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Hopf Bifurcation Theorem in Sobolev Spaces

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1 Introduction

The Hopf bifurcation theorem provides an effective criterion for finding out periodic solutions for ordinary differential equations. Although various proofs of this classical theorem are known, there seems to be no easy way to arrive at the goal. Among them, the idea of Ambrosetti and Prodi [1] is particularly noteworthy.

Maruyama [7] tries to establish the Hopf theorem in the framework of some Sobolev space instead of C^r. This approach seems to enable us to simplify the technical details in the course of the proof to some extent. The basic result due to Carleson [2] and Hunt [4] plays a crucial role in our theory.

Examples of its applications are seen in Flaschel et al. [3], Maruyama [6], and Mas-Colell [8].

2 Abstract Hopf Bifurcation Theorem

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be a couple of real Banach spaces. And $F(\omega,\mu,x)$ is assumed to be a function of class $C^2(\mathbb{R}^2\times\mathfrak{X},\mathfrak{Y})$ which satisfies

$$F(\omega,\mu,0)=0 \quad \text{for all} \quad (\omega,\mu)\in\mathbb{R}^2.$$

A point $(\omega^*,\mu^*)\in\mathbb{R}^2$ is called a *bifurcation point* of F if $(\omega^*,\mu^*,0)$ is in the closure of the set

$$S=\{(\omega,\mu,x)\in\mathbb{R}^2\times\mathfrak{X}\mid x\neq 0, F(\omega,\mu,x)=0\}. \tag{1}$$

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We shall use several notations for the sake of simplicity.

$$T = D_x F(\omega^*, \mu^*, 0),$$
$$\mathfrak{V} = \operatorname{Ker} T, \quad \mathfrak{R} = T(\mathfrak{X}),$$
$$M = D^2_{x,\mu} F(\omega^*, \mu^*, 0),$$
$$N = D^2_{x,\omega} F(\omega^*, \mu^*, 0).$$

T is the derivative of F with respect to x at $(\omega^*, \mu^*, 0)$. It is a bounded linear operator of $\mathfrak{X}$ into $\mathfrak{Y}$. $\mathfrak{V}$ and $\mathfrak{R}$ are the kernel and the image of T, respectively. M (resp.N) is the second derivative of F with respect to (x, μ) (resp. (x, ω)) at $(\omega^*, \mu^*, 0)$. Since $D^2_{x,\mu}F$ is the bounded linear operator of $\mathbb{R}$ into $\mathfrak{L}(\mathfrak{X}, \mathfrak{Y})$ (the Banach space of all the bounded linear operators of $\mathfrak{X}$ into $\mathfrak{Y}$), it can be identified with an element of $\mathfrak{L}(\mathfrak{X}, \mathfrak{Y})$. The same is true for $D^2_{x,\omega}F$.

The following theorem is an abstract version of the Hopf bifurcation theorem due to Ambrosetti and Prodi [1] (pp.136–139).

THEOREM 1 *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be real Banach spaces. Assume that $F(\omega, \mu, x)$ is a function of the class $C^2(\mathbb{R}^2 \times \mathfrak{X}, \mathfrak{Y})$ which satisfies the following two conditions.*

1. dim $\mathfrak{V} = 2$. $\mathfrak{R}$ *is closed and* codim$\mathfrak{R} = 2$.

   We represent $\mathfrak{X}$and $\mathfrak{Y}$ in the forms of topological direct sums:

$$\mathfrak{X} = \mathfrak{V} \oplus \mathfrak{W}, \quad \mathfrak{Y} = \mathfrak{Z} \oplus \mathfrak{R}.$$

   We denote by P the projection of $\mathfrak{Y}$ into $\mathfrak{Z}$, and by Q the projection of $\mathfrak{Y}$ into $\mathfrak{R}$.

2. *There exists some point $v* \in \mathfrak{V}$ such that PMv^* and PNv^* are linearly independent.*

   Then (ω^*, μ^*) is a bifurcation point of F.

The proof of this theorem is based upon the Ljapunov-Schmidt reduction method, which is also neatly explained in Ambrosetti and Prodi [1] (pp.89–91).

3 Classical Hopf Bifurcation for Ordinary Differential Equations

We now turn to the classical bifurcation phenomena of periodic solutions for some ordinary differential equation. Let $f(\mu, x)$ be a function of the class $C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. And consider the differential equation

$$\frac{dx}{ds} = f(\mu, x). \tag{1}$$

Changing the time variable s by the relation

$$t = \omega s \quad (\omega \neq 0), \tag{2}$$

we rewrite the equation (1) as

$$\frac{dx}{dt} = \frac{1}{\omega} f(\mu, x), \tag{3}$$

$$i.e. \quad \omega \frac{dx}{dt} = f(\mu, x). \tag{3'}$$

This is an ordinary differential equation with two real parameters, ω and μ. For the sake of simplicity, we assume that the function $f(\mu, x)$ satisfies

$$f(\mu, 0) = 0 \quad \text{for all} \quad \mu \in \mathbb{R}. \tag{4}$$

We denote by $\mathfrak{W}^{1,2}_{2\pi}(\mathbb{R}, \mathbb{R}^n)$ the set of all the 2π-periodic and absolutely continuous functions $x : \mathbb{R} \to \mathbb{R}^n$ such that $x|_{[0,2\pi]} \in \mathfrak{W}^{1,2}([0, 2\pi], \mathbb{R}^n)$, where $x|_{[0,2\pi]}$ denotes the restriction of x to the interval $[0, 2\pi]$; *i.e.*

$$\mathfrak{W}^{1,2}_{2\pi} = \{x : \mathbb{R} \to \mathbb{R}^n | x \text{ is } 2\pi\text{-periodic, absolutely continuous and } \dot{x}(\cdot)|_{[0,2\pi]} \in \mathfrak{L}^2([0, 2\pi], \mathbb{R}^n)\}. \tag{5}$$

$\mathfrak{W}^{1,2}_{2\pi}$ is a Banach space under the norm

$$\|x\|_{\mathfrak{W}^{1,2}_{2\pi}} = \left(\int_0^{2\pi} \|x(t)\|^2 dt\right)^{1/2} + \left(\int_0^{2\pi} \|\dot{x}(t)\|^2 dt\right)^{1/2}. \tag{6}$$

We also denote by $\mathfrak{L}^2_{2\pi}(\mathbb{R}, \mathbb{R}^n)$ the set of all the 2π-periodic measurable functions $y : \mathbb{R} \to \mathbb{R}^n$ such that $y|_{[0,2\pi]} \in \mathfrak{L}^2([0, 2\pi], \mathbb{R}^2)$; *i.e.*

$$\mathfrak{L}^2_{2\pi} = \{y : \mathbb{R} \to \mathbb{R}^n | y \text{ is } 2\pi\text{-periodic and } y|_{[0,2\pi]} \in \mathfrak{L}^2([0, 2\pi], \mathbb{R}^n)\}. \tag{7}$$

$\mathfrak{L}^2_{2\pi}$ is a Banach space under the norm

$$\|y\|_{\mathfrak{L}^2_{2\pi}} = \left(\int_0^{2\pi} \|y(t)\|^2 dt\right)^{1/2}. \tag{8}$$

In this section, we adopt $\mathfrak{W}^{1,2}_{2\pi}$ as $\mathfrak{X}$ and $\mathfrak{L}^2_{2\pi}$ as $\mathfrak{Y}$, respectively; *i.e.*

$$\mathfrak{X} = \mathfrak{W}^{1,2}_{2\pi}, \quad \mathfrak{Y} = \mathfrak{L}^2_{2\pi}. \tag{9}$$

Define the function $F : \mathbb{R}^2 \times \mathfrak{X} \to \mathfrak{Y}$ by (cf. (9))

$$F(\omega, \mu, x) = \omega \frac{dx}{dt} - f(\mu, x). \tag{10}$$

Then we can prove that F is a function of the class $C^2(\mathbb{R}^2 \times \mathfrak{X}, \mathfrak{Y})$ provided that the following assumptions are satisfied.

ASSUMPTION 1 (i) There exists some constants α and $\beta \in \mathbb{R}$ such that

$$\|f(\mu, x)\| \leqq \alpha + \beta \|x\|, \quad x \in \mathbb{R}^n.$$

(ii) There exists some constant ρ such that

$$\|D_x f(\mu, x)\|, \quad \|D^2 f(\mu, x)\| \leqq \rho \quad \text{for all} \quad x \in \mathbb{R}^2$$

It is ovbious that

$$F(\omega, \mu, 0) = 0 \quad \text{for all} \quad (\omega, \mu) \in \mathbb{R}^2. \tag{11}$$

Recall that $(\omega^*, \mu^*) \in \mathbb{R}^2$ is a bifurcation point of F if there exists a sequence (ω_n, μ_n, x_n) in $\mathbb{R}^2 \times \mathfrak{X}$ such that

$$\begin{cases} F(\omega_n, \mu_n, x_n) = 0, \\ (\omega_n, \mu_n) \to (\omega^*, \mu^*) \quad \text{as} \quad n \to \infty, \quad \text{and} \\ x_n \neq 0, \quad x_n \to 0 \quad \text{as} \quad n \to \infty. \end{cases}$$

Each x_n is a non-trivial (not identically zero) periodic solution with period 2π of the equation

$$\omega_n \frac{dx}{dt} - f(\mu_n, x) = 0.$$

Hence, changing the time-variable to s again, we obtain a periodic solution

$$X_n(s) = x_n(\omega_n s)$$

with period $\tau_n = 2\pi/\omega_n$ for the original equation (1). Here

$$\tau_n \to \tau^* = \frac{2\pi}{\omega^*}, \qquad \| X_n \|_{\mathfrak{W}^{1,2}_{2\pi/\omega_n}} \to 0 \quad \text{as} \quad n \to \infty.$$

The target of our investigations is to find out a bifurcation point of F according to the principle of Theorem 1. We have to note that the derivative

$$D_x F(\omega, \mu, 0) : \ x \mapsto \omega \dot{x} - D_x f(\mu, 0) x \tag{12}$$

is to play the most important role in the course of our discussions. ($\dot{x}$ means dx/dt.) Of course, $D_x F(\omega, \mu, 0)$ is a bounded linear operator of $\mathfrak{X}$ into $\mathfrak{Y}$. If we denote

$$A_\mu = D_x f(\mu, 0),$$

A_μ is $(n \times n)$-matrix and (7) can be rewritten in the form

$$D_x F(\omega, \mu, 0) x = \omega \dot{x} - A_\mu x. \tag{12'}$$

Here we need a couple of assumptions to be imposed upon A_μ at some (ω^*, μ^*).

ASSUMPTION 2 *A_{μ^*} is regular, and $\pm i\omega^*(\omega^* > 0)$ are simple eigenvalues of A_{μ^*}.*

ASSUMPTION 3 *None of $\pm ik\omega^*$ $(k \neq \pm 1)$ is an eigenvalue of A_{μ^*}.*

4 dim $\mathfrak{V}$, codim$\mathfrak{R}$, and All That

Denoting $T = D_xF(\omega^*, \mu^*, 0)$, we have

$$Tx = 0 \quad \text{if and only if} \quad \omega^*\dot{x} - A_{\mu^*}x = 0. \tag{1}$$

In order to apply Theorem 1 to our classical problem in section 3, we have to start with confirming that (a) the dimension of the kernel of T is 2, and (b) the codimension of the image of T is also 2. Henceforth, we denote Ker T by $\mathfrak{V}$ and $T(\mathfrak{X})$ by $\mathfrak{R}$, repectively.

In fact, we can prove the desined results thanks to Assumption 2; i.e.

$$\dim\mathfrak{V} = 2, \quad \operatorname{codim}\mathfrak{R} = 2.$$

It is here that the Carleson-Hunt theory plays an indispensable performance.

According to the Assumption 2, $\pm i\omega^*$ are simple eigenvalues of A_{μ^*}. Hence $\mathbb{C}^n$ can be expressed as a direct sum as

$$\mathbb{C}^n = \mathrm{Ker}[\pm i\omega^* I - A_{\mu^*}] \oplus [\pm i\omega^* I - A_{\mu^*}](\mathbb{C}^n). \tag{2}$$

We shall now concentrate on the case $+i\omega^*$. (The case $-i\omega^*$ can be discussed similarly.)

Let $\eta \in \mathbb{C}^n$ $(\eta \neq 0)$ be any vector which is orthogonal to $[i\omega^* I - A_{\mu^*}](\mathbb{C}^n)$. We define a function $g : \mathbb{R} \times \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}$ by

$$g(\mu, \lambda, \theta) = \begin{pmatrix} (\lambda I - A_\mu)(\xi + \theta) \\ \langle \eta, \theta \rangle \end{pmatrix}. \tag{3}$$

($\langle\cdot,\cdot\rangle$ denotes the inner product.) Then the function g is of the class C^1 and satisfies

$$g(\mu^*, i\omega^*, 0) = 0. \tag{4}$$

Applying the Implicit Function Theorem, we can prove the following lemma.

LEMMA *There exist a couple of functions, $\lambda(\mu)$ and $\theta(\mu)$ of the C^1-class which are defined in some neighborhood of μ^* and satisfy*

$$\begin{pmatrix} (\lambda(\mu)I - A_\mu)(\xi + \theta(\mu)) \\ \langle \eta, \theta(\mu) \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5}$$

and

$$\lambda(\mu^*) = i\omega^*, \quad \theta(\mu^*) = 0. \tag{6}$$

Dividing $\lambda(\mu)$ (obtained by the above lemma) into real and imaginary parts, we write

$$\lambda(\mu) = \alpha(\mu) + i\beta(\mu).$$

We also write

$$\lambda'(\mu) = \alpha'(\mu) + i\beta'(\mu).$$

Then we get a simple fact that PNv^* and PMv^* are linearly independent if and only if $\alpha'(\mu^*) \neq 0$.

All the requirements in Theorem 1 are fulfilled if we make an additional assumption that $\alpha'(\mu^*) \neq 0$.

THEOREM 2 Let $f(\mu, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *be a function of the class* $C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ *which satisfies* $f(\mu, 0) = 0$ *for all* $\mu \in \mathbb{R}$. *Suppose that Assumptions 1-3 as well as the condition* $\alpha'(\mu^*) \neq 0$ *are satisfied. Then* (ω^*, μ^*) *is a bifurcation point of* $F(\omega, \mu, x) = \omega dx/dt - f(\mu, x)$.

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