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Arithmetical rank of squarefree monomial ideals of height two whose quotient rings are Cohen–Macaulay

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1. Introduction

Let S be a polynomial ring over a field K and I a squarefree monomial ideal of S. We denote by $G(I)$ the minimal set of monomial generators of I. The *arithmetical rank* of I is defined by the minimum number of elements of S such that those generate I up to radical:

$$\operatorname{ara} I := \min \left\{ r \ : \ \text{there exist } g_1, \ldots, g_r \in S \text{ such that } \sqrt{(g_1, \ldots, g_r)} = \sqrt{I} \right\}.$$

By the result due to Lyubeznik [12], inequalities

$$\operatorname{height} I \leq \operatorname{pd}_S S/I \leq \operatorname{ara} I$$

hold, where $\operatorname{height} I$ is the height of I and $\operatorname{pd}_S S/I$ is the projective dimension of S/I over S. We sometimes write $\operatorname{pd}_S S/I$ as $\operatorname{pd} S/I$ if there is no fear of confusion. It is natural to ask when $\operatorname{ara} I = \operatorname{pd}_S S/I$ holds. Examples of squarefree monomial ideals those satisfy this equality are found in e.g., [1, 2, 3, 4, 5, 7, 9, 10, 11, 13, 14]. We say that I is a *set-theoretic complete intersection* if $\operatorname{ara} I = \operatorname{height} I$ holds. When this is the case, the equality $\operatorname{ara} I = \operatorname{pd}_S S/I$ holds. On the other hand, we say that S/I is *Cohen–Macaulay* if $\operatorname{pd}_S S/I = \operatorname{height} I$ holds. By definition, if I is a set-theoretic complete intersection, then S/I is Cohen–Macaulay. However for the converse, there are counterexamples (see [15, 10]), though these depend on the characteristic of K. Our main result on this report is the following theorem.

Theorem 1.1. *Let I be a squarefree monomial ideal of S. Suppose that S/I is Cohen–Macaulay. If* $\operatorname{height} I = 2$, *then I is a set-theoretic complete intersection. That is,*

$$\operatorname{ara} I = \operatorname{pd}_S S/I = \operatorname{height} I = 2.$$

This theorem says that the converse holds true when $\operatorname{height} I = 2$. Note that counterexamples found in [15, 10] are ideals of height 3.

The key in the proof of Theorem 1.1 is Proposition 4.2. In this report, we observe the outline of the proof of Theorem 1.1 and prove Proposition 4.2. For the detailed proof of Theorem 1.1, please see [8].

2. Preliminaries

In this section, we recall the notion of the Stanley–Reisner ring and the Alexander duality.

Let $X = \{x_1, \dots, x_n\}$ be a set of vertices. A collection Γ of subsets of X is called a simplicial complex on the vertex set X if (i) $\{x_i\} \in \Gamma$ for all $i = 1, \dots, n$; (ii) if $F \in \Gamma$, then $G \in \Gamma$ for all $G \subset F$. If Γ consists of all subsets of its vertex set, then Γ is called a simplex. An element $F \in \Gamma$ is called a face and a maximal face of Γ is called a facet. A simplicial complex is determined by its facets. When the set of facets of Γ is $\{G_1, \dots, G_s\}$, we write $\Gamma = \langle G_1, \dots, G_s \rangle$. The dimension of Γ is defined by $\dim \Gamma := \max\{|F| - 1 : F \in \Gamma\}$. Throughout this report, we assume $\dim \Gamma < |X| - 2$. The Alexander dual complex of Γ is defined by

$$\Gamma^* := \{F \subset X \ : \ X \setminus F \notin \Gamma\}.$$

This is also a simplicial complex on X. Note that $(\Gamma^*)^* = \Gamma$.

We identify the vertex set $X = \{x_1, \dots, x_n\}$ with the set of variables of $S = K[X] = k[x_1, \dots, x_n]$. The Stanley–Reisner ideal of Γ is defined by

$$I_\Gamma := (m_F \ : \ F \subset X, \ F \notin \Gamma), \quad \text{where} \quad m_F = \prod_{x_i \in F} x_i.$$

The quotient ring $K[\Gamma] := K[X]/I_\Gamma$ is called the Stanley–Reisner ring of Γ. The prime decomposition of I_Γ is

$$I_\Gamma = \bigcap_{G \in \Gamma:\ \text{a facet}} P_G, \quad \text{where} \quad P_G = (x_i \in X \ : \ x_i \notin G).$$

On the other hand, the Stanley–Reisner ideal I_{Γ^*} is minimally generated by

$$G(I_{\Gamma^*}) = \{m_{X \setminus G} \ : \ G \in \Gamma \text{ is a facet}\}.$$

In above, we construct a squarefree monomial ideal of $K[X]$ from a given simplicial complex Γ on X with $\dim \Gamma < |X| - 2$. On the contrary, we can construct a simplicial complex on X when a squarefree monomial ideal I of $K[X]$ with $\operatorname{indeg} I := \min\{\deg m : m \in G(I)\} \geq 2$. When $I = I_\Gamma$, then the ideal $I^* := I_{\Gamma^*}$ is called the Alexander dual ideal of I.

Example 2.1. Let Γ be the simplicial complex on $X = \{x_1, \dots, x_6\}$ whose facets are

$$\{x_1, x_2, x_3\}, \ \{x_3, x_4, x_5\}, \ \{x_3, x_5, x_6\}$$

(see Figure 1). Then

$$\begin{aligned} I = I_\Gamma &= (x_4, x_5, x_6) \cap (x_1, x_2, x_6) \cap (x_1, x_2, x_4), \\ I^* = I_{\Gamma^*} &= (x_4 x_5 x_6, x_1 x_2 x_6, x_1 x_2 x_4). \end{aligned}$$

Remark 2.2. By the Alexander duality, it is clear that $\operatorname{indeg} I^* = \operatorname{height} I$. Moreover, Eagon-Reiner [6] proved that I^* has a linear resolution if and only if the quotient ring is Cohen–Macaulay.

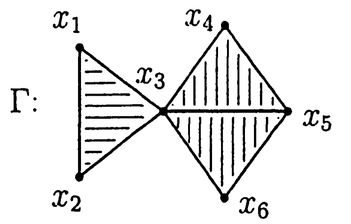

FIGURE 1. $\Gamma = \langle \{x_1, x_2, x_3\}, \{x_3, x_4, x_5\}, \{x_3, x_5, x_6\} \rangle$

3. GENERALIZED TREE

The generalized tree, which was introduced by Barile and Terai [4], is the notion on simplicial complexes. The definition is recursive: (i) a simplex is a generalized tree; (ii) if Γ is a generalized tree on X, then for an arbitrary face $F \in \Gamma$ and an arbitrary new vertex x_0, the union $\Gamma' := \Gamma \cup \mathrm{co}_{x_0} F$ is a generalized tree on $X' := X \cup \{x_0\}$, where $\mathrm{co}_{x_0} F$ is the simplex on $F \cup \{x_0\}$.

Example 3.1. As noted below, the simplicial complex Γ in Example 2.1 is a generalized tree (see also Figure 2).

First, $\Gamma_1 := \langle \{x_1, x_2, x_3\} \rangle$ is a simplex, thus it is a generalized tree. Second, set $F_1 = \{x_3\} \in \Gamma_1$. Then $\Gamma_2 := \Gamma_1 \cup \mathrm{co}_{x_4} F_1 = \langle \{x_1, x_2, x_3\}, \{x_3, x_4\} \rangle$ is a generalized tree. Third, set $F_2 = \{x_3, x_4\} \in \Gamma_2$. Then $\Gamma_3 := \Gamma_2 \cup \mathrm{co}_{x_5} F_2 = \langle \{x_1, x_2, x_3\}, \{x_3, x_4, x_5\} \rangle$ is a generalized tree. Last, set $F_3 = \{x_3, x_5\} \in \Gamma_3$. Then $\Gamma_4 := \Gamma_3 \cup \mathrm{co}_{x_6} F_3 = \langle \{x_1, x_2, x_3\}, \{x_3, x_4, x_5\}, \{x_3, x_5, x_6\} \rangle = \Gamma$ is a generalized tree.

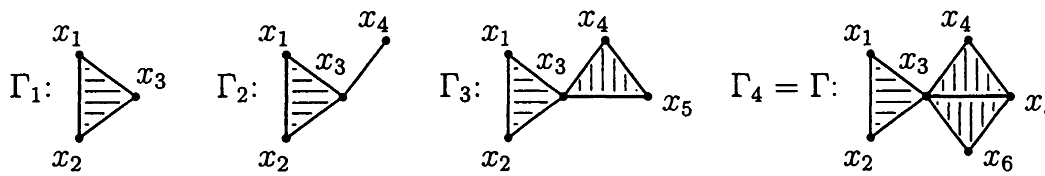

FIGURE 2. Generalized trees

The following lemma can be obtained by [4, Lemma 2] using the Alexander duality (see Remark 2.2).

Lemma 3.2. *Let* Γ *be a simplicial complex on* X *with* $\dim \Gamma < |X| - 2$. *Then* Γ *is a generalized tree if and only if* $\mathrm{height}\, I_{\Gamma^*} = 2$ *and* S/I_{Γ^*} *is Cohen–Macaulay.*

Barile and Terai [4] used the original form of this lemma to prove that if I has a 2-linear resolution, then $\mathrm{ara}\, I = \mathrm{pd}_S\, S/I$, which was first proved by Morales [13]. Thanks to the inductive definition of generalized tree, the proof due to Barile and Terai was proceeded by induction on $|X|$, and done by comparing the projective dimensions of $K[\Delta]$ and $K[\Delta']$, the arithmetical ranks of I_Δ and $I_{\Delta'}$ which is needed to guarantee the inductive step. In fact, our motivation is to consider the Alexander dual of these results.

4. Key result

In this section, we state the outline of the proof of Theorem 1.1.

Let Δ be a simplicial complex on X with $\dim \Delta < |X| - 2$. Set $\Gamma = \Delta^*$. Let F be a face of Γ and x_0 a new vertex. Set $\Gamma' = \Gamma \cup \mathrm{co}_{x_0} F$, $X' = X \cup \{x_0\}$, and $\Delta' = (\Gamma')^*$.

First we compare the projective dimensions of $K[\Delta]$ and $K[\Delta']$.

Lemma 4.1. *Using above notations, we have*

$$\operatorname{pd} K[\Delta'] = \operatorname{pd} K[\Delta].$$

For the proof of this lemma, please see [8].

Second we compare the arithmetical ranks of I_Δ and $I_{\Delta'}$.

Proposition 4.2. *We use the notations as above. If* $\operatorname{ara} I_\Delta = 2$, *then* $\operatorname{ara} I_{\Delta'} \leq 2$. *In particular, if* $\operatorname{ara} I_\Delta = \operatorname{pd} K[\Delta] = 2$, *then the same equalities hold for* Δ'.

The proof of Theorem 1.1 is done by induction on $|X|$ using Lemma 3.2. Proposition 4.2 guarantees the inductive step on the proof.

Proof of Proposition 4.2. Set

$$G(I_\Delta) = \{m_1, \ldots, m_\mu\}.$$

Then it is easy to see that

$$I_{\Delta'} = (m_0, x_0 m_1, \ldots, x_0 m_\mu), \tag{4.1}$$

where $m_0 = m_{X \setminus F}$. Let G be a facet of Γ which contains F. We may assume $m_1 = m_{X \setminus G}$. Then m_1 divides m_0.

Let $g_1, g_2 \in I_\Delta$ be elements which generate I_Δ up to radical. Since $m_1 \in I_\Delta = \sqrt{(g_1, g_2)}$, there exists an integer ℓ such that $m_1^\ell \in (g_1, g_2)$. Therefore we can write as

$$m_1^\ell = a_1 g_1 + a_2 g_2, \qquad a_1, a_2 \in K[X].$$

Set

$$g_1' = x_0 g_1 - a_2 m_0, \quad g_2' = x_0 g_2 + a_1 m_0.$$

We claim that g_1', g_2' generate $I_{\Delta'}$ up to radical. Set $J = (g_1', g_2')$. Since $g_1', g_2' \in I_{\Delta'}$, it is clear $\sqrt{J} \subset I_{\Delta'}$. We prove the opposite inclusion.

Since

$$a_1 g_1' + a_2 g_2' = x_0(a_1 g_1 + a_2 g_2) = x_0 m_1^\ell,$$

we have $x_0 m_1^\ell \in J$, thus $x_0 m_1 \in \sqrt{J}$. Since m_1 divides m_0, we have $x_0 m_0 \in \sqrt{J}$. Then we have $x_0 g_1, x_0 g_2 \in \sqrt{J}$ because $x_0 g_1', x_0 g_2' \in J$. This leads that $x_0 m_1, \ldots, x_0 m_\mu \in \sqrt{J}$. On the other hand, we also have $a_2 m_0, a_1 m_0 \in \sqrt{J}$. Since

$$g_1(a_1 m_0) + g_2(a_2 m_0) = m_0(a_1 g_1 + a_2 g_2) = m_0 m_1^\ell,$$

we have $m_0 m_1 \in \sqrt{J}$. Again since m_1 divides m_0, we have $m_0 \in \sqrt{J}$, as required. $\square$

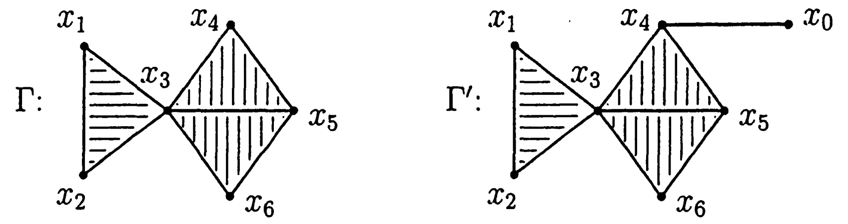

FIGURE 3. Γ and Γ'

Example 4.3. Let Γ be the simplicial complex as in Example 2.1. Set $F = \{x_4\}$ and $\Gamma' = \Gamma \cup \mathrm{co}_{x_0} F$. The vertex set of Γ' is $X' = X \cup \{x_0\}$. Then facets of Γ' are $\{x_0, x_4\}$ together with facets of Γ. Therefore

$$I_{\Gamma'} = (x_0, x_4, x_5, x_6) \cap (x_0, x_1, x_2, x_6) \cap (x_0, x_1, x_2, x_4) \cap (x_1, x_2, x_3, x_5, x_6),$$
$$I_{\Delta'} = (x_0x_4x_5x_6, x_0x_1x_2x_6, x_0x_1x_2x_4, x_1x_2x_3x_5x_6).$$

In this case, $m_0 = x_1x_2x_3x_5x_6$. Note that

$$I_\Gamma = (x_4, x_5, x_6) \cap (x_1, x_2, x_6) \cap (x_1, x_2, x_4),$$
$$I_\Delta = (x_4x_5x_6, x_1x_2x_6, x_1x_2x_4).$$

The facet of Γ which contains F is $\{x_3, x_4, x_5\}$. It corresponds to $m_1 := x_1x_2x_6 \in G(I_\Delta)$ and this divides m_0.

Set

$$\begin{cases} g_1 = x_1x_2x_6, \\ g_2 = x_4x_5x_6 + x_1x_2x_4, \end{cases} \quad \text{and} \quad \begin{cases} h_1 = x_1x_2x_4, \\ h_2 = x_4x_5x_6 + x_1x_2x_6. \end{cases}$$

Then $\sqrt{(g_1, g_2)} = \sqrt{(h_1, h_2)} = I_\Delta$ (this fact can be easily seen by [14, Lemma, p. 249]). Since $g_1 = m_1$, we can easily construct two elements g_1', g_2' which generate $I_{\Delta'}$ up to radical from g_1, g_2 as in the proof of Proposition 4.2:

$$\begin{cases} g_1' = x_0x_1x_2x_6, \\ g_2' = x_0x_4x_5x_6 + x_0x_1x_2x_4 + x_1x_2x_3x_5x_6. \end{cases}$$

(In this case, $a_1 = 1$, $a_2 = 0$. We can also prove that this g_1', g_2' generate $I_{\Delta'}$ up to radical by [14, Lemma, p. 249].) On the other hand, for h_1, h_2, the construction of two elements h_1', h_2' which generate $I_{\Delta'}$ up to radical is rather complicated. In this case,

$$m_1^2 = -x_5x_6^2h_1 + x_1x_2x_6h_2.$$

Thus $a_1 = -x_5x_6^2$, $a_2 = x_1x_2x_6$. Therefore

$$\begin{cases} h_1' = x_0x_1x_2x_4 - x_1x_2x_6 \cdot x_1x_2x_3x_5x_6, \\ h_2' = x_0x_4x_5x_6 + x_0x_1x_2x_6 - x_5x_6^2 \cdot x_1x_2x_3x_5x_6. \end{cases}$$

By (4.1), if I_Δ is generated by $g_1, \ldots, g_h$ up to radical, then $m_0, g_1, \ldots, g_h$ generate $I_{\Delta'}$ up to radical. Therefore the inequality $\operatorname{ara} I_{\Delta'} \leq \operatorname{ara} I_\Delta + 1$ always holds. Proposition 4.2 says that more precisely, the inequality $\operatorname{ara} I_{\Delta'} \leq \operatorname{ara} I_\Delta$

holds when $\operatorname{ara} I_\Delta = 2$. In general, does this inequality hold? By the similar technique to the proof of Proposition 4.2, we have the following corollary.

Corollary 4.4. *We use the notations as above. If* $\operatorname{ara} I_\Delta$ *is even, then*

$$\operatorname{ara} I_{\Delta'} \leq \operatorname{ara} I_\Delta.$$

Proof. Set $\operatorname{ara} I_\Delta = 2h$. We assume that $g_1, \ldots, g_{2h}$ generate I_Δ up to radical. Since $m_1 \in I_\Delta = \sqrt{(g_1, \ldots, g_{2h})}$, there is an integer ℓ such that $m_1^\ell \in (g_1, \ldots, g_{2h})$. Then we can write as

$$m_1^\ell = a_1 g_1 + \cdots + a_{2h} g_{2h}, \qquad a_1, \ldots, a_{2h} \in K[X].$$

Set

$$g'_{2i-1} = x_0 g_{2i-1} - a_{2i} m_0, \quad g'_{2i} = x_0 g_{2i} + a_{2i-1} m_0, \quad i = 1, \ldots .h.$$

We claim that $g'_1, \ldots, g'_{2h}$ generate $I_{\Delta'}$ up to radical.

Set $J = (g'_1, \ldots, g'_{2h})$. First we note that

$$x_0(a_{2i-1} g_{2i-1} + a_{2i} g_{2i}) = a_{2i-1} g'_{2i-1} + a_{2i} g'_{2i} \in J.$$

Then

$$x_0 m_1^\ell = x_0(a_1 g_1 + \cdots + a_{2h} g_{2h}) = \sum_{i=1}^{h} x_0(a_{2i-1} g_{2i-1} + a_{2i} g_{2i}) \in J.$$

Thus $x_0 m_1 \in \sqrt{J}$. Since m_1 divides m_0, we have $x_0 m_0 \in \sqrt{J}$. Then $x_0 g'_i \in J$ implies $x_0 g_i \in \sqrt{J}$ for $i = 1, \ldots, 2h$. Therefore $x_0 m_1, \ldots, x_0 m_\mu \in \sqrt{J}$. On the other hand, we also have $a_i m_0 \in \sqrt{J}$ for $i = 1, \ldots, 2h$. Since

$$m_0 m_1^\ell = m_0(a_1 g_1 + \cdots + a_{2h} g_{2h}) = g_1(a_1 m_0) + \cdots + g_{2h}(a_{2h} m_0) \in \sqrt{J},$$

we have $m_0 m_1 \in \sqrt{J}$. Again since m_1 divides m_0, we have $m_0 \in \sqrt{J}$. □

Then the following question occurs.

Question. *If* $\operatorname{ara} I_\Delta = 3$, *then does the inequality* $\operatorname{ara} I_{\Delta'} \leq \operatorname{ara} I_\Delta$ *hold?*

If this is true, then the same technique as in the proof of Corollary 4.4 would lead the inequality $\operatorname{ara} I_{\Delta'} \leq \operatorname{ara} I_\Delta$ with no condition on $\operatorname{ara} I_\Delta$.

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