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Commutation and Centralizers in Clone Theory

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Abstract

Commutation theory is one of the central areas of research in universal algebra and clone theory. After giving the definitions of commutation, centralizers and endoprimal monoids, we present some of the results in this field which the author obtained during the past decade as the joint work with Ivo G. Rosenberg.

Keywords: clone; centralizer; endoprimal monoid

1 Introduction

What we need for constructing clone theory is simple and elementary: A fixed set A and a set of (multi-variable) functions

$$f : A^n \longrightarrow A$$

defined on A. In universal algebra, an n-variable function on A is called an operation on A of arity n. We denote by $\mathcal{O}_A^{(n)}$ the set of all n-variable functions on A, i.e., $\mathcal{O}_A^{(n)} = A^{A^n}$. We also denote by $\mathcal{O}_A$ the set of all functions defined on A, i.e.,

$$\mathcal{O}_A = \bigcup_{n=1}^{\infty} \mathcal{O}_A^{(n)}.$$

For $1 \leq i \leq n$, the i-th *projection* e_i^n of n variables is defined by $e_i^n(x_1, \ldots, x_i, \ldots, x_n) = x_i$ for any $(x_1, \ldots, x_n) \in A^n$. $\mathcal{J}_A$ denotes the set of all projections e_i^n $(1 \leq i \leq n)$ on A.

We consider (functional) composition among functions defined on A, and define a *clone* on A as follows:

Definition 1.1 *For a subset $C \subseteq \mathcal{O}_A$, C is a* clone *if C satisfies the following:*

(1) *C is closed under (functional) composition.*

(2) *C contains all the projections, i.e., $J_A \subseteq C$.*

N.B. In order to avoid confusion, we remark that our clone has no relation to biology !!

For a fixed set A, $\mathcal{L}_A$ denotes the set of all clones on A. It is known and easy to see that for any set$A$, $\mathcal{L}_A$ forms a lattice with respect to inclusion.

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In this paper, we let A be a (non-empty) finite set E_k where $E_k = \{0, 1, \dots, k-1\}$ for $k > 1$. Then we write $\mathcal{O}_k^{(n)}$, $\mathcal{O}_k$, $\mathcal{J}_k$ and $\mathcal{L}_k$ instead of $\mathcal{O}_A^{(n)}$, $\mathcal{O}_A$, $\mathcal{J}_A$ and $\mathcal{L}_A$, respectively.

For the case where $k = 2$, that is, the case of Boolean functions, things are, in a sense, already settled.

Theorem 1.1 *(E. Post)*
The structure of $\mathcal{L}_2$ is completely determined. In particular, the cardinality of $\mathcal{L}_2$ is countable.

On the other hand, the structure of $\mathcal{L}_k$ for each $k > 2$ is still largely unknown, remains mysterious and waits for further investigations. The following is one of few facts that we know up to now.

Theorem 1.2 *(Janov and Muchnik)*
For any $2 < k < \omega$, $\mathcal{L}_k$ has the cardinality of the continuum.

In this paper we focus our attention on commutation theory of clones. Commutation theory is one of the central areas of research in universal algebra and clone theory, which attracts many researchers in these fields. After giving the definitions of those terms such as centralizers and endoprimal monoids, we present some of the results that were obtained during the past decade as the joint work of the author with Ivo G. Rosenberg (Montréal). For most of the results presented here the proofs are omitted. (For the proofs refer to the references given at the end of this manuscript.)

2 Commutation

The main concept of this paper is **commutation** between two functions in $\mathcal{O}_A$.

Definition 2.1 *For $f \in \mathcal{O}_k^{(m)}$ and $g \in \mathcal{O}_k^{(n)}$ we say that f* commutes *with g (or, f and g* commute*) if the following holds*

$$f(g(b_{11}, \dots, b_{1n}), \dots, g(b_{m1}, \dots, b_{mn})) \quad = \quad g(f(b_{11}, \dots, b_{m1}), \dots, f(b_{1n}, \dots, b_{mn}))$$

for every $m \times n$ matrix $B = (b_{ij})$ over E_k.

The definition may be better understood by the following picture.

$$\begin{array}{cccc|c}
b_{11} & b_{12} & \cdots & b_{1n} & g(\dots, b_{1j}, \dots) \\
b_{21} & b_{22} & \cdots & b_{2n} & g(\dots, b_{2j}, \dots) \\
\vdots & \vdots & & \vdots & \vdots \\
\vdots & \vdots & & \vdots & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{mn} & g(\dots, b_{m1}, \dots) \\
\hline
f(\dots, b_{i1}, \dots) & f(\dots, b_{i2}, \dots) & \cdots & f(\dots, b_{in}, \dots) & f(g, \dots) = g(f, \dots)
\end{array}$$

We use the notation $f \perp g$ to mean that f commutes with g. It is clear that $f \perp g$ is equivalent to $g \perp f$.

Example

(1) Let $f \in \mathcal{O}_k^{(1)}$ be any constant function and $g \in \mathcal{O}_k^{(n)}$ be any idempotent function. Then it is clear that f and g commute, i.e., $f \perp g$. Here, by definition, g is *idempotent* if $g(x, \dots, x) = x$ for

all $x \in E_k$.

(2) For $k = 3$ let $f,\ g \in \mathcal{O}_3^{(2)}$ be defined as follows:

$$f(x,y) = \begin{cases} 2 & \text{if } \ 2 \in \{x,y\} \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x,y) = \max\{x,y\}.$$

Then it is easily verified that f and g commute, i.e., $f \perp g$.

Definition 2.2 *For $F \subseteq \mathcal{O}_k$ define*

$$F^* \ = \ \{\, g \in \mathcal{O}_k \mid g \perp f \ \text{ for all } \ f \in F \,\}$$

F^* is called the centralizer *of F.*

Lemma 2.1 *For any $F \subseteq \mathcal{O}_k$, the centralizer F^* of F is a clone.*

The following properties of centralizers are easy but important.

Lemma 2.2 *For any $F,\ G \subseteq \mathcal{O}_k$ we have :*

(i) $F \subseteq F^{**}$

(ii) $F \subseteq G \implies F^* \supseteq G^*$

(iii) $F^{***} = F^*$

3 Centralizers of Monoids

For unary functions $f,\ g \in \mathcal{O}_A^{(1)}$ the composition $f \circ g$ is defined by setting

$$(f \circ g)(x) = f(g(x))$$

for all $x \in A$. The operation $\circ$ is associative and the identity function s_1 is the neutral element. Hence the algebra $\langle \mathcal{O}_A^{(1)};\ \circ,\ s_1 \rangle$ is a *monoid.* A subset M of $\mathcal{O}_A^{(1)}$ is a *submonoid* of $\mathcal{O}_A^{(1)}$ if $s_1 \in M$ and M is closed under the operation $\circ$.

In this section, we determine centralizers of monoids of unary functions containing the symmetric group S_k of E_k.

3.1 Results

Some years ago we posed the following problem.

Problem: For every $k \geq 3$, determine centralizers of all submonoids of $\mathcal{O}_k^{(1)}$ which contain the symmetric group S_k.

The complete solution to this problem was given in Machida and Rosenberg [MR 05]. It turned out that most of the centralizers of monoids containing the symmetric group are the same. This makes a clear contrast to the fact that, for any subgroups $G_1,\ G_2$ of S_k, $G_1^* \neq G_2^*$ whenever $G_1 \neq G_2$.

First we note a simple fact. Let $\mathcal{M}_k$ denote the set of submonoids of unary functions in $\mathcal{O}_k^{(1)}$.

Lemma 3.1 *For any submonoid $M \in \mathcal{M}_k$,*

$$S_k \subset M \quad \Longrightarrow \quad S_k \cup \mathrm{CONST} \subseteq M$$

Here, CONST *denotes the set of all unary constant functions in $\mathcal{O}_k^{(1)}$.*

We present the results from smaller submonoids.

Case 1: $M = S_k$

For n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n) \in E_k^n$ we say that $(x_1, \ldots, x_n)$ is *similar* to $(y_1, \ldots, y_n)$ if

$$x_i = x_j \iff y_i = y_j$$

for all $1 \leq i, j \leq n$.

Proposition 3.2 *(Marczewski)*
The centralizer S_k^* of S_k is the set of functions $f\ (\in \mathcal{O}_k^{(n)})$ satisfying the following conditions.

(1) If $|\{x_1, \ldots, x_n\}| \neq k-1$ then

(i) $f(x_1, \ldots, x_n) = x_\ell$ for some $1 \leq \ell \leq n$ and

(ii) $f(y_1, \ldots, y_n) = y_\ell$ for $\forall\,(y_1, \ldots, y_n) \in (E_k)^n$ which is similar to $(x_1, \ldots, x_n)$.

(2) If $|\{x_1, \ldots, x_n\}| = k-1$ and $f(x_1, \ldots, x_n) = u$ for some $u \in E_k$ then

(i) $u = x_\ell$ for some $1 \leq \ell \leq n$ implies $f(y_1, \ldots, y_n) = y_\ell$ for $\forall\,(y_1, \ldots, y_n) \in (E_k)^n$ which is similar to $(x_1, \ldots, x_n)$ and

(ii) $u \in E_k \setminus \{x_1, \ldots, x_n\}$ implies $f(y_1, \ldots, y_n) = v$ where $v \in E_k \setminus \{y_1, \ldots, y_n\}$ for $\forall\,(y_1, \ldots, y_n) \in (E_k)^n$ which is similar to $(x_1, \ldots, x_n)$.

Case 2: $M = S_k \cup \mathrm{CONST}$

Proposition 3.3 *(1) For $k = 2$, the centralizer $(S_2 \cup \mathrm{CONST})^*$ is the clone*

$$\{\, f \in {S_2}^* \mid f : \text{idempotent} \,\}.$$

(2) For every $k \geq 3$, the centralizer $(S_k \cup \mathrm{CONST})^*$ is the clone ${S_k}^*$.

Case 3: $M \supset S_k \cup \mathrm{CONST}$

As an exceptional case for $k = 4$, we need to consider a submonoid which we call M_2.
For $u \in \mathcal{O}_k^{(1)}$ the *kernel* of u is defined by

$$\ker u = \{(x, y) \in \boldsymbol{k}^2 \mid u(x) = u(y)\}.$$

Clearky, $\ker u$ is an equivalence relation on E_k. An equivalence class is called a block.

Let $k = 4$. We define M_2 as the submonoid consisting of $u \in \mathcal{O}_4^{(1)}$ satisfying one of the following conditions:

(i) $E_4 / \ker u$ has four singleton blocks, i.e., u is a permutation on E_4.

(ii) $E_4 / \ker u$ has one block, i.e., u is a constant function on E_4.

(iii) $E_4 / \ker u$ has two blocks of size 2, i.e., u sends two elements in E_4 to an element in E_4 and the other two to another element in E_4.

Here, $E_4 \,/\, \ker u$ is the quotient set over E_4 induced by the equivalence relation $\ker u$. It is clear that $M_2 \supset S_4 \cup \mathrm{CONST}_4$.

Proposition 3.4 *If $M \supset S_k \cup \mathrm{CONST}$ then the following holds.*

(i) *For $k = 3$:* $\quad M^* = \mathcal{J}_3$

(ii) *For $k = 4$:* $\quad$ *If $M \neq M_2$ then $M^* = \mathcal{J}_4$*

(iii) *For $k \geq 5$:* $\quad M^* = \mathcal{J}_k$

Note: As mentioned below, the centralizer M_2^* of M_2 for $k = 4$ is not equal to $\mathcal{J}_4$.

3.2 A Sufficient Condition for a Trivial Centralizer

We start with two properties of functions.

I. (Separation Property)
For all $a, b, c, d \in E_k$, if $\{a, b\} \neq \{c, d\}$ and $c \neq d$ then M contains $f\ (= f_{cd}^{ab})$ which satisfies

$$f(a) = f(b) \quad \text{and} \quad f(c) \neq f(d).$$

II. (Fixed-Point-Free Property)
For every $i \in E_k$, M contains g_i which satisfies $g_i(i) \neq i$.

The following fact appears in Machida and Rosenberg [MR 04a] and [MR 04b].

Lemma 3.5 *For any $M \in \mathcal{M}_k$, if M satisfies the above conditions I and II then $M^* = \mathcal{J}_k$.*

It is an easy task to verify Proposition 3.4 from Lemma 3.5.

For the submonoid M_2 in the case $k = 4$, we can show that the centralizer of M_2 is not the least clone. Let the ternary function $m(x_1, x_2, x_3)$ $(\in \mathcal{O}_4^{(3)})$ be defined as follows:

$$m(x_1, x_2, x_3) = \begin{cases} x_1 & \text{if } x_1 = x_2 = x_3 \\ x_1 & \text{if } x_1 \neq x_2 = x_3 \\ x_2 & \text{if } x_2 \neq x_1 = x_3 \\ x_3 & \text{if } x_3 \neq x_1 = x_2 \\ y & \text{if } \{x_1, x_2, x_3, y\} = E_4 \end{cases}$$

It is readily verified that m commutes with every member in M_2, i.e., $m \in M_2^*$. Hence, we have:

Lemma 3.6 *M_2^* is* not *the least clone $\mathcal{J}_4$.*

4 Kuznetsov Criterion

Kuznetsov Criterion was discovered by Kuznetsov in 1960's, and is an extremely useful tool ([Da 77]).

Definition 4.1 *For $f \in \mathcal{O}_k^{(n)}$ and $\Sigma \subseteq \mathcal{O}_k$, f is* parametrically expressible *(*p-expressible*) by Σ if there exist $m \geq 1$, $\ell \geq 0$ and $g_i, h_i \in \mathcal{O}_k^{(n+\ell+1)}$ $(i = 1, \ldots, m)$ such that $g_i, h_i \in \langle \Sigma \rangle$ and*

$$f^{\Box} = \{ (x_1, \ldots, x_n, x_{n+1}) \mid \exists x_{n+2}, \ldots, x_{n+\ell+1} \in E_k,\ \forall i \in \{1, \ldots, m\},\ g_i(x_1, \ldots, x_{n+\ell+1}) = h_i(x_1, \ldots, x_{n+\ell+1}) \}.$$

Here, $f^{\Box}$ means the graph of f, i.e., $f^{\Box} = \{(x_1, \ldots, x_n, x_{n+1}) \mid f(x_1, \ldots, x_n) = x_{n+1}\}$

Kuznetsov criterion states as follows:

Theorem 4.1 *(Kuznetsov criterion)*
For $f \in \mathcal{O}_k$ and $\Sigma \subseteq \mathcal{O}_k$, f is p-expressible by Σ if and only if $\Sigma^* \subseteq \{f\}^*$.

Equivalently, it can be expressed as:

Corollary 4.2 *(Kuznetsov criterion)*
For $f \in \mathcal{O}_k$ and $\Sigma \subseteq \mathcal{O}_k$, f is p-expressible by Σ if and only if $f \in \Sigma^{**}$.

Example. Let unary functions $j_0, j_1, s_3 \in \mathcal{O}_3^{(1)}$ be given below.

	j_0	j_1	s_3
0	1	0	1
1	0	1	0
2	0	0	2

From j_0 and j_1 we get s_3 in the following sense:

$$s_3^\square = \{(x,y) \in (E_3)^2 \mid j_0(x) = j_1(y),\ j_1(x) = j_0(y)\ \}$$

Hence s_3 is p-expressible by $\{j_0, j_1\}$. Then, due to Kuznetsov Criterion, we have

$$s_3 \in \{ j_0, j_1 \}^{**}$$

4.1 Centralizers of Subgroups of S_k

Lemma 4.3 *For any subgroup H of S_k and any $s \in S_k$, s is p-expressible by H if and only if $s \in H$.*

Proof ($\Leftarrow$) Trivial.
($\Rightarrow$) Suppose that s is p-expressible by H. Then, by definition, $s^\square = \{(x,y) \mid t(x) = u(y)\}$ for some $t, u \in H$. This is equivalent to $s^\square = \{(x,y) \mid (u^{-1}t)(x) = y\}$ for some $t, u \in H$, which implies that $s = u^{-1}t \in H$. □

Theorem 4.4 *(Machida and Rosenberg)*
The $*$-operator is injective over S_k, that is, for subgroups H_1 and H_2 of S_k,

$$H_1^* = H_2^* \quad \Longrightarrow \quad H_1 = H_2.$$

Proof Suppose $H_1^* = H_2^*$ and $H_1 \neq H_2$. Then, w.l.o.g., we may take $s \in H_2 - H_1$. Now $H_1^* = H_2^*$ implies that

$$H_1^* \subseteq \{s\}^* (= \operatorname{Pol} s^\square)$$

since $H_2^* = \bigcap_{t \in H_2} \operatorname{Pol} t^\square$. By Kuznetsov criterion, s is p-expressible by H_1. Hence, by Lemma 4.3, we have $s \in H_1$. Contradiction. □

5 Endoprimal Monoids

In this section, we consider "endoprimal monoids", that is, the unary part of the centralizer of some set. Most of the results which will be presented in this section appeared in Machida and Rosenberg [MR 09] and [MR 10].

Definition 5.1 *Let $\mathcal{A} = (A;\ F)$ be an algebra. For a map $\varphi : A \longrightarrow A$, φ is an* endomorphism *of $\mathcal{A}$ if*

$$f(\varphi(x_1), \ldots, \varphi(x_n)) \;=\; \varphi(f(x_1, \ldots, x_n))$$

holds for any $f \in F$ and all $(x_1, \ldots, x_n) \in A^n$.

An endomorphism is naturally connected to commutation. Remember that for $f \in \mathcal{O}_k^{(1)}$ and $F \subseteq \mathcal{O}_k$, the fact that f commutes with F, i.e., $f \perp F$, means that

$$g(f(x_1), \ldots, f(x_n)) \;=\; f(g(x_1, \ldots, x_n))$$

for any $g \in F$.

Lemma 5.1 *For a map $\varphi : A \longrightarrow A$, the following are equivalent.*

(1) *φ is an endomorphism of $\mathcal{A}$.*

(2) *$\varphi \perp F$, that is, $\varphi \perp f$ for all $f \in F$.*

(3) *$\varphi \in F^*$*

Definition 5.2 *For a submonoid $M \subseteq \mathcal{O}_k^{(1)}$, M is an* endoprimal monoid *if there exists $F \subseteq \mathcal{O}_k$ which satisfies $M = F^* \cap \mathcal{O}_k^{(1)}$.*

In other words, M is an endoprimal monoid if M is the unary part of a centralizer of some set $F \subseteq \mathcal{O}_k$.

Lemma 5.2 *For a submonoid $M \subseteq \mathcal{O}_k^{(1)}$, M is an endoprimal monoid if and only if $M = M^{**} \cap \mathcal{O}_k^{(1)}$.*

Proof
($\Leftarrow$) : Trivial.
($\Rightarrow$) : Suppose $M = F^* \cap \mathcal{O}_k^{(1)}$ for some $F \subseteq \mathcal{O}_k$. Then, since $M \subseteq F^*$, we have $M^{**} \subseteq F^{***} = F^*$. Taking the unary part, $M^{**} \cap \mathcal{O}_k^{(1)} \subseteq F^* \cap \mathcal{O}_k^{(1)} = M$. On the other hand, from $M \subseteq M^{**}$ it follows that $M = M \cap \mathcal{O}_k^{(1)} \subseteq M^{**} \cap \mathcal{O}_k^{(1)}$. Therefore, $M = M^{**} \cap \mathcal{O}_k^{(1)}$ as desired. □

For a submonoid $M \subseteq \mathcal{O}_k^{(1)}$ we sometimes write M^+ to mean $M^+ = M^{**} \cap \mathcal{O}_k^{(1)}$.

Lemma 5.3 *For a submonoid $M \subseteq \mathcal{O}_k^{(1)}$, M^+ satisfies the following properties.*

(1) *M^+ is an endoprimal monoid.*

(2) *$M \subseteq M^+$*

(3) *M^+ is the largest submonoid consisting of "endomorphisms" of the algebra $\langle E_k\,;\, M\rangle$*

Up to now, not many examples of endoprimal monoids are known. In the sequel, we shall mostly concentrate on the ternary case, that is, the case where the base set is $E_3 = \{0, 1, 2\}$.

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5
0	1	0	0	1	1	0	2	0	0	2	2	0	2	1	1	2	2	1
1	0	1	0	1	0	1	0	2	0	2	0	2	1	2	1	2	1	2
2	0	0	1	0	1	1	0	0	2	0	2	2	1	1	2	1	2	2

	c_0	c_1	c_2
0	0	1	2
1	0	1	2
2	0	1	2

	s_1	s_2	s_3	s_4	s_5	s_6
0	0	0	1	1	2	2
1	1	2	0	2	0	1
2	2	1	2	0	1	0

Table 1: Unary Functions in $\mathcal{O}_3^{(1)}$

5.1 Unary Functions and Submonoids on $\{0, 1, 2\}$

As is well-known, the number of unary functions over E_3 is 27. They are shown in Table 1. Much less known is the number of submonoids of unary functions over E_3. Due to D. Lau ([La 84], [La 06]), the number of submonoids of unary functions over E_3 is 700.

Let us search for an endoprimal monoid containing both j_0 and j_1. Repeated applications of "Kuznetsov Criterion" imply the following. (We omit the details here.)

Lemma 5.4 *If* $M\,(\subseteq \mathcal{O}_3^{(1)})$ *is an endoprimal monoid and* $\{j_0, j_1\} \in M$ *then*

$$P_2 \subseteq M \;\; (= M^+)$$

where $P_2 = \{c_0, c_1, c_2\} \cup \{j_0, j_1, j_4, j_5\} \cup \{u_0, u_1, u_4, u_5\} \cup \{v_0, v_1, v_4, v_5\} \cup \{s_1, s_3\}$.

Actually, P_2 is the submonoid #1227 in Lau's list. At this point, we do not know if P_2 is endoprimal or not. The following "witness lemma" will tell us that, in fact, P_2 is endoprimal.

5.2 Witness Lemma

The following lemma wa given in Machida and Rosenberg [MR 10].

Lemma 5.5 *(Witness Lemma)*
For a submonoid $M \subseteq \mathcal{O}^{(1)}$ *of unary functions and a subset* $S \subseteq \mathcal{O}$, *suppose the following conditions (i) and (ii) hold:*

(i) *For any* $f \in M$ *and any* $u \in S$ *it holds that* $f \perp u$.

(ii) *For any* $g \in \mathcal{O}^{(1)} \setminus M$ *there exists* $w \in S$ *such that* $g \not\perp w$.

Then M *is endoprimal.*

Definition 5.3 *S in the lemma will be called a* witness *for an endoprimal monoid* M.

The proof is straightforward, but, for the reader's sake, we give it below.

Proof of Lemma Condition (i) implies $S \subseteq M^*$, from which it follows that $M^{**} \subseteq S^*$. Condition (ii) asserts that, for all $g \in (\mathcal{O}^{(1)} \setminus M)$, it holds that $g \notin S^*$. Then it follows that, for all $g \in (\mathcal{O}^{(1)} \setminus M)$, $g \notin M^{**}$, because we have $M^{**} \subseteq S^*$ as stated above. Hence $(\mathcal{O}^{(1)} \setminus M) \cap M^{**} = \emptyset$.
On the other hand, $M \subseteq M^{**}$, in general. Therefore $M = M^{**} \cap \mathcal{O}^{(1)}$, i.e., $M = M^+$. □

Corollary 5.6 *(Special Case where S is a singleton)*
For a submonoid $M \subseteq \mathcal{O}^{(1)}$ of unary functions and a function $f \in \mathcal{O}$, if $f \perp M$ and $f \not\perp (\mathcal{O}^{(1)} \backslash M)$ then M is endoprimal.

5.3 Some Endoprimal Monoids on $\{0, 1, 2\}$

We show two applications of the witness lemma.

5.3.1 Application of Witness Lemma (1)

Let $m \in \mathcal{O}_3^{(3)}$ be a witness, which is defined as follows:

$$m(x,y,z) = \begin{cases} x & \text{if } \; x = y \; \text{ or } \; x = z \\ y & \text{if } \; y = z \\ 2 & \text{if } \; \{x,y,z\} = \{0,1,2\} \end{cases}$$

In other words, m is the majority and totally symmetric function satisfying the following.

(i) $\quad m(a,a,b) = a \quad$ for all $a, b \in E_3$

(ii) $\quad m(0,1,2) = 2$

Then it is easily verified that (1) the function m commutes with all functions in P_2, i.e., $m \in P_2{}^*$ and (2) m does not commute with any function in $\mathcal{O}_3^{(1)} \backslash P_2$. Therefore, the witness lemma implies:

Proposition 5.7 *P_2 is an endoprimal monoid.*

Moreover, we note that P_2 is shown to be a maximal endoprimal monoid.

5.3.2 Application of Witness Lemma (2)

For each subset S of unary functions, i.e., $S \subseteq \mathcal{O}_3^{(1)}$, one can construct an endoprimal monoid which has S as its witness.

Example 1. For $c_0 \in \mathcal{O}_3^{(1)}$ take $S = \{c_0\}$ as a (singleton) witness. It is easy to check that the set of unary functions which commute with c_0 is $\{c_0, j_1, j_2, j_5, u_1, u_2, u_5, s_1, s_2\}$. Hence, by the witness lemma, we see that

$$M(c_0) = \{\, c_0, j_1, j_2, j_5, u_1, u_2, u_5, s_1, s_2 \,\}$$

is an endoprimal monoid.

Example 2. Let $S = \{c_0, j_1\}$ be a doubleton consisting of c_0 and $j_1 \in \mathcal{O}_3^{(1)}$. It is readily verified that the set of unary functions which commute with j_1 is $\{c_0, c_1, j_1, j_4, u_2, s_1\}$. Together with the result given in Example 1, we see that the set of unary functions which commute with both c_0 and j_1 is $\{c_0, j_1, u_2, s_1\}$. Hence, by the witness lemma, it follows that

$$M(c_0, j_1) = \{\, c_0, j_1, u_2, s_1 \,\}$$

is an endoprimal monoid.

We have the complete list of the endoprimal monoids having subsets ($\subseteq \mathcal{O}_3^{(1)}$) of unary functions as their witnesses. Below we give the summary of this list. For more precise description the reader is referred to [MR 10].

Proposition 5.8 *Over E_3, there are* 51 *endoprimal monoids each having a subset of $\mathcal{O}_3^{(1)}$ as its witness.*

(1) *(Singleton witnesses) Out of 27 unary functions f in $\mathcal{O}_3^{(1)}$, there are 26 different endoprimal monoids $M(f)$ each having singleton witness $\{f\}$. An exception is for s_4 and s_5, where we have $M(s_4) = M(s_5)$.*

(2) *(Doubleton witnesses) There are 25 endoprimal monoids which have doubleton witnesses (and have no singleton witnesses).*

(3) *(Larger witnesses) There is no endoprimal monoid over E_3 which requires a witness, consisting of unary functions, whose size is greater than* 2.

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