

Note on the center of generalized quantum groups

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1 Introduction

Recently study of generalized quantum groups defined for wider class of bi-characters has been achieved marvelously. It can be said that the study was initiated by Andruskiewitsch and Schneider's suggestion [2] of classification program of pointed Hopf algebras. It should be mentioned that the Drinfeld-Jimbo's original quantum groups, the Lusztig small quantum groups at roots of unity, the quantum superalgebras of type $A-G$, the ones at roots of unity, $\mathbb{Z}/3\mathbb{Z}$ -quantum groups (cf. [8]), and multi-parameter quantum groups are generalized quantum groups. Under the idea, Heckenberger achieved studies of Nichols algebras of diagonal type and their quantum doubles, including classification of those of finite type [3], [4], [5]. Under influence of the program, he and the author obtained a Matsumoto type theorem of the Weyl groupoids associated to finite type generalized root systems [6]. Algebras mentioned above admit generalized root systems. They also obtained a factorization formula of the Shapovalov determinants of finite type generalized quantum groups [7].

We consider that it is very important to formulate a Harish-Chandra theorem for generalized quantum groups. In this note, we make preliminary study of the Harish-Chandra maps of generalized quantum groups defined for symmetric bi-characters. For original results for quantum groups, see [1].

2 Multi-parameter generalized quantum groups

For $x, y \in \mathbb{Z}$, let $J_{x,y} := \{z \in \mathbb{Z} | x \leq z \leq y\}$. Let $SJ_{x,y}$ be the set of finite sequences in $J_{x,y}$; we assume that $SJ_{x,y}$ has a 0-sequence ϕ . Namely $SJ_{x,y} = \cup_{r=0}^{+\infty} SJ_{x,y}^{(r)}$ (disjoint), where $SJ_{x,y}^{(0)} = \{\phi\}$, and if $r \in \mathbb{N}$, we mean $SJ_{x,y}^{(r)} = \{(i_1, i_2, \dots, i_r) | i_{r'} \in J_{x,y} (r' \in J_{x,y})\}$. For $\bar{x} = (i_1, i_2, \dots, i_r) \in SJ_{x,y}^{(r)}$, let $\|\bar{x}\| = (j_1, j_2, \dots, j_r) \in SJ_{x,y}^{(r)}$ be such that $j_1 \leq j_2 \leq \dots \leq j_r$ and $|\{k \in J_{1,r} | j_k = z\}| = |\{l \in J_{1,r} | i_l = z\}|$ for all $z \in J_{x,y}$; we also let $\|\phi\| := \phi$.

Let $N \in \mathbb{N}$. Let $q \in \mathbb{C} \setminus \bar{\mathbb{Q}}$. Fix $q^{\frac{1}{2}} \in \mathbb{C} \setminus \bar{\mathbb{Q}}$ with $q = (q^{\frac{1}{2}})^2$. For $r \in \mathbb{Z}$, we write $q^{\frac{r}{2}} := (q^{\frac{1}{2}})^r$. Let $\Pi'' = \{\epsilon_i, \epsilon'_i | i \in J_{1,N}\}$ be a finite set with $2N = |\Pi''|$. Let $\mathbb{Z}\Pi''$ be a rank- $2N$ free \mathbb{Z} -module with the base Π'' . Let $\sqrt{\chi} : \mathbb{Z}\Pi'' \times \mathbb{Z}\Pi'' \rightarrow \mathbb{C}^\times$ be a map such that

$$(2.1) \quad \sqrt{\chi}(a, b+c) = \sqrt{\chi}(a, b)\sqrt{\chi}(a, c), \quad \sqrt{\chi}(a+b, c) = \sqrt{\chi}(a, c)\sqrt{\chi}(b, c)$$

for all $a, b, c \in \mathbb{Z}\Pi''$, and

$$(2.2) \quad \sqrt{\chi}(\epsilon_i, \epsilon'_j) = \sqrt{\chi}(\epsilon'_i, \epsilon_j) = q^{\frac{\delta_{ij}}{2}}, \quad \sqrt{\chi}(\epsilon'_i, \epsilon'_j) = 1$$

for all $i, j \in J_{1,N}$. Define a map $\chi : \mathbb{Z}\Pi'' \times \mathbb{Z}\Pi'' \rightarrow \mathbb{C}^\times$ by

$$(2.3) \quad \chi(a, b) := \sqrt{\chi}(a, b)^2.$$

Let $\Pi' := \{\epsilon_j | j \in J_{1,N}\}$, so $\Pi' \subset \Pi''$. Let $\ell \in J_{1,N}$. Let $\Pi = \{\alpha_i | i \in J_{1,\ell}\}$ be a subset of $\mathbb{Z}\Pi'$ such that $\mathbb{Z}\Pi$ is a rank- ℓ submodule of $\mathbb{Z}\Pi'$.

Let $\tilde{B}^+ = \tilde{B}^+(\chi)$ be the unital \mathbb{C} -algebra defined with generators

$$(2.4) \quad L_a \ (a \in \mathbb{Z}\Pi''), \ E_i \ (i \in J_{1,\ell})$$

and relations

$$(2.5) \quad L_0 = 1, \ L_a L_b = L_{a+b}, \ L_a E_i = \chi(a, \alpha_i) E_i L_a.$$

For $\phi \in SJ_{1,\ell}^{(0)}$, let $E_\phi := 1 \in \tilde{B}^+$, and for $\bar{x} := (i_1, \dots, i_r) \in SJ_{1,\ell}^{(r)}$ with $r \in \mathbb{N}$, let $\bar{E}_{\bar{x}} := E_{i_1} \cdots E_{i_r}$. Then using a standard argument, we see

Lemma 1. *As a \mathbb{C} -linear space, \tilde{B}^+ has a \mathbb{C} -basis*

$$(2.6) \quad \{\bar{E}_{\bar{x}} L_a \mid \bar{x} \in SJ_{1,\ell}, \ a \in \mathbb{Z}\Pi''\}.$$

The \mathbb{C} -algebra \tilde{B}^+ can be regarded as a Hopf algebra $(\tilde{B}^+, \Delta, S, \varepsilon)$ with $\Delta(L_a) = L_a \otimes L_a$, $S(L_a) = L_{-a}$, $\varepsilon(L_a) = 1$, $\Delta(E_i) = E_i \otimes 1 + L_{\alpha_i} \otimes E_i$, $S(E_i) = -L_{-\alpha_i} E_i$, $\varepsilon(E_i) = 0$.

Let $(\tilde{B}^+)^*$ be the dual linear space of \tilde{B}^+ . We regard $(\tilde{B}^+)^*$ as a \tilde{B}^+ -module by $X.f(Y) = f(YX)$ for all $f \in (\tilde{B}^+)^*$, and all $X, Y \in \tilde{B}^+$. Let

$$(2.7) \quad (\tilde{B}^+)^{\circ} := \{f \in (\tilde{B}^+)^* \mid \dim \tilde{B}^+.f < +\infty\}.$$

Let $f \in (\tilde{B}^+)^{\circ} \setminus \{0\}$. Let $r := \dim \tilde{B}^+.f$ and let $\{f_i \mid i \in J_{1,r}\}$ be a \mathbb{C} -basis of $\tilde{B}^+.f$. Assume that $f_1 = f$. Define $\rho_{ij} \in (\tilde{B}^+)^*$ ($i, j \in J_{1,r}$) by $X.f_j = \sum_{i \in J_{1,r}} \rho_{ij}(X) f_i$. Then $\rho_{ij}(XY) = \sum_{k \in J_{1,r}} \rho_{ik}(X) \rho_{kj}(Y)$, so $\rho_{ij} \in (\tilde{B}^+)^{\circ}$. We have $f = \sum_{i \in J_{1,r}} f_i(1) \rho_{ij}$. We regard $(\tilde{B}^+)^{\circ}$ as a unital \mathbb{C} -algebra with the unit ε , by the multiplication defined by $fg(X) := \sum_k f(X_k^{(1)})g(X_k^{(2)})$ for all $f, g \in (\tilde{B}^+)^{\circ}$, and all $X \in \tilde{B}^+$ with $\Delta(X) = \sum_k X_k^{(1)} \otimes X_k^{(2)}$; we note that $fg \in (\tilde{B}^+)^{\circ}$ since $X.(fg) = \sum_k (X_k^{(1)}.f)(X_k^{(2)}.g)$. We regard $(\tilde{B}^+)^{\circ}$, $(\tilde{B}^+)^{\circ} \otimes (\tilde{B}^+)^{\circ}$, $(\tilde{B}^+)^* \otimes (\tilde{B}^+)^*$ as subspaces of $(\tilde{B}^+)^*$, $(\tilde{B}^+)^* \otimes (\tilde{B}^+)^*$, $(\tilde{B}^+ \otimes \tilde{B}^+)^*$ respectively in a natural way. Define the linear maps $\Delta^{\circ} : (\tilde{B}^+)^{\circ} \rightarrow (\tilde{B}^+)^{\circ} \otimes (\tilde{B}^+)^{\circ} \subset (\tilde{B}^+)^* \otimes (\tilde{B}^+)^* \subset (\tilde{B}^+ \otimes \tilde{B}^+)^*$, $S^{\circ} : (\tilde{B}^+)^{\circ} \rightarrow (\tilde{B}^+)^{\circ} \subset (\tilde{B}^+)^*$ and $\varepsilon^{\circ} : (\tilde{B}^+)^{\circ} \rightarrow \mathbb{C}$ by $\Delta^{\circ}(f)(X \otimes Y) = f(XY)$, $S^{\circ}(f)(X) = f(S(X))$ and $\varepsilon^{\circ}(f) = f(1)$ respectively, where we

note that $\Delta^\circ(\rho_{ij}) = \sum_{k \in J_{1,r}} \rho_{ik} \otimes \rho_{kj}$, $X \cdot (S^\circ(\rho_{ij})) = \sum_{k \in J_{1,r}} \rho_{ik}(S(X))S^\circ(\rho_{kj})$, and $\sum_{k \in J_{1,r}} \varepsilon^\circ(\rho_{ik})\rho_{kj} = \sum_{k \in J_{1,r}} \varepsilon^\circ(\rho_{kj})\rho_{ik} = \rho_{ij}$ hold for the above ρ_{ij} . Then

$$(2.8) \quad (\tilde{B}^+)^\circ = ((\tilde{B}^+)^\circ, \Delta^\circ, S^\circ, \varepsilon^\circ).$$

can be regarded as a Hopf algebra. For the above χ , define the map $\chi^\vee : \mathbb{Z}\Pi'' \times \mathbb{Z}\Pi'' \rightarrow \mathbb{C}^\times$ by $\chi^\vee(a, b) := \chi(b, a)$, and let $(\tilde{B}^+)^\vee := \tilde{B}^+(\chi^\vee)$. We denote the elements L_a , E_i and $\bar{E}_{\bar{x}}$ of $(\tilde{B}^+)^\vee$ by L_a^\vee , E_i^\vee and $\bar{E}_{\bar{x}}^\vee$ respectively. By a natural argument, we see

Lemma 2. *There exists a unique Hopf algebra homomorphism $\varphi : (\tilde{B}^+)^\vee \rightarrow (\tilde{B}^+)^\circ$ such that $\varphi(L_a^\vee)(\bar{E}_{\bar{x}}L_b) = \delta_{\bar{x},\phi}\chi(b, a)$ and $\varphi(E_i^\vee)(\bar{E}_{\bar{x}}L_b) = \delta_{\bar{x},(i)}$.*

Define the bi-linear map

$$(2.9) \quad (,) : \tilde{B}^+ \times (\tilde{B}^+)^\vee \rightarrow \mathbb{C}$$

by $(X, X^\vee) := \varphi(X^\vee)(X)$. We denote the maps Δ , S and ε for $(\tilde{B}^+)^\vee$ by Δ^\vee , S^\vee and ε^\vee respectively. As a bi-linear map, $(,)$ is characterized by

$$(2.10) \quad \begin{aligned} (L_a, L_b^\vee) &= \chi(a, b), (E_i, E_j^\vee) = \delta_{ij}, (L_a, E_j^\vee) = (E_i, L_b^\vee) = 0, \\ (XY, X^\vee) &= \sum_r (X, (X^\vee)_r^{(1)})(Y, (X^\vee)_r^{(2)}), \\ (X, X^\vee Y^\vee) &= \sum_k (X_k^{(1)}, X^\vee)(X_k^{(2)}, Y^\vee), \end{aligned}$$

for all $a, b \in \mathbb{Z}\Pi''$, all $i, j \in J_{1,\ell}$, all $X, Y \in \tilde{B}^+$ with $\Delta(X) = \sum_k X_k^{(1)} \otimes X_k^{(2)}$, and all $X^\vee, Y^\vee \in (\tilde{B}^+)^\vee$ with $\Delta^\vee(X^\vee) = \sum_k (X^\vee)_k^{(1)} \otimes (X^\vee)_k^{(2)}$. We also have

$$(2.11) \quad (\bar{E}_{\bar{x}}L_a, \bar{E}_{\bar{y}}^\vee L_b^\vee) = \delta_{\|\bar{x}\|, \|\bar{y}\|} \cdot \chi(a, b)(\bar{E}_{\bar{x}}, \bar{E}_{\bar{y}}^\vee)$$

for all $\bar{x}, \bar{y} \in SJ_{1,\ell}$ and all $a, b \in \mathbb{Z}\Pi''$. We also have

$$(2.12) \quad (S(X), X^\vee) = (X, S^\vee(X^\vee)), (1, X^\vee) = \varepsilon^\vee(X^\vee), (X, 1) = \varepsilon(X)$$

for all $X \in \tilde{B}^+$ and all $X^\vee \in (\tilde{B}^+)^\vee$. Let $\tilde{C}^+ := \{X \in \tilde{B}^+ | (X, (\tilde{B}^+)^\vee) = \{0\}\}$ and $(\tilde{C}^+)^\vee := \{X^\vee \in (\tilde{B}^+)^\vee | (\tilde{B}^+, X^\vee) = \{0\}\}$. Let B^+ and $(B^+)^\vee$ denote the quotient Hopf algebras \tilde{B}^+/\tilde{C}^+ and $(\tilde{B}^+)^\vee/(\tilde{C}^+)^\vee$ respectively. By abuse of notation, we shall use the same symbols for objects L_a , E_i , L_a^\vee , E_i^\vee , $(,)$ etc. and the objects defined as those modulo \tilde{C}^+ or/and $(\tilde{C}^+)^\vee$.

By a cerebrate argument due to Drinfeld, we have a Hopf algebra $D = D(\sqrt{\chi}) = (D = D(\sqrt{\chi}), \Delta = \Delta_D, S = S_D, \varepsilon = \varepsilon_D)$ such that

(1) As a \mathbb{C} -linear space, $D = B^+ \otimes (B^+)^\vee = B^+(\chi) \otimes B^+(\chi^\vee)$. By abuse of notation, for $X \in B^+$ and $X^\vee \in (B^+)^\vee$, we denote the elements $X \otimes 1$ and $1 \otimes X^\vee$ of D by X and X^\vee respectively. The linear map $B^+ \rightarrow D$, $X \mapsto X$, is a Hopf algebra homomorphism. The linear map $(B^+)^\vee \rightarrow D$, $X^\vee \mapsto X^\vee$, is a \mathbb{C} -algebra

homomorphism. For $X^\vee \in (B^+)^\vee$ with $\Delta^\vee(X^\vee) = \sum_r (X^\vee)_r^{(1)} \otimes (X^\vee)_r^{(2)}$, we have $\Delta_D(X^\vee) = \sum_r (X^\vee)_r^{(2)} \otimes (X^\vee)_r^{(1)}$, $S_D(X^\vee) = (S^\vee)^{-1}(X^\vee)$, and $\varepsilon_D(X^\vee) = \varepsilon^\vee(X^\vee)$.

(2) As for the multiplication of D , for $X \in B^+$ and $X^\vee \in (B^+)^\vee$, with $((1 \otimes \Delta) \circ \Delta)(X) = \sum_r X_r^{(1)} \otimes X_r^{(2)} \otimes X_r^{(3)}$ and $((1 \otimes \Delta^\vee) \circ \Delta^\vee)(X^\vee) = \sum_k (X^\vee)_k^{(1)} \otimes (X^\vee)_k^{(2)} \otimes (X^\vee)_k^{(3)}$, we have

$$(2.13) \quad X^\vee \cdot X = \sum_{r,k} (S^{-1}(X_r^{(1)}), (X^\vee)_k^{(3)})(X_r^{(3)}, (X^\vee)_k^{(1)}) X_r^{(2)} \cdot (X^\vee)_k^{(2)}.$$

For X and X^\vee in the above (2), we also have

$$(2.14) \quad X \cdot X^\vee = \sum_{r,k} (S^{-1}(X_r^{(3)}), (X^\vee)_k^{(1)})(X_r^{(1)}, (X^\vee)_k^{(3)})(X^\vee)_k^{(2)} \cdot X_r^{(2)}.$$

Further we have

$$(2.15) \quad \begin{aligned} L_a L_b^\vee &= L_b^\vee L_a, \quad L_a E_j^\vee = \chi(-a, \alpha_j) E_j^\vee L_a, \quad L_b^\vee E_i = \chi(\alpha_i, -b) E_i L_b^\vee, \\ E_i E_j^\vee - E_j^\vee E_i &= \delta_{ij} (-L_{\alpha_i} + L_{-\alpha_i}^\vee). \end{aligned}$$

Note that $L_0 = L_0^\vee = 1$ holds in D . Let $\mathbb{Z}_{\geq 0}\Pi = \{\sum_{i \in J_{1,\ell}} n_i \alpha_i \in \mathbb{Z}\Pi \mid n_i \in \mathbb{Z}, n_i \geq 0\}$. For $\beta \in \mathbb{Z}\Pi$, define subspaces U_β^+ and $U_{-\beta}^-$ of D in the following way. If $\beta \in \mathbb{Z}\Pi \setminus \mathbb{Z}_{\geq 0}\Pi$, let $U_\beta^+ := U_{-\beta}^- := \{0\}$. Let $U_0^+ := U_0^- := \mathbb{C}L_0$. If $\beta \in \mathbb{Z}_{\geq 0}\Pi$, let $U_\beta^+ := \sum_{i \in J_{1,\ell}} E_i U_{\beta - \alpha_i}^+$, and $U_{-\beta}^- := \sum_{i \in J_{1,\ell}} E_i^\vee U_{-\beta + \alpha_i}^-$. Let $U^+ := \sum_{\beta \in \mathbb{Z}_{\geq 0}\Pi} U_\beta^+$ and $U^- := \sum_{\beta \in \mathbb{Z}_{\geq 0}\Pi} U_{-\beta}^-$. Let $D^0 := \sum_{\gamma, \theta \in \mathbb{Z}\Pi''} \mathbb{C}L_\gamma L_\theta^\vee$. Then $D = \text{Span}_{\mathbb{C}}(U^+ D^0 U^-) = \text{Span}_{\mathbb{C}}(U^- D^0 U^+)$. Further, by (2.11), we have

Lemma 3. (1) For any $\gamma, \theta \in \mathbb{Z}\Pi''$, $L_\gamma L_\theta^\vee \neq 0$ holds in D . In particular, $\dim U_0^+ = \dim U_0^- = 1$

(2) $\dim U_{\alpha_i}^+ = \dim U_{-\alpha_i}^- = 1$ holds for any $i \in J_{1,\ell}$.

(3) $U^+ = \bigoplus_{\beta \in \mathbb{Z}_{\geq 0}\Pi} U_\beta^+$, $U^- = \bigoplus_{\beta \in \mathbb{Z}_{\geq 0}\Pi} U_{-\beta}^-$, and $D^0 = \bigoplus_{\gamma, \theta \in \mathbb{Z}\Pi''} \mathbb{C}L_\gamma L_\theta^\vee$ hold as \mathbb{C} -linear spaces.

(4) The linear maps $U^+ \otimes D^0 \otimes U^- \rightarrow D$, $X \otimes L_\gamma L_\theta^\vee \otimes X^\vee \mapsto X L_\gamma L_\theta^\vee X^\vee$, and $U^- \otimes D^0 \otimes U^+ \rightarrow D$, $X^\vee \otimes L_\gamma L_\theta^\vee \otimes X \mapsto X^\vee L_\gamma L_\theta^\vee X$, are bijective.

3 Rosso form

From now on, except for Section 8, we assume that $\sqrt{\chi}$ is symmetric, that is,

$$(3.1) \quad \text{we assume that } \sqrt{\chi}(a, b) = \sqrt{\chi}(b, a) \text{ for all } a, b \in \mathbb{Z}\Pi''.$$

Define the subgroup T of $\mathbb{Z}\Pi'$ by

$$(3.2) \quad T := \{\omega \in \mathbb{Z}\Pi' \mid \sqrt{\chi}(\omega, \mathbb{Z}\Pi') = \{1\}\}.$$

Let D' be the subalgebra of D generated by E_i, E_i^\vee ($i \in J_{1,\ell}$) and L_θ, L_θ^\vee ($\theta \in \mathbb{Z}\Pi'$). Then D' is a Hopf subalgebra of D . Let G be the ideal of D' (as a \mathbb{C} -algebra) generated by $L_\theta L_\theta^\vee - 1$ ($\theta \in \mathbb{Z}\Pi'$) and $L_\omega - 1$ ($\omega \in T$). Let $U = U(\sqrt{\chi}) := D'/G$ (as a \mathbb{C} -algebra). Then U can be regarded as a quotient Hopf algebra of D' . Let

$$(3.3) \quad \overline{\mathbb{Z}\Pi'} := \mathbb{Z}\Pi'/T.$$

For $\lambda \in \mathbb{Z}\Pi'$, let $\bar{\lambda} := \lambda + T \in \overline{\mathbb{Z}\Pi'}$, and let $L_{\bar{\lambda}} := L_\lambda + G \in U$. For any $\eta \in \overline{\mathbb{Z}\Pi'}$, $L_\eta \neq 0$ holds in U . Let $U^0 := \sum_{\eta \in \overline{\mathbb{Z}\Pi'}} L_\eta$. Then $U^0 = \bigoplus_{\eta \in \overline{\mathbb{Z}\Pi'}} L_\eta$ holds. The U^+ and U^- in the previous section can be regarded as subalgebras of U . Further, the linear maps $U^+ \otimes U^0 \otimes U^- \rightarrow D$, $X \otimes L_\eta \otimes X^\vee \mapsto XL_\eta X^\vee$, and $U^- \otimes U^0 \otimes U^+ \rightarrow D$, $X^\vee \otimes L_\eta \otimes X \mapsto X^\vee L_\eta X$, are bijective. We have a \mathbb{C} -algebra automorphism Ω of U such that $\Omega(E_i) = E_i^\vee$, $\Omega(E_i^\vee) = E_i$, and $\Omega(L_\eta) = L_{-\eta}$. Then $\Omega^2 = 1$. Let $U^{\geq 0} := \text{Span}(U^+ U^0)$. Define the bi-linear form $(,) : U^{\geq 0} \times U^{\geq 0} \rightarrow \mathbb{C}$ by $(XL_{\bar{\lambda}}, \tilde{X}L_{\bar{\mu}}) := (XL_\lambda, \Omega(\tilde{X})L_{-\mu}^\vee)$ for all $X, \tilde{X} \in U^+$, and all $\lambda, \mu \in \mathbb{Z}\Pi'$. Then $(,)$ is symmetric. Define the non-degenerate bi-linear form

$$(3.4) \quad \langle , \rangle : U \times U \rightarrow \mathbb{C}$$

by

$$(3.5) \quad \langle XL_{\bar{\lambda}}S(Y), \tilde{Y}L_{\bar{\mu}}S(\tilde{X}) \rangle := \sqrt{\chi}(-\lambda, \mu)(X, \Omega(\tilde{Y}))(\tilde{X}, \Omega(Y))$$

for all $X, \tilde{X} \in U^+$, all $Y, \tilde{Y} \in U^-$, and all $\lambda, \mu \in \mathbb{Z}\Pi'$.

Define the left action ad and the right action $\widetilde{\text{ad}}$ of U on U by

$$(3.6) \quad \text{ad}(u) \cdot v := \sum_r u_r^{(1)} v S(u_r^{(2)}) \quad \text{and} \quad v \cdot \widetilde{\text{ad}}(u) := \sum_r S(u_r^{(1)}) v u_r^{(2)}$$

respectively for all $u, v \in U$ with $\Delta(u) = \sum_r u_r^{(1)} \otimes u_r^{(2)}$.

Theorem 4. *We have*

$$(3.7) \quad \langle \text{ad}(u) \cdot v_1, v_2 \rangle = \langle v_1, v_2 \cdot \widetilde{\text{ad}}(u) \rangle$$

for all $u, v_1, v_2 \in U$.

Proof. We may assume that

$$(3.8) \quad v_1 = XL_{\bar{\lambda}}S(Y) \quad \text{and} \quad v_2 = \tilde{Y}L_{\bar{\mu}}S(\tilde{X})$$

with $\lambda, \mu \in \mathbb{Z}\Pi'$, $X \in U_\theta^+$, $Y \in U_{-\gamma}^-$, $\tilde{X} \in U_\omega^+$, $\tilde{Y} \in U_{-\delta}^-$, and $\theta, \gamma, \omega, \delta \in \mathbb{Z}_{\geq 0}\Pi$.

Case-1. Assume $u = L_\nu$ with $\nu \in \mathbb{Z}\Pi'$. Then we have

$$\begin{aligned} \langle \text{ad}(u) \cdot v_1, v_2 \rangle &= \chi(\nu, \theta - \gamma) \langle v_1, v_2 \rangle \\ &= \chi(\nu, \theta - \gamma) \delta_{\theta, \delta} \delta_{\gamma, \omega} \langle v_1, v_2 \rangle = \chi(-\nu, \omega - \delta) \langle v_1, v_2 \rangle \\ &= \langle v_1, v_2 \cdot \widetilde{\text{ad}}(u) \rangle, \end{aligned}$$

as desired.

Case-2. Assume $u \in U_{\beta}^+$ with $\beta \in \mathbb{Z}_{\geq 0}\Pi$. We write:

$$(3.9) \quad \Delta(u) = \sum_{r'} u_{r'}^{(1)} \otimes u_{r'}^{(2)},$$

$$\begin{aligned} & ((1 \otimes 1 \otimes \Delta) \circ (1 \otimes \Delta) \circ \Delta)(u) \\ &= \sum_{r''} u_{r''}^{(1)} \otimes u_{r''}^{(2)} \otimes u_{r''}^{(3)} \otimes u_{r''}^{(4)} \\ &= \sum_{\vec{\beta}, r} u_{1,r}^{(\beta_1)} L_{\beta_2+\beta_3+\beta_4} \otimes u_{2,r}^{(\beta_2)} L_{\beta_3+\beta_4} \otimes u_{3,r}^{(\beta_3)} L_{\beta_4} \otimes u_{4,r}^{(\beta_4)}, \end{aligned}$$

where $\vec{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4) \in (\mathbb{Z}_{\geq 0}\Pi)^4$ with $\beta_1 + \beta_2 + \beta_3 + \beta_4 = \beta$, and $u_{x,r}^{(\beta_x)} \in U_{\beta_x}^+$. We also write:

$$(3.10) \quad \begin{aligned} & ((1 \otimes \Delta) \circ \Delta)(Y) \\ &= \sum_{s'} Y_{s'}^{(1)} \otimes Y_{s'}^{(2)} \otimes Y_{s'}^{(3)} \\ &= \sum_{\vec{\gamma}, s} Y_{1,s}^{(\gamma_1)} \otimes Y_{2,s}^{(\gamma_2)} L_{-\gamma_1} \otimes Y_{3,s}^{(\gamma_3)} L_{-\gamma_1-\gamma_2}, \end{aligned}$$

where $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{Z}_{\geq 0}\Pi)^3$ with $\gamma_1 + \gamma_2 + \gamma_3 = \gamma$, and $Y_{y,s}^{(\gamma_y)} \in U_{-\gamma_y}^-$. Then

we have

$$\begin{aligned}
\text{ad}(u) \cdot v_1 &= \sum_{r'} u_{r'}^{(1)} v_1 S(u_{r'}^{(2)}) \\
&= \sum_{r'} u_{r'}^{(1)} X L_{\bar{\lambda}} S(Y) S(u_{r'}^{(2)}) = \sum_{r'} u_{r'}^{(1)} X L_{\bar{\lambda}} S(u_{r'}^{(2)}) Y \\
&= \sum_{r'', s'} u_{r''}^{(1)} X L_{\bar{\lambda}} S\left((u_{r''}^{(2)}, \Omega(Y_{s'}^{(1)})) (S^{-1}(u_{r''}^{(4)}), \Omega(Y_{s'}^{(3)})) Y_{s'}^{(2)} u_{r''}^{(3)}\right) \\
&= \sum_{\vec{\beta}, r, \vec{\gamma}, s} (u_{2,r}^{(\beta_2)} L_{\beta_3 + \beta_4}, \Omega(Y_{1,s}^{(\gamma_1)})) (S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{-\gamma_1 - \gamma_2})) \\
&\quad \cdot u_{1,r}^{(\beta_1)} L_{\beta_2 + \beta_3 + \beta_4} X L_{\bar{\lambda}} S(Y_{2,s}^{(\gamma_2)} L_{-\gamma_1} u_{3,r}^{(\beta_3)} L_{\beta_4}) \\
&= \sum_{\vec{\beta}, r, \vec{\gamma}, s} \chi(\beta_2 + \beta_3 + \beta_4, \theta) \chi(-\beta_4, \beta_3) \\
&\quad \cdot (u_{2,r}^{(\beta_2)} L_{\beta_3 + \beta_4}, \Omega(Y_{1,s}^{(\gamma_1)})) (S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{-\gamma_1 - \gamma_2})) \\
&\quad \cdot u_{1,r}^{(\beta_1)} X L_{\beta_2 + \beta_3 + \beta_4 + \lambda} S(Y_{2,s}^{(\gamma_2)} L_{-\gamma_1 + \beta_4} u_{3,r}^{(\beta_3)}) \\
&= \sum_{\vec{\beta}, r, \vec{\gamma}, s} \chi(\beta_2 + \beta_3 + \beta_4, \theta) \chi(-\beta_4, \beta_3) \chi(\beta_2 + \beta_3 + \beta_4 + \lambda, \beta_3) \\
&\quad \cdot (u_{2,r}^{(\beta_2)} L_{\beta_3 + \beta_4}, \Omega(Y_{1,s}^{(\gamma_1)})) (S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{-\gamma_1 - \gamma_2})) \\
&\quad \cdot (u_{1,r}^{(\beta_1)} X S(u_{3,r}^{(\beta_3)}) L_{\beta_3}) L_{\beta_2 + \lambda + \gamma_1} S(Y_{2,s}^{(\gamma_2)}) \\
&= \sum_{\vec{\beta}, r, \vec{\gamma}, s} \chi(\beta_2 + \beta_3 + \beta_4, \theta) \chi(\beta_2 + \beta_3 + \lambda, \beta_3) \\
&\quad \cdot (u_{2,r}^{(\beta_2)} L_{\beta_3 + \beta_4}, \Omega(Y_{1,s}^{(\gamma_1)})) (S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{-\gamma_1 - \gamma_2})) \\
&\quad \cdot (u_{1,r}^{(\beta_1)} X S(u_{3,r}^{(\beta_3)}) L_{\beta_3}) L_{\beta_2 + \lambda + \gamma_1} S(Y_{2,s}^{(\gamma_2)}).
\end{aligned}$$

Then we have

$$\begin{aligned}
& \langle \text{ad}(u) \cdot v_1, v_2 \rangle \\
&= \sum_{\vec{\beta}, r, \vec{\gamma}, s} \chi(\beta_2 + \beta_3 + \beta_4, \theta) \chi(\beta_2 + \beta_3 + \lambda, \beta_3) \sqrt{\chi}(-(\beta_2 + \lambda + \gamma_1), \mu) \\
&\quad \cdot (u_{2,r}^{(\beta_2)} L_{\beta_3 + \beta_4}, \Omega(Y_{1,s}^{(\gamma_1)})) (S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{-\gamma_1 - \gamma_2})) \\
&\quad \cdot (u_{1,r}^{(\beta_1)} X S(u_{3,r}^{(\beta_3)}) L_{\beta_3}, \Omega(\tilde{Y})) (\tilde{X}, \Omega(Y_{2,s}^{(\gamma_2)})) \\
&= \sum_{\vec{\beta}, r, \vec{\gamma}, s} \chi(\beta_2 + \beta_3 + \beta_4, \theta) \chi(\beta_2 + \beta_3 + \lambda, \beta_3) \sqrt{\chi}(-(\beta_2 + \lambda + \gamma_1), \mu) \\
&\quad \cdot \delta_{\beta_2, \gamma_1} \delta_{\beta_4, \gamma_3} \delta_{\beta_1 + \theta + \beta_3, \delta} \delta_{\omega, \gamma_2} \\
&\quad \cdot (u_{2,r}^{(\beta_2)}, \Omega(Y_{1,s}^{(\gamma_1)})) (S^{-1}(u_{4,r}^{(\beta_4)}), \Omega(Y_{3,s}^{(\gamma_3)} L_{-\gamma_1 - \gamma_2})) \\
&\quad \cdot (u_{1,r}^{(\beta_1)} X S(u_{3,r}^{(\beta_3)}) L_{\beta_3}, \Omega(\tilde{Y})) (\tilde{X}, \Omega(Y_{2,s}^{(\gamma_2)} L_{-\gamma_1})) \\
&= \sum_{\vec{\beta}, r} \chi(\beta_2 + \beta_3 + \beta_4, \theta) \chi(\beta_2 + \beta_3 + \lambda, \beta_3) \chi(-\beta_2, \mu) \sqrt{\chi}(-\lambda, \mu) \\
&\quad \cdot (S^{-1}(u_{4,r}^{(\beta_4)}) \tilde{X} u_{2,r}^{(\beta_2)}, \Omega(Y)) (u_{1,r}^{(\beta_1)} X S(u_{3,r}^{(\beta_3)}) L_{\beta_3}, \Omega(\tilde{Y})).
\end{aligned}$$

We write:

$$\begin{aligned}
& ((1 \otimes \Delta) \circ \Delta)(\tilde{Y}) \\
&= \sum_{t'} \tilde{Y}_{t'}^{(1)} \otimes \tilde{Y}_{t'}^{(2)} \otimes \tilde{Y}_{t'}^{(3)} \\
&= \sum_{\vec{\delta}, t} \tilde{Y}_{1,t}^{(\delta_1)} \otimes \tilde{Y}_{2,t}^{(\delta_2)} L_{-\delta_1} \otimes \tilde{Y}_{3,t}^{(\delta_3)} L_{-\delta_1 - \delta_2},
\end{aligned}$$

where $\vec{\delta} = (\delta_1, \delta_2, \delta_3) \in (\mathbb{Z}_{\geq 0} \Pi)^3$ with $\delta_1 + \delta_2 + \delta_3 = \delta$, and $\tilde{Y}_{y,s}^{(\delta_z)} \in U_{-\delta_z}^-$. Then

we have

$$\begin{aligned}
v_1 \cdot \widetilde{\text{ad}}(u) &= \sum_{r'} S(u_{r'}^{(1)}) v_2 u_{r'}^{(2)} \\
&= \sum_{r'} S(u_{r'}^{(1)}) \tilde{Y} L_{\bar{\mu}} S(\tilde{X}) u_{r'}^{(2)} \\
&= \sum_{r'', t'} (S(u_{r''}^{(3)}), \Omega(\tilde{Y}_{t'}^{(1)})) (S^{-1}(S(u_{r''}^{(1)})), \Omega(\tilde{Y}_{t'}^{(3)})) \tilde{Y}_{t'}^{(2)} S(u_{r''}^{(2)}) L_{\bar{\mu}} S(\tilde{X}) u_{r''}^{(4)} \\
&= \sum_{\vec{\beta}, r, \vec{\delta}, t} (S(u_{3,r}^{(\beta_3)} L_{\bar{\beta}_4}), \Omega(\tilde{Y}_{1,t}^{(\delta_1)})) (u_{1,r}^{(\beta_1)} L_{\bar{\beta}_2 + \beta_3 + \beta_4}, \Omega(\tilde{Y}_{3,t}^{(\delta_3)} L_{-\delta_1 - \delta_2})) \\
&\quad \cdot \tilde{Y}_{2,t}^{(\delta_2)} L_{-\delta_1} S(u_{2,r}^{(\beta_2)} L_{\bar{\beta}_3 + \beta_4}) L_{\bar{\mu}} S(\tilde{X}) u_{4,r}^{(\beta_4)} \\
&= \sum_{\vec{\beta}, r, \vec{\delta}, t} (S(u_{3,r}^{(\beta_3)} L_{\bar{\beta}_4}), \Omega(\tilde{Y}_{1,t}^{(\delta_1)})) (u_{1,r}^{(\beta_1)} L_{\bar{\beta}_2 + \beta_3 + \beta_4}, \Omega(\tilde{Y}_{3,t}^{(\delta_3)} L_{-\delta_1 - \delta_2})) \\
&\quad \cdot \tilde{Y}_{2,t}^{(\delta_2)} L_{-\delta_1} S(u_{2,r}^{(\beta_2)} L_{\bar{\beta}_3 + \beta_4}) L_{\bar{\mu}} S(\tilde{X}) L_{\bar{\beta}_4} S(S^{-1}(u_{4,r}^{(\beta_4)}) L_{\bar{\beta}_4}) \\
&= \sum_{\vec{\beta}, r, \vec{\delta}, t} \chi(-\mu, \beta_2) \chi(-\beta_4, \beta_2 + \omega) \\
&\quad \cdot (S(u_{3,r}^{(\beta_3)} L_{\bar{\beta}_4}), \Omega(\tilde{Y}_{1,t}^{(\delta_1)})) (u_{1,r}^{(\beta_1)} L_{\bar{\beta}_2 + \beta_3 + \beta_4}, \Omega(\tilde{Y}_{3,t}^{(\delta_3)} L_{-\delta_1 - \delta_2})) \\
&\quad \cdot \tilde{Y}_{2,t}^{(\delta_2)} L_{-\delta_1 - \beta_3 + \mu} S(S^{-1}(u_{4,r}^{(\beta_4)}) L_{\bar{\beta}_4} \tilde{X} u_{2,r}^{(\beta_2)}).
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \langle v_1, v_2 \cdot \widetilde{\text{ad}}(u) \rangle \\
&= \sum_{\vec{\beta}, r, \vec{\delta}, t} \chi(-\mu, \beta_2) \chi(-\beta_4, \beta_2 + \omega) \sqrt{\chi}(-\lambda, -\delta_1 - \beta_3 + \mu) \\
&\quad \cdot (S(u_{3,r}^{(\beta_3)} L_{\beta_4}), \Omega(\tilde{Y}_{1,t}^{(\delta_1)})) (u_{1,r}^{(\beta_1)} L_{\beta_2 + \beta_3 + \beta_4}, \Omega(\tilde{Y}_{3,t}^{(\delta_3)} L_{-\delta_1 - \delta_2})) \\
&\quad \cdot (X, \Omega(\tilde{Y}_{2,t}^{(\delta_2)})) (S^{-1}(u_{4,r}^{(\beta_4)}) L_{\beta_4} \tilde{X} u_{2,r}^{(\beta_2)}, \Omega(Y)) \\
&= \sum_{\vec{\beta}, r, \vec{\delta}, t} \chi(-\mu, \beta_2) \chi(-\beta_4, \beta_2 + \omega) \sqrt{\chi}(-\lambda, -\delta_1 - \beta_3 + \mu) \\
&\quad \cdot \delta_{\beta_3, \delta_1} \delta_{\beta_1, \delta_3} \delta_{\theta, \delta_2} \delta_{\beta_4 + \omega + \beta_2, \gamma} \\
&\quad \cdot (S(u_{3,r}^{(\beta_3)} L_{\beta_4}), \Omega(\tilde{Y}_{1,t}^{(\delta_1)})) (u_{1,r}^{(\beta_1)} L_{\beta_2 + \beta_3 + \beta_4}, \Omega(\tilde{Y}_{3,t}^{(\delta_3)} L_{-\delta_1 - \delta_2})) \\
&\quad \cdot (X, \Omega(\tilde{Y}_{2,t}^{(\delta_2)} L_{-\delta_1})) (S^{-1}(u_{4,r}^{(\beta_4)}) L_{\beta_4} \tilde{X} u_{2,r}^{(\beta_2)}, \Omega(Y)) \\
&= \sum_{\vec{\beta}, r} \chi(-\mu, \beta_2) \chi(-\beta_4, \beta_2 + \omega) \chi(\lambda, \beta_3) \sqrt{\chi}(-\lambda, \mu) \\
&\quad \cdot (u_{1,r}^{(\beta_1)} L_{\beta_2 + \beta_3 + \beta_4} X S(u_{3,r}^{(\beta_3)} L_{\beta_4}), \Omega(\tilde{Y})) (S^{-1}(u_{4,r}^{(\beta_4)}) L_{\beta_4} \tilde{X} u_{2,r}^{(\beta_2)}, \Omega(Y)) \\
&= \sum_{\vec{\beta}, r} \chi(-\mu, \beta_2) \chi(-\beta_4, \beta_2 + \omega) \chi(\lambda, \beta_3) \sqrt{\chi}(-\lambda, \mu) \\
&\quad \cdot \chi(\beta_4, \theta) \chi(\beta_2 + \beta_3, \theta + \beta_3) \chi(\beta_4, \omega + \beta_2) \\
&\quad \cdot (u_{1,r}^{(\beta_1)} X S(u_{3,r}^{(\beta_3)}), \Omega(\tilde{Y})) (S^{-1}(u_{4,r}^{(\beta_4)}) \tilde{X} u_{2,r}^{(\beta_2)}, \Omega(Y)) \\
&= \langle \text{ad}(u) \cdot v_1, v_2 \rangle,
\end{aligned}$$

as desired.

Case-3. Assume $u = E_i^\vee$ with $i \in J_{1,\ell}$. Note that $\Omega S \Omega = S^{-1}$. Then

$$\begin{aligned}
& \langle \Omega(v_2), \Omega(v_1) \rangle \\
&= \langle \Omega(\tilde{Y} L_{\vec{\mu}} S(\tilde{X})), \Omega(X L_{\vec{\lambda}} S(Y)) \rangle \\
&= \langle \Omega(\tilde{Y}) L_{-\vec{\mu}} S(S^{-2}(\Omega(\tilde{X}))), \Omega(X) L_{-\vec{\lambda}} S(S^{-2}(\Omega(Y))) \rangle \\
&= \sqrt{\chi}(\mu, -\lambda) (\Omega(\tilde{Y}), X) (S^{-2}(\Omega(Y)), \Omega(S^{-2}(\Omega(\tilde{X})))) \\
&= \sqrt{\chi}(\mu, -\lambda) (\Omega(\tilde{Y}), X) (S^{-2}(\Omega(Y)), S^2(\tilde{X})) \\
&= \sqrt{\chi}(\mu, -\lambda) (\Omega(\tilde{Y}), X) (\Omega(Y), \tilde{X}) \\
&= \langle v_1, v_2 \rangle.
\end{aligned}$$

We have

$$\begin{aligned}
& \Omega(\text{ad}(E_i) \cdot v_1) \\
&= \Omega(E_i v_1 - L_{\bar{\alpha}_i} v_1 L_{-\bar{\alpha}_i} E_i) \\
&= \Omega(E_i v_1 - \chi(\alpha_i, \theta - \gamma) v_1 E_i) \\
&= -\chi(\alpha_i, \theta - \gamma) (-E_i^\vee L_{\bar{\alpha}_i} \Omega(v_1) L_{-\bar{\alpha}_i} + \Omega(v_1) E_i^\vee) \\
&= -\chi(\alpha_i, \theta - \gamma) \Omega(v_1) \cdot \widetilde{\text{ad}}(E_i^\vee),
\end{aligned}$$

and

$$\begin{aligned}
& \Omega(v_2 \cdot \widetilde{\text{ad}}(E_i)) \\
&= \Omega(-L_{-\bar{\alpha}_i} E_i v_2 + L_{-\bar{\alpha}_i} v_2 E_i) \\
&= \chi(-\alpha_i, \omega - \delta + \alpha_i) \Omega(-E_i v_2 L_{-\bar{\alpha}_i} + v_2 E_i L_{-\bar{\alpha}_i}) \\
&= -\chi(-\alpha_i, \omega - \delta + \alpha_i) (E_i^\vee \Omega(v_2) L_{\bar{\alpha}_i} - \Omega(v_2) E_i^\vee L_{\bar{\alpha}_i}) \\
&= -\chi(-\alpha_i, \omega - \delta + \alpha_i) \text{ad}(E_i^\vee) \cdot \Omega(v_2).
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \langle \text{ad}(E_i^\vee) \cdot \Omega(v_2), \Omega(v_1) \rangle \\
&= -\chi(\alpha_i, \omega - \delta + \alpha_i) \langle \Omega(v_2 \cdot \widetilde{\text{ad}}(E_i)), \Omega(v_1) \rangle \\
&= -\chi(\alpha_i, \omega - \delta + \alpha_i) \langle v_1, v_2 \cdot \widetilde{\text{ad}}(E_i) \rangle \\
&= -\chi(\alpha_i, \omega - \delta + \alpha_i) \langle \text{ad}(E_i) \cdot v_1, v_2 \rangle \\
&= -\chi(\alpha_i, \omega - \delta + \alpha_i) \langle \Omega(v_2), \Omega(\text{ad}(E_i) \cdot v_1) \rangle \\
&= \chi(\alpha_i, \omega - \delta + \alpha_i + \theta - \gamma) \langle \Omega(v_2), \Omega(v_1) \cdot \widetilde{\text{ad}}(E_i^\vee) \rangle \\
&= \chi(\alpha_i, \omega - \delta + \alpha_i + \theta - \gamma) \delta_{-(\omega - \delta), -(\theta - \gamma) - \alpha_i} \langle \Omega(v_2), \Omega(v_1) \cdot \widetilde{\text{ad}}(E_i^\vee) \rangle \\
&= \delta_{-(\omega - \delta), -(\theta - \gamma) - \alpha_i} \langle \Omega(v_2), \Omega(v_1) \cdot \widetilde{\text{ad}}(E_i^\vee) \rangle \\
&= \langle \Omega(v_2), \Omega(v_1) \cdot \widetilde{\text{ad}}(E_i^\vee) \rangle,
\end{aligned}$$

as desired. This completes the proof. \square

4 Harish-Chandra map

We define the \mathbb{C} -linear map $\Phi : U \rightarrow U^0$ by $\Phi(X^\vee L_\mu X) := \varepsilon(X^\vee) \varepsilon(X) L_\mu$ for all $X \in U^+$, all $\mu \in \mathbb{Z}\Pi'$, and all $X^\vee \in U^-$. For $\lambda \in \mathbb{Z}\Pi'$, define $\sqrt{\chi_\lambda} \in (U^0)^*$ by $\sqrt{\chi_\lambda}(L_\mu) := \sqrt{\chi}(\lambda, \mu)$ for all $\mu \in \mathbb{Z}\Pi'$. Define the \mathbb{C} -linear monomorphism $\zeta : U \rightarrow U^*$ by $\zeta(v_2)(v_1) := \langle v_1, v_2 \rangle$. Define the right action of U on U^* by $f \cdot u(v) := f(\text{ad}(u) \cdot v)$ for and $f \in U^*$, all $u, v \in u$. By (3.7), we have $\zeta(v) \cdot u = \zeta(v \cdot \text{ad}(u))$ for all $u, v \in u$. Let $\mathfrak{z}(U)$ be the center of U , that is $\mathfrak{z}(U) := \{u \in U \mid \forall v \in U, uv = vu\}$. It is easy to see that $\mathfrak{z}(U) = \{u \in U \mid \forall v \in U, u \cdot \text{ad}(v) = \varepsilon(v)u\}$.

$$(4.1) \quad \text{Assume that } \exists \tilde{\rho} \in \mathbb{Z}\Pi', \forall i \in J_{1,\ell}, \chi(\tilde{\rho}, \alpha_i) = \chi(\alpha_i, \alpha_i).$$

Then $S^2(u) = L_{-\tilde{\rho}}uL_{\tilde{\rho}}$ hold for all $u \in U$.

Let V be a finite dimensional left U -module. Define $f_V \in U^*$ by $f_V(u) := \text{Tr}(uL_{-\tilde{\rho}}; V)$. Then for all $u, v \in U$ with $\Delta(u) = \sum_r u_r^{(1)} \otimes u_r^{(2)}$ we have

$$\begin{aligned} (f_V \cdot u)(v) &= f_V(\text{ad}(u) \cdot v) \\ &= \sum_r f_V(u_r^{(1)}vS(u_r^{(2)})) = \sum_r \text{Tr}(u_r^{(1)}vS(u_r^{(2)})L_{-\tilde{\rho}}; V) \\ &= \sum_r \text{Tr}(vS(u_r^{(2)})L_{-\tilde{\rho}}u_r^{(1)}; V) = \sum_r \text{Tr}(vS(u_r^{(2)})S^2(u_r^{(1)})L_{-\tilde{\rho}}; V) \\ &= \sum_r \text{Tr}(vS(S(u_r^{(1)})u_r^{(2)})L_{-\tilde{\rho}}; V) = \text{Tr}(vS(\varepsilon(u))L_{-\tilde{\rho}}; V) \\ &= \text{Tr}(\varepsilon(u)vL_{-\tilde{\rho}}; V) = \varepsilon(u)f_V(v), \end{aligned}$$

so we have $f_V \cdot u = \varepsilon(u)f_V$. Hence we see that

$$(4.2) \quad f_V \in \text{Im}(\zeta) \implies \zeta^{-1}(f_V) \in \mathfrak{z}(U).$$

$$(4.3) \quad \begin{aligned} &\text{Assume that } \exists \lambda \in \mathbb{Z}\Pi', \exists v_{\tilde{\lambda}} \in V \setminus \{0\}, V = \bigoplus_{\beta \in \mathbb{Z}_{\geq 0}\Pi} U_{-\beta}^- v_{\tilde{\lambda}}, \\ &\forall \mu \in \mathbb{Z}\Pi', L_{\tilde{\mu}}v_{\tilde{\lambda}} = \sqrt{\chi}(\mu, \lambda)v_{\tilde{\lambda}}, \forall i \in J_{1,\ell}, E_i v_{\tilde{\lambda}} = 0. \end{aligned}$$

Lemma 5. *We have $f_V \in \text{Im}(\zeta)$. In particular, $\zeta^{-1}(f_V) \in \mathfrak{z}(U)$. Further we have*

$$(4.4) \quad \Phi(\zeta^{-1}(f_V)) = \sum_{\beta \in \mathbb{Z}_{\geq 0}\Pi} (\dim U_{-\beta}^- v_{\tilde{\lambda}}) \sqrt{\chi}(\tilde{\rho}, 2\beta - \lambda) L_{\frac{2\beta - \lambda}{2}},$$

and

$$\begin{aligned} &\zeta^{-1}(f_V) - \Phi(\zeta^{-1}(f_V)) \\ &\in \sum_{\omega \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}, \beta \in \mathbb{Z}_{\geq 0}\Pi, \dim U_{-\beta-\omega}^- v_{\tilde{\lambda}} \neq 0} \text{Span}_{\mathbb{C}}(U_{-\omega}^- L_{\frac{\omega+2\beta-\lambda}{2}} U_{\omega}^+). \end{aligned}$$

Proof. Let $v_1 = XL_{\tilde{\nu}}Y, v_2 = \tilde{Y}L_{\tilde{\mu}}\tilde{X} \in U$ with $\nu, \mu \in \mathbb{Z}\Pi', X \in U_{\theta}^+, Y \in U_{-\gamma}^-,$

$\tilde{X} \in U_{\omega}^+$, $\tilde{Y} \in U_{-\delta}^-$, and $\theta, \gamma, \omega, \delta \in \mathbb{Z}_{\geq 0}\Pi$. Then we have

$$\begin{aligned}
\zeta(v_2)(v_1) &= \langle v_1, v_2 \rangle = \langle XL_{\bar{\nu}}Y, \tilde{Y}L_{\bar{\mu}}\tilde{X} \rangle \\
&= \langle XL_{\bar{\nu}}S(S^{-1}(Y)L_{-\bar{\gamma}}L_{\bar{\gamma}}), \tilde{Y}L_{\bar{\mu}}S(S^{-1}(\tilde{X})L_{\bar{\omega}}L_{-\bar{\omega}}) \rangle \\
&= \langle XL_{\bar{\nu}-\bar{\gamma}}S(S^{-1}(Y)L_{-\bar{\gamma}}), \tilde{Y}L_{\bar{\mu}+\bar{\omega}}S(S^{-1}(\tilde{X})L_{\bar{\omega}}) \rangle \\
&= \sqrt{\bar{\chi}}(-\nu + \gamma, \mu + \omega)(X, \Omega(\tilde{Y}))(S^{-1}(\tilde{X})L_{\bar{\omega}}, \Omega(S^{-1}(Y)L_{-\bar{\gamma}})) \\
&= \sqrt{\bar{\chi}}(-\nu + \gamma, \mu + \omega)(X, \Omega(\tilde{Y}))(S^{-1}(L_{-\bar{\omega}}\tilde{X}), \Omega(S^{-1}(L_{\bar{\gamma}}Y))) \\
&= \sqrt{\bar{\chi}}(-\nu + \gamma, \mu + \omega)(X, \Omega(\tilde{Y}))(S^{-1}(L_{-\bar{\omega}}\tilde{X}), S(\Omega(L_{\bar{\gamma}}Y))) \\
&= \sqrt{\bar{\chi}}(-\nu + \gamma, \mu + \omega)(X, \Omega(\tilde{Y}))(S(S^{-1}(L_{-\bar{\omega}}\tilde{X})), \Omega(L_{\bar{\gamma}}Y)) \\
&= \sqrt{\bar{\chi}}(-\nu + \gamma, \mu + \omega)\chi(-\omega, \omega)\chi(-\gamma, \gamma)(X, \Omega(\tilde{Y}))(\tilde{X}L_{-\bar{\omega}}, \Omega(Y)L_{-\bar{\gamma}}) \\
&= \sqrt{\bar{\chi}}(-\nu + \gamma, \mu + \omega)\chi(-\omega, \omega)\chi(-\gamma, \gamma)\chi(-\omega, -\gamma)(X, \Omega(\tilde{Y}))(\tilde{X}, \Omega(Y)) \\
&= \sqrt{\bar{\chi}}(\omega, \mu - \omega)\sqrt{\bar{\chi}_{-\mu-\omega}}(\bar{\nu})\delta_{\theta, \delta}\delta_{\omega, \gamma}(X, \Omega(\tilde{Y}))(\tilde{X}, \Omega(Y)).
\end{aligned}$$

Let $\beta \in \mathbb{Z}_{\geq 0}\Pi$. Let $m_{\beta} := \dim U_{\beta}^+$. Let $X_{\beta, x} \in U_{\beta}^+$ and $Y_{-\beta, x} \in U_{-\beta}^-$ ($x \in J_{1, m_{\beta}}$) be such that $(X_{\beta, x}, \Omega(Y_{-\beta, y})) = \delta_{xy}$. Then $\{X_{-\beta, x} | x \in J_{1, m_{\beta}}\}$ and $\{Y_{-\beta, x} | x \in J_{1, m_{\beta}}\}$ is \mathbb{C} -bases of U_{β}^+ and $U_{-\beta}^-$ respectively. Let $k_{\beta} := \dim U_{-\beta}^- v_{\bar{\lambda}}$. Let $\{Z_{-\beta, r} v_{\bar{\lambda}} | r \in J_{1, k_{\beta}}\}$ be a \mathbb{C} -basis of $U_{-\beta}^- v_{\bar{\lambda}}$. For $r \in J_{1, k_{\beta}}$, Define $t_{-\beta, r} \in V^*$ by $t_{-\beta, r}(Z_{-\beta, r'} v_{\bar{\lambda}}) := \delta_{\beta, \beta'} \delta_{r, r'}$.

Let $v_1 \in U$ be as above. Then we have

$$\begin{aligned}
f_V(v_1) &= \sum_{\beta \in \mathbb{Z}_{\geq 0}\Pi, r \in J_{1, k_{\beta}}} t_{-\beta, r}(v_1 L_{-\bar{\rho}} Z_{-\beta, r} v_{\bar{\lambda}}) \\
&= \sum_{\beta, r} \chi(-\bar{\rho}, -\beta) \sqrt{\bar{\chi}}(-\bar{\rho}, \lambda) t_{-\beta, r}(v_1 Z_{-\beta, r} v_{\bar{\lambda}}) \\
&= \sum_{\beta, r} \chi(-\bar{\rho}, -\beta) \sqrt{\bar{\chi}}(-\bar{\rho}, \lambda) t_{-\beta, r}(XL_{\bar{\nu}}Y Z_{-\beta, r} v_{\bar{\lambda}}) \\
&= \sum_{\beta, r} \chi(-\bar{\rho}, -\beta) \sqrt{\bar{\chi}}(-\bar{\rho}, \lambda) \chi(\nu, -\gamma - \beta) \sqrt{\bar{\chi}}(\nu, \lambda) t_{-\beta, r}(XY Z_{-\beta, r} v_{\bar{\lambda}}) \\
&= \sum_{\beta, r} \sqrt{\bar{\chi}}(\bar{\rho}, 2\beta - \lambda) \sqrt{\bar{\chi}_{-2\gamma-2\beta+\lambda}}(\bar{\nu}) \delta_{\theta, \gamma} t_{-\beta, r}(XY Z_{-\beta, r} v_{\bar{\lambda}}).
\end{aligned}$$

Hence we have

$$\begin{aligned}
f_V &= \sum_{\beta \in \mathbb{Z}_{\geq 0}\Pi, r \in J_{1, k_{\beta}}} \sum_{\omega \in \mathbb{Z}_{\geq 0}\Pi, x, y \in J_{1, m_{\beta}}} \\
&\quad \cdot \sqrt{\bar{\chi}}(\bar{\rho} - \omega, 2\beta - \lambda) t_{-\beta, r}(X_{\omega, x} Y_{-\omega, y} Z_{-\beta, r} v_{\bar{\lambda}}) \\
&\quad \cdot \zeta(Y_{-\omega, y} L_{\bar{\omega}+2\beta-\bar{\lambda}} X_{\omega, x}),
\end{aligned}$$

as desired. \square

Let $\lambda \in \mathbb{Z}\Pi'$. Let $M(\lambda)$ be the left U -module satisfying (4.3) and satisfying that $\dim U_{-\beta}^- v_{\bar{\lambda}} = \dim U_{-\beta}^-$ for all $\beta \in \mathbb{Z}_{\geq 0}\Pi$. Let $I(\lambda)$ be the proper ideal of $M(\lambda)$ defined as the sum of proper ideals I' of $M(\lambda)$ with $I' \subset \bigoplus_{\beta \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}} U_{\beta}^+ v_{\bar{\lambda}}$. Let $V(\lambda) := M(\lambda)/I(\lambda)$. Note that $V(\lambda)$ satisfies (4.3).

$$(4.5) \quad \text{For } r \in \mathbb{Z}_{\geq 0} \text{ and } t \in \mathbb{C}, \text{ let } \{r\}_t := \sum_{k \in J_{1,r}} t^{k-1}, \text{ and } \{r\}_t! := \prod_{k \in J_{1,r}} \{k\}_t.$$

$$(4.6) \quad \text{For } i \in J_{1,\ell}, \text{ let } q_i := \sqrt{\chi(\alpha_i, \alpha_i)}, \text{ so } q_i^2 := \chi(\alpha_i, \alpha_i).$$

Let $i \in J_{1,\ell}$ and $r \in \mathbb{Z}_{\geq 0}$. Then we have

$$\begin{aligned} (E_i^r, E_i^r) &= \sum_{k \in J_{1,r}} (E_i^{k-1} L_{\bar{\alpha}_i} E_i^{r-k}, E_i^{r-1}) = \{r\}_{q_i^2} (E_i^{r-1} L_{\bar{\alpha}_i}, E_i^{r-1}) \\ &= \{r\}_{q_i^2} (E_i^{r-1}, E_i^{r-1}) = \{r\}_{q_i^2}!, \end{aligned}$$

which implies

$$(4.7) \quad \{r\}_{q_i^2}! = 0 \Leftrightarrow E_i^r = 0 \Leftrightarrow (E_i^{\vee})^r = 0,$$

since $E_i^{\vee} = \Omega(E_i)$. We also have

$$\begin{aligned} (4.8) \quad E_i (E_i^{\vee})^r - (E_i^{\vee})^r E_i &= \sum_{k \in J_{1,r}} (E_i^{\vee})^{r-k} (-L_{\bar{\alpha}_i} + L_{-\bar{\alpha}_i}) (E_i^{\vee})^{k-1} \\ &= (E_i^{\vee})^{r-1} \sum_{k \in J_{1,r}} (-q_i^{-2(k-1)} L_{\bar{\alpha}_i} + q_i^{2(k-1)} L_{-\bar{\alpha}_i}) \\ &= \{r\}_{q_i^2} (E_i^{\vee})^{r-1} (-q_i^{-2(r-1)} L_{\bar{\alpha}_i} + L_{-\bar{\alpha}_i}). \end{aligned}$$

Applying Ω , we have

$$(4.9) \quad E_i^r E_i^{\vee} - E_i^{\vee} E_i^r = \{r\}_{q_i^2} (-q_i^{-2(r-1)} L_{\bar{\alpha}_i} + L_{-\bar{\alpha}_i}) E_i^{r-1}.$$

By (4.8), we have

$$\begin{aligned} (4.10) \quad E_i (E_i^{\vee})^r v_{\bar{\lambda}} &= \{r\}_{q_i^2} (E_i^{\vee})^{r-1} (-q_i^{-2(r-1)} \sqrt{\chi(\alpha_i, \lambda)} + \sqrt{\chi(\alpha_i, -\lambda)}) v_{\bar{\lambda}} \\ &= -\{r\}_{q_i^2} \sqrt{\chi(\alpha_i, -\lambda)} (\chi(\alpha_i, \lambda) q_i^{-2(r-1)} - 1) (E_i^{\vee})^{r-1} v_{\bar{\lambda}}. \end{aligned}$$

5 Rank one case

In this section, we assume that $\ell = 1$. Let V be a U -module satisfying (4.3). Let $r \in \mathbb{N}$ be such that

$$(5.1) \quad (\chi(\alpha_1, \lambda)q_1^{-2(r-1)} - 1)\{r\}_{q_1^2} = 0, (E_1^\vee)^{r-1}v_{\bar{\lambda}} \neq 0 \text{ and } (E_1^\vee)^r v_{\bar{\lambda}} = 0.$$

Then $\dim V = r$. We have

$$(5.2) \quad \begin{aligned} & \zeta^{-1}(f_V) \\ &= \sum_{k \in J_{0,r-1}, m \in J_{0,r-1-k}} \sqrt{\chi}(\tilde{\rho} - m\alpha_1, 2k\alpha_1 - \lambda) \\ & \quad \cdot t_{-\beta, r}(E_1^m \left(\frac{1}{\{m\}_{q_1^2}!} (E_1^\vee)^m \right) (E_1^\vee)^k v_{\bar{\lambda}}) \\ & \quad \cdot \frac{1}{\{m\}_{q_1^2}!} (E_1^\vee)^m L_{\frac{(m+2k)\alpha_1 - \lambda}{1}} E_1^m \\ &= \sqrt{\chi}(\tilde{\rho}, -\lambda) \sum_{k \in J_{0,r-1}, m \in J_{0,r-1-k}} \sqrt{\chi}(\alpha_1, \lambda)^m \frac{q_1^{2(1-m)k}}{(\{m\}_{q_1^2}!)^2} \\ & \quad \cdot (-1)^m \left(\prod_{t \in J_{1,m}} (\chi(\alpha_1, \lambda)q_1^{-2(m+k-t)} - 1) \right) \\ & \quad \cdot \sqrt{\chi}(\alpha_1, -\lambda)^m \frac{\{m+k\}_{q_1^2}!}{\{k\}_{q_1^2}!} (E_1^\vee)^m L_{\frac{(m+2k)\alpha_1 - \lambda}{1}} E_1^m \\ &= \sqrt{\chi}(\tilde{\rho}, -\lambda) \sum_{m \in J_{0,r-1}} (E_1^\vee)^m \left((-1)^m \sum_{k \in J_{0,r-m-1}} \frac{q_1^{2(1-m)k} \{m+k\}_{q_1^2}!}{(\{m\}_{q_1^2}!)^2 \{k\}_{q_1^2}!} \right. \\ & \quad \left. \cdot \left(\prod_{t \in J_{1,m}} (\chi(\alpha_1, \lambda)q_1^{-2(m+k-t)} - 1) \right) L_{\frac{(m+2k)\alpha_1 - \lambda}{1}} \right) E_1^m, \end{aligned}$$

which implies

$$(5.3) \quad \Phi(\zeta^{-1}(f_V)) = \sqrt{\chi}(\tilde{\rho}, -\lambda) \sum_{k \in J_{0,r-1}} q_1^{2k} L_{\frac{2k\alpha_1 - \lambda}{1}}.$$

If $\{r\}_{q_1^2} \neq 0$, then $\chi(\alpha_1, \lambda) = q_1^{2(r-1)}$, which implies

(5.4)

$$\begin{aligned} \zeta^{-1}(f_V) &= \sqrt{\chi}(\tilde{\rho}, -\lambda) L_{\overline{(r-1)\alpha_1 - \lambda}} \\ &\cdot \sum_{m \in J_{0,r-1}} (E_1^\vee)^m \left((-1)^m \sum_{k \in J_{0,r-m-1}} \frac{q_1^{2(1-m)k} \{m+k\}_{q_1^2}!}{(\{m\}_{q_1^2}!)^2 \{k\}_{q_1^2}!} \right. \\ &\cdot \left. \left(\prod_{t \in J_{1,m}} (q_1^{2((r-1)-(m+k)+t)} - 1) \right) L_{\overline{(m+2k-r+1)\alpha_1}} \right) E_1^m, \end{aligned}$$

which implies

$$(5.5) \quad \Phi(\zeta^{-1}(f_V)) = L_{\overline{(r-1)\alpha_1 - \lambda}} \sqrt{\chi}(\tilde{\rho}, -\lambda) \sum_{k \in J_{0,r-1}} q_1^{2k} L_{\overline{(2k-r+1)\alpha_1}}.$$

Theorem 6. Assume that $\ell = 1$ and $q_1^2 \neq 1$. Let $\{\nu_p \in \mathbb{Z}\Pi' \mid p \in P\}$ be a set of representatives of $\{\bar{\nu} \in \overline{\mathbb{Z}\Pi'} \mid \nu \in \mathbb{Z}\Pi', \chi(\alpha_1, \nu) = 1\}$. Assume that for all $x \in \mathbb{N}$, $x\bar{\alpha}_1 \neq \bar{0}$, so $L_{\bar{\alpha}_1}^x \neq 1$.

(1) Assume that $\{k\}_{q_1^2} \neq 0$ for all $k \in \mathbb{N}$. Then

$$(5.6) \quad \mathfrak{z}(U) = \bigoplus_{p \in P, k \in \mathbb{Z}_{\geq 0}} \mathbb{C} \zeta^{-1}(f_{V(k\alpha_1)}) L_{\bar{\nu}_p},$$

as a \mathbb{C} -linear space. In particular, as a \mathbb{C} -algebra, $\mathfrak{z}(U)$ is generated by $L_{\bar{\nu}_p}$ ($p \in P$), and $-(q_1^2 - 1)E_1^\vee E_1 + L_{-\bar{\alpha}_1} + q_1^2 L_{\bar{\alpha}_1}$.

(2) Assume that there exists $r \in \mathbb{N}$ such that $\{r-1\}_{q_1^2} \neq 0$ and $\{r\}_{q_1^2} = 0$. Let $\mathcal{R} := \{\bar{\mu} \in \overline{\mathbb{Z}\Pi'} \mid \mu \in \mathbb{Z}\Pi', \chi(\alpha_1, \mu) \notin \{q_1^s \mid s \in J_{0,r-2}\}\}$. Then we have

$$(5.7) \quad \mathfrak{z}(U) = \left(\bigoplus_{p \in P, k \in J_{0,r-2}} \mathbb{C} \zeta^{-1}(f_{V(k\alpha_1)}) L_{\bar{\nu}_p} \right) \oplus \left(\bigoplus_{\eta \in \mathcal{R}} \mathbb{C} \zeta^{-1}(f_{V(\eta)}) \right),$$

as a \mathbb{C} -linear space, where let $V(\eta) := V(\bar{\mu})$ if $\eta = \bar{\mu}$.

Proof. Let $C := \sum_{m \in J_{0,k}} (E_1^\vee)^m z_m E_1^m \in \mathfrak{z}(U)$ with $z_m \in U^0$. Define the \mathbb{C} -algebra automorphism $g : U^0 \rightarrow U^0$ by $f(L_\lambda) = \chi(\alpha_1, -\lambda) L_\lambda$ for all $\lambda \in \mathbb{Z}\Pi'$. By

(4.8)-(4.9), we have

(5.8)

$$\begin{aligned}
0 &= CE_1 - E_1C \\
&= \sum_{m \in J_{0,k}} ((E_1^\vee)^m (z_m - g(z_m))) E_1^{m+1} \\
&\quad - (E_1^\vee)^{m-1} \{m\}_{q_1^2} (-q_1^{-2(m-1)} L_{\bar{\alpha}_1} + L_{-\bar{\alpha}_1} z_m) E_1^m \\
&= \left(\sum_{m \in J_{0,k-1}} (E_1^\vee)^m (z_m - g(z_m) - \{m+1\}_{q_1^2} (-q_1^{-2m} L_{\bar{\alpha}_1} + L_{-\bar{\alpha}_1} z_{m+1})) E_1^{m+1} \right) \\
&\quad + E_1^{k+1} (z_k - g(z_k)) (E_1^\vee)^k,
\end{aligned}$$

and

(5.9)

$$\begin{aligned}
0 &= E_1^\vee C - CE_1^\vee \\
&= \sum_{m \in J_{0,k}} ((E_1^\vee)^{m+1} (z_m - g(z_m))) E_1^m \\
&\quad - (E_1^\vee)^m \{m\}_{q_1^2} (-q_1^{-2(m-1)} L_{\bar{\alpha}_1} + L_{-\bar{\alpha}_1} z_m) E_1^{m-1} \\
&= \left(\sum_{m \in J_{0,k-1}} (E_1^\vee)^{m+1} (z_m - g(z_m) - \{m+1\}_{q_1^2} (-q_1^{-2m} L_{\bar{\alpha}_1} + L_{-\bar{\alpha}_1} z_{m+1})) E_1^m \right) \\
&\quad + E_1^k (z_k - g(z_k)) (E_1^\vee)^{k+1}.
\end{aligned}$$

Hence we have

$$(5.10) \quad E_1^{k+1} \neq 0 \Rightarrow z_k = g(z_k),$$

and

$$(5.11) \quad \forall m' \in J_{0,k-1}, \quad E_1^{m'} \neq 0 \Rightarrow z_{m'} - g(z_{m'}) = \{m'+1\}_{q_1^2} (-q_1^{-2m'} L_{\bar{\alpha}_1} + L_{-\bar{\alpha}_1}).$$

(1) Let C be as above. By (5.10), we have $z_k \in \oplus_{p \in P} \mathbb{C} L_{\bar{\nu}_p}$. By (5.4), the last term of $\zeta^{-1}(f_{V(k\alpha_1)})$ is $b(E_1^\vee)^k E_1^k$ for some $b \in \mathbb{C}^\times$. Then we can see (5.6).

(2) Let C be as above and assume that $k = r - 1$. Assume that C is not in RHS of (5.7). By (5.8)-(5.9), we may assume that there exists $\lambda \in \mathbb{Z}\Pi'$ such that $z_m \in \oplus_{y \in \mathbb{Z}} \mathbb{C} L_{\bar{\lambda} + 2y\bar{\alpha}_1}$ for all $m \in J_{0,r-1}$. By the same argument as in (1), we may assume $z_{r-1} \neq 0$. For $\mu \in \mathbb{Z}\Pi'$ with $\bar{\mu} \notin \mathcal{R}$, $V(\mu) = M(\mu)$ and, by (5.2), for $\mu \in \mathbb{Z}\Pi'$, the last term of $\zeta^{-1}(f_{M(\mu)})$ is $c(E_1^\vee)^{r-1} L_{\overline{(r-1)\alpha_1 - \mu}} E_1^{r-1}$ for some $c \in \mathbb{C}^\times$. Hence we may assume that $\chi(\lambda, \alpha_1) = 1$ and $z_{r-1} \in \oplus_{x \in \mathbb{Z}_{\geq 0}, y \in J_{1,r-1}} \mathbb{C} L_{\bar{\lambda} + (rx+y)\bar{\alpha}_1}$. However this contradicts (5.11) since $z' - g(z') \in \oplus_{x_1 \in \mathbb{Z}, y_1 \in J_{1,r-1}} \mathbb{C} L_{\bar{\lambda} + (rx_1+y_1)\bar{\alpha}_1}$ for all $z' \in \oplus_{x_2 \in \mathbb{Z}} \mathbb{C} L_{\bar{\lambda} + rx_2\bar{\alpha}_1}$. \square

6 Higher rank case

Assume that $\ell \in \mathbb{N}$. From now on,

$$(6.1) \quad \text{assume that } q_i^2 \neq 1 \text{ for all } i \in J_{1,\ell},$$

and

$$(6.2) \quad \text{assume that there exist } \omega_i \in \mathbb{Z}\Pi' \text{ (} i \in J_{1,\ell} \text{) such that } \sqrt{\chi}(\omega_i, \alpha_i)^{r_i} \neq 1 \text{ for all } r_i \in \mathbb{N} \text{ and } \sqrt{\chi}(\omega_i, \alpha_j) = 1 \text{ for } i \neq j.$$

Then $L_{\bar{\gamma}} \neq 1$ for all $\bar{\gamma} \in \mathbb{Z}\Pi \setminus \{0\}$. Further

$$(6.3) \quad \mathfrak{z}(U) \subset \bigoplus_{\beta \in \mathbb{Z}_{\geq 0}\Pi} \text{Span}_{\mathbb{C}}(U_{-\beta}^- U^0 U_{\beta}^+).$$

Moreover we have

Lemma 7. *Let $z \in \mathfrak{z}(U)$. Assume that $\Phi(z) = 0$. Then $z = 0$.*

Proof. Let $\beta \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$. Let $X_r \in U_{\beta}^+$, and $Y_r \in U_{-\beta}^-$ ($r \in J_{1, \dim U_{\beta}^+}$) be \mathbb{C} -base elements of U_{β}^+ , and $U_{-\beta}^-$ respectively such that $(X_r, \Omega(Y_k)) = \delta_{rk}$. By (2.14), and formulas similar to (3.10), we have

$$(6.4) \quad \Phi(X_r Y_k) \in \delta_{rk} L_{-\beta} + \sum_{\bar{\gamma} \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}} \mathbb{C} L_{-\beta + \bar{\gamma}}.$$

Then we can easily see that the statement holds. \square

Let $i \in J_{1,\ell}$. Let $(\mathbb{Z}\Pi')_i := \{\lambda \in \mathbb{Z}\Pi' \mid \exists t \in \mathbb{Z}, \chi(\alpha_i, \lambda) = q_i^{2t}\}$, and $\overline{(\mathbb{Z}\Pi')_i} := \{\bar{\lambda} \in \mathbb{Z}\Pi'/T \mid \lambda \in (\mathbb{Z}\Pi')_i\}$. If $q_i^m \neq 1$ for all $m \in \mathbb{N}$, define the map $\sigma_i : (\mathbb{Z}\Pi')_i \rightarrow (\mathbb{Z}\Pi')_i$ by letting $\sigma_i(\lambda)$ be such that $\sigma_i(\lambda) \in \lambda + \mathbb{Z}\alpha_i$ and $\chi(\lambda + \sigma_i(\lambda), \alpha_i) = 1$ for all $\lambda \in (\mathbb{Z}\Pi')_i$; we also denote the map $\overline{(\mathbb{Z}\Pi')_i} \rightarrow \overline{(\mathbb{Z}\Pi')_i}$ induced from σ_i by the same symbol. Assume that there exists $r \in \mathbb{N}$ such that $\{r-1\}_{q_i^2} \neq 0$ and $\{r\}_{q_i^2} = 0$. Define the map $\tau_i : (\mathbb{Z}\Pi')_i \rightarrow (\mathbb{Z}\Pi')_i$ as follows. Let $\lambda \in (\mathbb{Z}\Pi')_i$. Let $y \in J_{0,r-1}$ be such that $\chi(\lambda + y\alpha_i, \alpha_i) = 1$. Then we let $\tau_i(\lambda) := \lambda + 2y\alpha_i$. We also denote the map $\overline{(\mathbb{Z}\Pi')_i} \rightarrow \overline{(\mathbb{Z}\Pi')_i}$ induced from τ_i by the same symbol.

Theorem 8. *Let $i \in J_{1,\ell}$. Let $\sum_{\eta \in \overline{(\mathbb{Z}\Pi')_i}} a_{\eta} L_{\eta} \in \Phi(\mathfrak{z}(U))$ with $a_{\eta} \in \mathbb{C}$.*

(1) *Assume that $q_i^m \neq 1$ for all $m \in \mathbb{N}$. Then we have $\Phi(\mathfrak{z}(U)) \subset \bigoplus_{\eta \in \overline{(\mathbb{Z}\Pi')_i}} \mathbb{C} L_{\eta}$. Further we have*

$$(6.5) \quad \forall \eta_1 \in \overline{(\mathbb{Z}\Pi')_i}, \sqrt{\chi_{\bar{\beta}}}(-\sigma_i(\eta_1)) a_{\sigma_i(\eta_1)} = \sqrt{\chi_{\bar{\beta}}}(-\eta_1) a_{\eta_1}.$$

(2) *Assume that there exists $r \in \mathbb{N}$ such that $\{r-1\}_{q_i^2} \neq 0$ and $\{r\}_{q_i^2} = 0$. Then for $\lambda \in \mathbb{Z}\Pi' \setminus (\mathbb{Z}\Pi')_i$, we have*

$$(6.6) \quad \forall m \in \{0\} \cup J_{2,r-1}, \sum_{k \in \mathbb{Z}} q_i^{2mk} a_{\bar{\lambda} + 2k\bar{\alpha}_i} = 0.$$

Further for $\mu \in \mathbb{Z}\Pi'$ with $\chi(\mu, \alpha_i) = q_i^{2d}$ for some $d \in J_{1,r-1}$, we have

$$(6.7) \quad \sum_{k \in \mathbb{Z}} (-q_i^{-2d})^k a_{r_i^k(\bar{\mu})} = 0.$$

Proof. Let $z = \sum_{\beta \in \mathbb{Z}_{\geq 0}\Pi} z_\beta \in \mathfrak{z}(U)$ with $z_\beta \in \text{Span}_{\mathbb{C}}(U_{-\beta}^- U^0 U_\beta^+)$. Let $i \in J_{1,\ell}$. We see that $E_i(z_0 + z_{\alpha_i}) - (z_0 + z_{\alpha_i})E_i = E_i^\vee(z_0 + z_{\alpha_i}) - (z_0 + z_{\alpha_i})E_i^\vee = 0$. Then this theorem follows from Theorem 6, and (5.3), (5.5). \square

7 $U_q(\mathfrak{gl}(2|1))$

Assume that $N = 4$ and $\ell = 2$. Let $p_1 := p_2 := 0$, and let $p_3 := 1$. Define the symmetric bi-additive map $((,)) : \mathbb{Z}\Pi' \times \mathbb{Z}\Pi' \rightarrow \mathbb{Z}$ by $((\epsilon_i, \epsilon_j)) := (1 - \delta_{i,4})(1 - \delta_{j,4})(-1)^{\delta_{ij}p_i}$. Assume that $\sqrt{\chi}(\epsilon_4, \epsilon_4) = \sqrt{-1}$, $\sqrt{\chi}(\epsilon_4, \epsilon_r) = \sqrt{\chi}(\epsilon_r, \epsilon_4) = 1$ hold for all $r \in J_{1,3}$, and $\sqrt{\chi}(\epsilon_i, \epsilon_j) = q^{\frac{((\epsilon_i, \epsilon_j))}{2}}$ for all $i, j \in J_{1,3}$. Then there exists an additive group isomorphism $\mathbb{Z}^3 \times (\mathbb{Z}/4\mathbb{Z}) \rightarrow \overline{\mathbb{Z}\Pi'}$, $(m_1, m_2, m_3, m_4 + 4\mathbb{Z}) \mapsto \sum_{t \in J_{1,4}} m_t \bar{\epsilon}_t$, where $m_t \in \mathbb{Z}$. Assume that $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2 - \epsilon_3 + \epsilon_4$. We also denote this U by $U_q(\mathfrak{gl}(2|1))$. Let $E_{12}^\vee := E_1^\vee E_2^\vee - qE_2^\vee E_1^\vee$. It is well-known that

$$(7.1) \quad U^- = \bigoplus_{n_1 \in \mathbb{Z}_{\geq 0}, n_2, n_{12} \in J_{0,1}} \mathbb{C}(E_2^\vee)^{n_2} (E_{12}^\vee)^{n_{12}} (E_1^\vee)^{n_1},$$

as a \mathbb{C} -linear space.

Let $\lambda = \sum_{y \in J_{1,4}} x_y \epsilon_y \in \mathbb{Z}\Pi'$ with $x_y \in \mathbb{Z}$. Let $k := \frac{x_1 - x_2}{2}$. Assume that $k \in \mathbb{Z}_{\geq 0}$. We have $\chi(\lambda, \alpha_1) = q_1^{2k}$. By (4.10), we have a left U -module $K(\lambda)$ satisfying (4.3) and satisfying:

$$(7.2) \quad K(\lambda) = \bigoplus_{n_1 \in J_{0,k+1}, n_2, n_{12} \in J_{0,1}} \mathbb{C}(E_2^\vee)^{n_2} (E_{12}^\vee)^{n_{12}} (E_1^\vee)^{n_1} v_{\bar{\lambda}},$$

as a \mathbb{C} -linear space. By (4.4), we have

$$\begin{aligned} & \Phi(\zeta^{-1}(f_{K(\lambda)})) \\ &= \sqrt{\chi}(\bar{\rho}, -\lambda) \left(\left(\sum_{m_1 \in J_{0,k}} (q_1^{2m_1} L_{-\bar{\lambda}+2m_1\bar{\alpha}_1} + q_1^{2(m_1+1)} L_{-\bar{\lambda}+2(m_1+1)\bar{\alpha}_1+2\bar{\alpha}_2}) \right) \right. \\ & \quad \left. - L_{-\bar{\lambda}+2\bar{\alpha}_2} - \left(\sum_{m_2 \in J_{1,k-1}} 2q_1^{2m_2} L_{-\bar{\lambda}+2m_2\bar{\alpha}_1+2\bar{\alpha}_1} \right) - L_{-\bar{\lambda}+2(k+1)\bar{\alpha}_1+2\bar{\alpha}_1} \right). \end{aligned}$$

Note that for $m_1, m_2 \in \mathbb{Z}$, we have $\frac{1}{2}((\alpha_1, -\lambda + 2m_1\alpha_1 + 2m_2\alpha_2)) = -k + 2m_1 - m_2$.

For $k \in \mathbb{Z}$, let $[\mathbb{Z}\Pi']_k := \{\mu \in \mathbb{Z}\Pi' \mid \frac{1}{2}((\alpha_1, \mu)) = k\}$, and let $[\overline{\mathbb{Z}\Pi'}]_k := \{\bar{\nu} \in \overline{\mathbb{Z}\Pi'} \mid \nu \in [\mathbb{Z}\Pi']_k\}$. By the above argument, we see that for $\mu \in [\mathbb{Z}\Pi']_k$ with $k \in \mathbb{N}$,

we have

$$(7.3) \quad \Phi(\zeta^{-1}(f_{K(-\mu+2(k+1)\alpha_1+2\alpha_2)})) \in \mathbb{C}^\times L_{\bar{\mu}} \oplus \bigoplus_{m \in J_{-k, k-1}} \bigoplus_{\eta \in \overline{[\mathbb{Z}\Pi']_m}} \mathbb{C}L_\eta.$$

Let $z \in \mathfrak{z}(U)$. By Theorem 8 (1),

$$(7.4) \quad \Phi(z) = \sum_{\eta \in \overline{[\mathbb{Z}\Pi']_0}} a_\eta L_\eta + \sum_{k \in \mathbb{N}} \sum_{\omega \in \overline{[\mathbb{Z}\Pi']_k}} a_\omega (L_\omega + q_1^{-2k} L_{\omega-2k\bar{\alpha}_1}),$$

where $a_\eta, a_\omega \in \mathbb{C}$. Assume that $a_\omega = 0$ for all $\omega \in \cup_{k \in \mathbb{N}} \overline{[\mathbb{Z}\Pi']_k}$. By Theorem 8 (2), we see that for $\nu \in \overline{[\mathbb{Z}\Pi']_k}$, if $a_\nu \neq 0$, then $\nu = x(\epsilon_1 + \epsilon_2 - \epsilon_3) + 2y\epsilon_4$ for some $x, y \in \mathbb{Z}$. By Lemma 7, we have

Theorem 9. *Let U be as above. Then we have*

$$\begin{aligned} & \mathfrak{z}(U_q(\mathfrak{gl}(2|1))) \\ &= \bigoplus_{y_1 \in \mathbb{Z}, y_2 \in J_{0,1}} \mathbb{C}L_{y_1(\epsilon_1 + \epsilon_2 - \epsilon_3) + 2y_2\epsilon_4} \\ & \quad \oplus \bigoplus_{x_1 \in \mathbb{Z}_{\geq 0}, x_2, x_3 \in \mathbb{Z}, x_4 \in J_{0,3}} \mathbb{C}\zeta^{-1}(f_{K((2x_1+x_2+2)\epsilon_1+x_2\epsilon_2+x_3\epsilon_3+x_4\epsilon_4)}). \end{aligned}$$

8 Lusztig isomorphisms

In this section we may not assume that $\sqrt{\chi}$ is symmetric. For $n \in \mathbb{N}$ and $a, b \in \mathbb{C}$, let $\{n; a, b\} := a^{n-1}b - 1$, and $\{n; a, b\}! = \prod_{J_{1,m}} \{m; a, b\}$. Let $D = D(\sqrt{\chi}) = \text{Span}_{\mathbb{C}}(U^+D^0U^-) = \text{Span}_{\mathbb{C}}(U^-D^0U^+)$ be as above. For $\alpha \in \mathbb{Z}\Pi$, let $D_\alpha := \bigoplus_{\gamma \in \mathbb{Z}\Pi} \text{Span}_{\mathbb{C}}(U_\gamma^+D^0U_{\alpha-\gamma}^-)$.

For $\alpha \in \mathbb{Z}\Pi$ and $X \in D_\alpha$, define four \mathbb{C} -linear map $\text{ad}_+^L X, \text{ad}_-^L X, \text{ad}_+^R X, \text{ad}_-^R X : D \rightarrow D$ by letting

$$(8.1) \quad \begin{aligned} \text{ad}_\pm^L X(Y) &:= XY - \chi(\pm\alpha, \beta)YX, \\ \text{ad}_\pm^R X(Y) &:= XY - \chi(\beta, \pm\alpha)YX \end{aligned}$$

for all $\beta \in \mathbb{Z}\Pi$ and all $Y \in D_\beta$.

Let $q_{ij} := \chi(\alpha_i, \alpha_j)$ for all $i, j \in J_{1,\ell}$. Then $q_{ii} = q_i^2$. We have assumed that $q_{ii} \neq 1$ for all $i \in J_{1,\ell}$.

Lemma 10. (see [5]) *For $i, j \in J_{1,\ell}$ with $i \neq j$ and $m, n \in \mathbb{Z}_{\geq 0}$, in D , we have*

$$(8.2) \quad \begin{aligned} & (\text{ad}_+^L E_i)^m (E_j) (\text{ad}_+^R E_i^\vee)^n (E_j^\vee) - (\text{ad}_+^R E_i^\vee)^n (E_j^\vee) (\text{ad}_+^L E_i)^m (E_j) \\ &= \begin{cases} (-1)^{m+1} \frac{\{m\}_{q_{ii}}!}{\{m-n\}_{q_{ii}}!} \{m; q_{ii}, q_{ij}q_{ji}\}! E_i^{m-n} L_{n\alpha_i + \alpha_j} & \text{if } m > n, \\ (-1)^n \{n\}_{q_{ii}}! \{n; q_{ii}, q_{ij}q_{ji}\}! (-L_{n\alpha_i + \alpha_j} + L_{n\alpha_i + \alpha_j}^\vee) & \text{if } m = n, \\ (-1)^n \frac{\{n\}_{q_{ii}}!}{\{n-m\}_{q_{ii}}!} \{n; q_{ii}, q_{ij}q_{ji}\}! (E_i^\vee)^{n-m} L_{m\alpha_i + \alpha_j}^\vee & \text{if } m < n. \end{cases} \end{aligned}$$

Let $\sqrt{\chi}$ be as above. Let $i \in I$. We say that $\sqrt{\chi}$ is *i-finite*, if for any $j \in I \setminus \{i\}$, there exists $n \in \mathbb{N}$ such that $\{n\}_{q_{ii}}! \{n; q_{ii}, q_{ij}q_{ji}\}! = 0$.

Assume that $\sqrt{\chi}$ is *i-finite*. Define $c_{ij} = c(\alpha_i, \alpha_j) \in \mathbb{Z}$ by

$$(8.3) \quad c_{ij} := \begin{cases} 2 & \text{if } j = i, \\ -\min\{n \in \mathbb{Z}_{\geq 0} \mid \{n\}_{q_{ii}}! \{n; q_{ii}, q_{ij}q_{ji}\}! \neq 0\}. & \text{if } j \neq i. \end{cases}$$

Let

$$(8.4) \quad \alpha_j^{i\triangleright} := \alpha_j - c_{ij}\alpha_i.$$

Then $\alpha_j = \alpha_j^{i\triangleright} - c_{ij}\alpha_i^{i\triangleright}$. Further we have $c(\alpha_j^{i\triangleright}, \alpha_k^{i\triangleright}) = c(\alpha_j, \alpha_k)$, and $\alpha_j^{i\triangleright i\triangleright} = \alpha_j$. Let $\Pi^{i\triangleright} := \{\alpha_j^{i\triangleright} \mid j \in J_{1,\ell}\}$, and $(\Pi')^{i\triangleright} := \Pi'$, $(\Pi'')^{i\triangleright} := \Pi''$. Let $D^{i\triangleright}$ the Hopf algebra defined in the same way as for D with $\Pi^{i\triangleright}$, $(\Pi')^{i\triangleright}$, $(\Pi'')^{i\triangleright}$ in place of Π , Π' , Π'' . Define the \mathbb{Z} -module map $s_i : \mathbb{Z}(\Pi'')^{i\triangleright} \rightarrow \mathbb{Z}\Pi''$ to be the *identity map*. Then $s_i(\alpha_j^{i\triangleright}) = \alpha_j - c_{ij}\alpha_i$.

By [5], we have:

Theorem 11. (see [5]) Assume that $\sqrt{\chi}$ is *i-finite*. Let $q_{i',j'} := \chi(\alpha_{i'}, \alpha_{j'})$ and $q_{i',j'}^{i\triangleright} := \chi(\alpha_{i'}^{i\triangleright}, \alpha_{j'}^{i\triangleright})$ for all $i', j' \in J_{1,\ell}$. Then there exists a \mathbb{C} -algebra isomorphism

$$(8.5) \quad T_i : D^{i\triangleright} \rightarrow D$$

such that

$$(8.6) \quad \begin{aligned} T_i(L_\alpha) &= L_{s_i(\alpha)}, & T_i(L_\alpha^\vee) &= L_{s_i(\alpha)}^\vee, \\ T_i(E_i) &= E_i^\vee L_{-\alpha_i}^\vee, & T_i(E_i^\vee) &= L_{-\alpha_i} E_i, \\ T_i(E_j) &= (\text{ad}_+^L E_i)^{-c_{ij}}(E_j), \\ T_i(E_j^\vee) &= \frac{(-1)^{-c_{ij}}}{\{-c_{ij}\}_{q_{ii}}! \{-c_{ij}; q_{ii}, q_{ij}q_{ji}\}!} (\text{ad}_+^R E_i^\vee)^{-c_{ij}}(E_j^\vee), \end{aligned}$$

and

$$(8.7) \quad \begin{aligned} T_i^{-1}(L_\alpha) &= L_{s_i(\alpha)}, & T_i^{-1}(L_\alpha^\vee) &= L_{s_i(\alpha)}^\vee, \\ T_i^{-1}(E_i) &= L_{-\alpha_i} E_i^\vee, & T_i^{-1}(E_i^\vee) &= E_i L_{-\alpha_i}^\vee, \\ T_i^{-1}(E_j) &= \frac{(q_{ii}^{i\triangleright})^{(-c_{ij})(-c_{ij}-1)} (q_{ij}^{i\triangleright} q_{ji}^{i\triangleright})^{(-c_{ij})}}{\{-c_{ij}\}_{q_{ii}^{i\triangleright}}! \{-c_{ij}; q_{ii}^{i\triangleright}, q_{ij}^{i\triangleright} q_{ji}^{i\triangleright}\}!} (\text{ad}_-^R E_i)^{-c_{ij}}(E_j), \\ T_i^{-1}(E_j^\vee) &= (-1)^{c_{ij}} (\text{ad}_-^L E_i^\vee)^{-c_{ij}}(E_j^\vee), \end{aligned}$$

for all $\alpha \in \mathbb{Z}\Pi''$ and all $j \in J_{1,\ell} \setminus \{i\}$.

9 Lusztig isomorphisms and $\Phi(\mathfrak{z}(U^{i\triangleright}))$

From now on, we again assume that $\sqrt{\chi}$ is symmetric. Assume that $\sqrt{\chi}$ is *i-finite*. Let $U^{i\triangleright}$ the Hopf algebra defined in the same way as for U with $\Pi^{i\triangleright}$, $(\Pi')^{i\triangleright}$, $(\Pi'')^{i\triangleright}$ in place of Π , Π' , Π'' . Denote the \mathbb{C} -algebra isomorphism $U^{i\triangleright} \rightarrow U$ induced from $T_i : D^{i\triangleright} \rightarrow D$ by the same symbol. Denote the \mathbb{Z} -module isomorphism $\overline{\mathbb{Z}(\Pi')^{i\triangleright}} \rightarrow \overline{\mathbb{Z}\Pi'}$ induced from $s_i : \mathbb{Z}\Pi'' \rightarrow \mathbb{Z}(\Pi'')^{i\triangleright}$ by the same symbol; in this stage, s_i is the identity map.

Theorem 12. Assume that $\sqrt{\chi}$ is i -finite. Let $Z \in \mathfrak{z}(U^{i\flat})$. Assume that $\Phi(Z) = \sum_{\theta \in \overline{\mathbb{Z}(\Pi')^{i\flat}}} b_\theta L_\theta$ with $b_\theta \in \mathbb{C}$. Assume that $\Phi(T_i(Z)) = \sum_{\eta \in \overline{\mathbb{Z}\Pi'}} a_\eta L_\eta$ with $a_\eta \in \mathbb{C}$.

$$(9.1) \quad r := \begin{cases} \min\{r' \in \mathbb{N} | q_i^{2r'} = 1\} & \text{if there exists } r'' \in \mathbb{N} \text{ such that } q_i^{2r''} = 1, \\ 0 & \text{otherwise,} \end{cases}$$

(cf. (6.1)). Then we have

$$(9.2) \quad a_{\bar{\lambda}} = \chi(\alpha_i, \lambda)^{1-r} b_{\overline{s_i(\lambda)}},$$

for all $\lambda \in \mathbb{Z}\Pi'$.

Proof. Assume $\ell = 1$. Let $\lambda \in (\mathbb{Z}\Pi')^{1\flat}$ and $r \in \mathbb{N}$. Assume that $\prod_{0 \in J_{0,r-2}} (\chi(\alpha_1^{1\flat}, \lambda) q_1^{-2t} - 1) \neq 0$. Assume that $\{r-1\}_{q_1^2!} \neq 0$ and $\{r\}_{q_1^2!} = 0$. Let

$$(9.3) \quad Y^{1\flat} := \sqrt{\chi}(\alpha_1^{1\flat}, -\lambda) (-1)^{r-1} \left(\prod_{t \in J_{0,r-2}} (\chi(\alpha_1^{1\flat}, \lambda) q_1^{-2t} - 1) \right) (E_1^\vee)^{r-1} L_{\frac{\quad}{(r-1)\alpha_1^{1\flat} - \lambda}} E_1^{r-1},$$

and

$$(9.4) \quad \begin{aligned} Y &:= \sqrt{\chi}(\alpha_1, -(s_1(\lambda) + 2(r-1)\alpha_1)) (-1)^{r-1} \\ &\quad \cdot \left(\prod_{t \in J_{0,r-2}} (\chi(\alpha_1, s_1(\lambda) + 2(r-1)\alpha_1) q_1^{-2t} - 1) \right) \\ &\quad \cdot (E_1^\vee)^{r-1} L_{\frac{\quad}{(r-1)\alpha_1 - (s_1(\lambda) + 2(r-1)\alpha_1)}} E_1^{r-1} \\ &= \sqrt{\chi}(\alpha_1, s_1(\lambda))^{-1} q_1^2 (-1)^{r-1} \\ &\quad \cdot \left(\prod_{t \in J_{0,r-2}} (\chi(\alpha_1, s_1(\lambda)) q_1^{4(r-1)-2t} - 1) \right) \\ &\quad \cdot (E_1^\vee)^{r-1} L_{\frac{\quad}{(r-1)\alpha_1 - (s_1(\lambda) + 2(r-1)\alpha_1)}} E_1^{r-1} \\ &= \sqrt{\chi}(\alpha_1, s_1(\lambda))^{-1} q_1^2 (-1)^{r-1} \\ &\quad \cdot q_1^{4(r-1)^2 - (r-2)(r-1)} (-1)^{r-1} \chi(\alpha_1, s_1(\lambda))^{r-1} \\ &\quad \cdot \left(\prod_{t \in J_{0,r-2}} (\chi(\alpha_1, s_1(\lambda)) q_1^{2(t+2)} - 1) \right) \\ &\quad \cdot (E_1^\vee)^{r-1} L_{\frac{\quad}{(r-1)\alpha_1 - (s_1(\lambda) + 2(r-1)\alpha_1)}} E_1^{r-1} \\ &= (-1)^{r-1} q_1^{4(r-1)^2 - r(r-3)} \chi(\alpha_1, s_1(\lambda))^{r-2} \\ &\quad \cdot \sqrt{\chi}(\alpha_1, s_1(\lambda)) (-1)^{r-1} \left(\prod_{t \in J_{0,r-2}} (\chi(\alpha_1, s_1(\lambda)) q_1^{-2t} - 1) \right) \\ &\quad \cdot (E_1^\vee)^{r-1} L_{\frac{\quad}{(r-1)\alpha_1 - (s_1(\lambda) + 2(r-1)\alpha_1)}} E_1^{r-1}. \end{aligned}$$

Then we have

$$\begin{aligned}
T_1(Y) &= \\
T_1(\sqrt{\chi}(\alpha_1^{1\flat}, -\lambda)(-1)^{r-1}(\prod_{t \in J_{0,r-2}} (\chi(\alpha_1^{1\flat}, \lambda)q_1^{-2t} - 1))(E_1^\vee)^{r-1}L_{\frac{(r-1)\alpha_1^{1\flat}-\lambda}{(r-1)\alpha_1^{1\flat}-\lambda}}E_1^{r-1}) \\
&= \sqrt{\chi}(-\alpha_1, -s_1(\lambda))(-1)^{r-1}(\prod_{t \in J_{0,r-2}} (\chi(-\alpha_1, s_1(\lambda))q_1^{-2t} - 1)) \\
&\quad \cdot (L_{-\alpha_1}E_1)^{r-1}L_{\frac{-(r-1)\alpha_1-s_1(\lambda)}{-(r-1)\alpha_1-s_1(\lambda)}}(E_1^\vee L_{\alpha_1})^{r-1} \\
&\equiv \sqrt{\chi}(\alpha_1, s_1(\lambda))(-1)^{r-1}(\prod_{t \in J_{0,r-2}} (\chi(-\alpha_1, s_1(\lambda))q_1^{-2t} - 1)) \\
&\quad \cdot \chi(\alpha_1, (r-1)\alpha_1 + s_1(\lambda))^{2(r-1)} \\
&\quad \cdot (E_1^\vee)^{r-1}L_{\frac{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}}E_1^{r-1} \pmod{\oplus_{k \in J_{0,r-2}} (E_1^\vee)^k U^0 E_1^k} \\
&= (-1)^{r-1}q_1^{r(r-3)}\chi(\alpha_1, s_1(\lambda))^r Y \\
&= \chi(\alpha_1, s_1(\lambda))^r Y.
\end{aligned}$$

Assume that $\{r-1\}_{q_1^2}! \neq 0$ and $\chi(\alpha_1^{1\flat}, \lambda) = q_1^{2(r-1)}$. Note that $s_1(\lambda) = \lambda$, since in this stage, s_1 is the identity map. Let

$$(9.5) \quad X^{1\flat} := \sqrt{\chi}(\alpha_1^{1\flat}, -\lambda)(-1)^{r-1}(\prod_{t \in J_{1,r-1}} (q_1^{2t} - 1))(E_1^\vee)^{r-1}L_{\frac{(r-1)\alpha_1^{1\flat}-\lambda}{(r-1)\alpha_1^{1\flat}-\lambda}}E_1^{r-1},$$

and

$$\begin{aligned}
(9.6) \quad X &:= \sqrt{\chi}(\alpha_1, -(s_1(\lambda) + 2(r-1)\alpha_1))(-1)^{r-1}(\prod_{t \in J_{1,r-1}} (q_1^{2t} - 1)) \\
&\quad \cdot (E_1^\vee)^{r-1}L_{\frac{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}{(r-1)\alpha_1-(s_1(\lambda)+2(r-1)\alpha_1)}}E_1^{r-1} \\
&= \sqrt{\chi}(\alpha_1^{1\flat}, -\lambda)(-1)^{r-1}(\prod_{t \in J_{1,r-1}} (q_1^{2t} - 1))(E_1^\vee)^{r-1}L_{\frac{(r-1)\alpha_1-\lambda}{(r-1)\alpha_1-\lambda}}E_1^{r-1}.
\end{aligned}$$

Then by the same argument as above, we have

$$(9.7) \quad T_1(X^{1\flat}) = X \pmod{\oplus_{k \in J_{0,r-2}} (E_1^\vee)^k U^0 E_1^k}.$$

Then the statement of this theorem follows by an argument similar to that for Proof of Theorem 8. \square

10 $\mathfrak{osp}(3|2)$

Assume that $N = 3$ and $\ell = 2$. Let $p_1 := 0$, and $p_2 := 1$. Define the symmetric bi-additive map $((,)) : \mathbb{Z}\Pi' \times \mathbb{Z}\Pi' \rightarrow \mathbb{Z}$ by $((\epsilon_i, \epsilon_j)) := (1 - \delta_{i,3})(1 - \delta_{j,3})(-1)^{\delta_{ij}p_i}$. Assume that $\sqrt{\chi}(\epsilon_3, \epsilon_3) = \sqrt{-1}$, $\sqrt{\chi}(\epsilon_3, \epsilon_r) = \sqrt{\chi}(\epsilon_r, \epsilon_3) = 1$ hold for all $r \in J_{1,2}$, and $\sqrt{\chi}(\epsilon_i, \epsilon_j) = q^{\frac{((\epsilon_i, \epsilon_j))}{2}}$ for all $i, j \in J_{1,3}$. Then there exists an additive group isomorphism $\mathbb{Z}^2 \times (\mathbb{Z}/4\mathbb{Z}) \rightarrow \overline{\mathbb{Z}\Pi'}$, $(m_1, m_2, m_3 + 4\mathbb{Z}) \mapsto \sum_{t \in J_{1,3}} m_t \bar{\epsilon}_t$, where

$m_t \in \mathbb{Z}$. Assume that $\alpha_1 = \epsilon_1 - \epsilon_2 + \epsilon_3$ and $\alpha_2 = \epsilon_2$. Assume that $\tilde{\rho} = -\epsilon_1 + \epsilon_2 + \epsilon_3$. We also denote this U by $U_q(\mathfrak{osp}(3|2))$.

We have $\alpha_1^{1\flat} = -\alpha_1 = -\epsilon_1 + \epsilon_2 - \epsilon_3$, and $\alpha_2^{1\flat} = \alpha_1 + \alpha_2 = \epsilon_1 + \epsilon_3$. Let $\tilde{\rho}^{1\flat} := \epsilon_1 - \epsilon_2 - \epsilon_3$. Then $\chi(\tilde{\rho}^{1\flat}, \alpha_i^{1\flat}) = \chi(\alpha_i^{1\flat}, \alpha_i^{1\flat})$.

Let $z \in \mathfrak{z}(U)$. Assume that $\Phi(z) = \sum_{m_1, m_2 \in \mathbb{Z}, m_3 \in J_{0,3}} x_{m_1, m_2, m_3} L_{m_1 \bar{\epsilon}_1 + m_2 \bar{\epsilon}_2 + m_3 \bar{\epsilon}_3}$ with $x_{m_1, m_2, m_3} \in \mathbb{C}$. By Theorem 8 (1), we have $q^{\frac{m_2}{2}} x_{m_1, -m_2, m_3} = q^{-\frac{m_2}{2}} x_{m_1, m_2, m_3}$. By Theorem 8 (2), we conclude that if $m_1 + m_2 \neq 0$ or $m_3 \in \{1, 3\}$, then $\sum_{k \in \mathbb{Z}} x_{m_1+2k, m_2-2k, m_3+2k} = 0$. By Theorem 12, we have $m_3 - m_1 \in 2\mathbb{Z}$, and we have $x_{-m_1, m_2, -m_1+2m_3} = q^{m_1} x_{m_1, m_2, m_1+2m_3}$. Let $n_1 \in \mathbb{N} + 1$, $n_2 \in \mathbb{Z}_{\geq 0}$ and $n_3 \in J_{0,1}$. Let $\lambda := n_1 \epsilon_1 + n_2 \epsilon_2 + (n_1 + 2n_3) \epsilon_3$. Then $\chi(\alpha_1, \lambda) = q^{n_1+n_2} (-1)^{n_1} \neq 1$, $\chi(\alpha_2, \lambda) = q^{-n_2} = \chi(\alpha_2, \alpha_2)^{n_2}$, and $\chi(\alpha_2^{1\flat}, \lambda - \alpha_1) = q^{n_1-1} (-1)^{n_1-1} = \chi(\alpha_2^{1\flat}, \alpha_2^{1\flat})^{n_1-1}$. Recall $T_1 : U^{1\flat} \rightarrow U$. Recall $v_{\bar{\lambda}} \in V(\lambda)$. Then $E_1^{\vee} v_{\bar{\lambda}} \neq 0$, and $U_{\mu}^{+} v_{\bar{\lambda}} = T_1((U^{1\flat})_{s_i(\mu) + \alpha_i^{1\flat}}^{+}) E_1^{\vee} v_{\bar{\lambda}}$ hold for all $\mu \in \mathbb{Z}\Pi$ (in this stage, s_i means the identity map). Then in $V(\lambda)$, we have $(E_2^{\vee})^k v_{\bar{\lambda}} \neq 0$ ($k \in J_{0, n_2}$), $(E_2^{\vee})^{n_2+1} v_{\bar{\lambda}} = 0$, and $T_1(E_2^{\vee})^r E_1^{\vee} v_{\bar{\lambda}} \neq 0$ ($r \in J_{0, n_1-1}$), $T_1(E_2^{\vee})^{n_1} E_1^{\vee} v_{\bar{\lambda}} = 0$. Let $\alpha \in \mathbb{Z}_{\geq 0}\Pi$ be such that $U_{-\alpha}^{-} v_{\lambda} \neq \{0\}$ holds in $V(\lambda)$. Then $\alpha = x_1 \alpha_1 + x_2 \alpha_2 = \alpha_1 + y_1 \alpha_1^{1\flat} + y_2 \alpha_2^{1\flat} = (1 - y_1 + y_2) \alpha_1 + y_2 \alpha_2$ for some $x_1, x_2, y_1, y_2 \in \mathbb{Z}_{\geq 0}$. Then $x_2 = y_2 = x_1 + y_1 - 1$. Then $x_2 \geq x_1 - 1 + \delta_{x_1, 0}$. Note $\alpha = x_1 \epsilon_1 + (x_2 - x_1) \epsilon_2 + x_1 \epsilon_3$. By (4.4), we conclude

$$\begin{aligned} & \Phi(\zeta^{-1}(f_{V(\lambda)})) \\ & \in \sum_{k_1 \in J_{0, n_1-2}} (\mathbb{C}^{\times} L_{(n_1-2-2k_1)\bar{\epsilon}_1 + (n_2+2)\bar{\epsilon}_2 + (n_1+2n_3)\bar{\epsilon}_3} \\ & \quad + \mathbb{C}^{\times} L_{(n_1+2-2k_1)\bar{\epsilon}_1 + (-n_2-2)\bar{\epsilon}_2 + (n_1+2n_3)\bar{\epsilon}_3}) \\ & \quad \sum_{k_2 \in J_{0, n_2}, k_3 \in J_{0, n_1}} \mathbb{C}^{\times} L_{(n_1-2k_3)\bar{\epsilon}_1 + (n_2-2k_2)\bar{\epsilon}_2 + (n_1+2n_3-2k_3)\bar{\epsilon}_3}. \end{aligned}$$

Finally we have

Theorem 13. *Let U be as above. Then we have*

$$\begin{aligned} & \mathfrak{z}(U_q(\mathfrak{osp}(3|2))) \\ & = \bigoplus_{y \in J_{0,1}} \mathbb{C} L_{2y\bar{\epsilon}_3} \oplus \bigoplus_{n_1 \in \mathbb{N}+1, n_2 \in \mathbb{Z}_{\geq 0}, n_3 \in J_{0,1}} \mathbb{C} \zeta^{-1}(f_{V(n_1 \bar{\epsilon}_1 + n_2 \bar{\epsilon}_2 + (n_1+2n_3)\bar{\epsilon}_3)}). \end{aligned}$$

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