)(x, y, z) = (1 - \beta)(y, x, z)$, $(1 - \alpha)(\beta + 2\gamma)(\overline{y} \cdot x - x \cdot \overline{y}) = 0$, need not be considered so as not to give new constraints. However, this case is treated already in Theorem 1.2, since a $(1,-1,1)$ TS is equivalent to a generalized Jordan triple system.

However, if we modify the unital condition for the system by some additional ansatz, we can have the following new result:

Theorem 5.1

Let V be a $(-1,1)$ FKTS over a field F of characteristic $\neq 2$, $\neq 3$. Suppose that V has a privileged element $e \in V$ satisfying a modified unital condition of

$$\text{(i)} \qquad eex = x\,, \tag{33}$$

$$\text{(ii)} \qquad exe + 2xee = 3x\,. \tag{34}$$

for any $x \in V$. We then introduce a linear map $x \to \overline{x}$ and a bilinear product $x \cdot y$ in V by

$$\text{(i)} \qquad \overline{x} := 2x - xee\,, \tag{35}$$

$$\text{(ii)} \qquad x \cdot y := yex - \overline{x}\,\overline{y}e + \overline{x}ey\,. \tag{36}$$

The resulting bilinear algebra A is then a unital involutive algebra with the unit element e and with the involution given by $x \to \overline{x}$. Moreover, the original triple product is expressed as

$$xyz = (z \cdot \overline{y}) \cdot x - (z \cdot \overline{x}) \cdot y + (x \cdot \overline{y}) \cdot z\,, \tag{37}$$

in terms of the bilinear product, implying that the algebra A is a structurable algebra. Conversely, if A is a structurable algebra with the unit element e and with the involution map $x \to \overline{x}$, then the triple product given by Eq. (37) defines a $(-1,1)$ FKTS, satisfying Eqs. (33) and (34).

6 Peirce decompositions

In this section, we will descrive about Peirce decompositions of several triple systems over the field F of characteristic not 2.

Theorem 6.1

Let V be a (α,β,γ) TS. Then we have

$$V = V^{+} \oplus V^{-},$$

where $V^{\pm} = \{x \in V | Q_e(x) := exe = \pm x\}$.

We will now denote V_{ij} by $V_{ij} = \{x|eex = ix, xee = jx\}$.

Theorem 6.2 ([6])

Let V be a $(-1,1)$-F-K.t.S. with $eex = x$. Then we have

$$V = V_{11}^{+} \oplus V_{11}^{-} \oplus V_{13}^{+} \oplus V_{13}^{-},$$

where

$$V_{11}^{\pm} = \{x|Q_e(x) = \pm x\}, V_{13}^{\pm} = \{x|Q_e(x) = \pm 3x\}.$$

Cor.

Let V be a structurable algebra. Then we have

$$V = V_{11}^{+} \oplus V_{11}^{-} \oplus V_{13}^{+} \oplus V_{13}^{-}.$$

Theorem 6.3 ([4])

Let V be a $(-1,-1)$-F-K.t.s. with $eex = x$. Then we have

$$V = V_{11}^{+} \oplus V_{11}^{-} \oplus V_{1-1}^{+} \oplus V_{1-1}^{-},$$

where

$$V^{\pm} = \{x|Q_e(x) = \pm x\}.$$

Remark. For $(+,+,-)$, if we set $x \cdot y = xey$, Theorem 6.1 implies a Peirce decomposition of alternative algebras.
Remark. For $(-,+,+)$, if we set $x \cdot y = xey$, Theorem 6.1 implies a Peirce decomposition of associative algebras.
Remark. For a weakly commutative $(-1,1)$-F-K.t.s, we have 6-components Peirce decomposition:

$$V = V_{00} \oplus V_{01} \oplus V_{\frac{1}{2}\frac{1}{2}} \oplus V_{11}^{+} \oplus V_{11}^{-} \oplus V_{13}^{-}$$

For the definition of an anti-structurable algebra, we refer in ([5]).

Theorem 6.4

Let V be an anti-structurable algebra. Then we have

$$V = V^{+} \oplus V^{-}$$

where $V^{\pm} = \{x|R(x) = xee = \pm x\}$.

Proof. From the definition of an anti-structurable algebra, it follows that

$$xyz = (x\overline{y})z - (z\overline{y}) + (z\overline{x})y, \quad eez = z, \quad eze = z, \quad zee = \overline{z}.$$

On the other hand, we have

$$(zee)ee = ze(eee) + e(eze)e - ee(zee), \quad R^2(z) = z.$$

This completes the proof.

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