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HOPF ALGEBRAS AND POLYNOMIAL IDENTITIES

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ABSTRACT. This is a survey of results obtained jointly with E. Aljadeff and published in [2]. We explain how to set up a theory of polynomial identities for comodule algebras over a Hopf algebra, and concentrate on the universal comodule algebra constructed from the identities satisfied by a given comodule algebra. All concepts are illustrated with various examples.

KEY WORDS: Polynomial identity, Hopf algebra, comodule, localization

MATHEMATICS SUBJECT CLASSIFICATION (2010): 16R50, 16T05, 16T15, 16T20, 16S40, 16S85

INTRODUCTION

As has been stressed many times (see, e.g., [19]), Hopf Galois extensions can be viewed as non-commutative analogues of principal fiber bundles (also known as G-torsors), where the role of the structural group is played by a Hopf algebra. Such extensions abound in the world of quantum groups and of non-commutative geometry. The problem of constructing systematically all Hopf Galois extensions of a given algebra for a given Hopf algebra and of classifying them up to isomorphism has been addressed in a number of papers, such as [4, 7, 9, 12, 13, 14, 15, 18] to quote but a few.

A new approach to the classification problem of Hopf Galois extensions was recently advanced by Eli Aljadeff and the present author in [2]; this approach uses classical techniques from non-commutative algebra such as *polynomial identities* (such techniques had previously been used in [1] for group-graded algebras). In [2] we developed a theory of identities for any comodule algebra over a given Hopf algebra H, hence for any Hopf Galois extension. As a result, out of the identities for an H-comodule algebra A, we obtained a *universal H-comodule algebra* $\mathcal{U}_H(A)$. It turns out that if A is a cleft H-Galois object (i.e., a comodule algebra obtained from H by twisting its product with the help of a two-cocycle) with trivial center, then a suitable central localization of $\mathcal{U}_H(A)$ is an H-Galois extension of its center. We thus obtain a "non-commutative principal fiber bundle" whose base space is the spectrum of some localization of the center of $\mathcal{U}_H(A)$.

This survey is organized as follows. After a preliminary section on comodule algebras, we define the concept of an H-identity for such algebras in § 2. We illustrate this concept with a few examples and we attach a universal H-comodule algebra $\mathcal{U}_H(A)$ to each H-comodule algebra A.

In § 3 turning to the special case where $A = {}^{\alpha}H$ is a twisted comodule algebra, we exhibit a universal comodule algebra map that allows us to detect the H-identities for A.

In § 4 we construct a commutative domain $\mathcal{B}_H^\alpha$ and we state that under some natural extra condition, $\mathcal{B}_H^\alpha$ is the center of a suitable central localization of $\mathcal{U}_H(A)$; moreover after localization, $\mathcal{U}_H(A)$ becomes a free module over its center.

Lastly in § 5, we illustrate all previous constructions with the help of the four-dimensional Sweedler algebra, thus giving complete answers in this simple, but non-trivial example. We end the paper with an open question on Taft algebras.

The material of the present text is mainly taken from [2], for which it provides an easy access. The reader is advised to complement it with [10, 11].

1. Hopf algebras and coactions

1.1. **Standing assumption.** We fix a field k over which all our constructions are defined. In particular, all linear maps are supposed to be k-linear and unadorned tensor products mean tensor products over k. Throughout the survey we assume that the ground field k is *infinite.*

By algebra we always mean an associative unital k-algebra. We suppose the reader familiar with the language of Hopf algebra, as expounded for instance in [20]. As is customary, we denote the coproduct of a Hopf algebra by Δ, its counit by ε, and its antipode by S. We also make use of a Heyneman-Sweedler-type notation for the image

$$\Delta(x) = x_1 \otimes x_2$$

of an element x of a Hopf algebra H under the coproduct, and we write

$$\Delta^{(2)}(x) = x_1 \otimes x_2 \otimes x_3$$

for the iterated coproduct $\Delta^{(2)} = (\Delta \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \Delta) \circ \Delta$, and so on.

1.2. **Comodule algebras.** Let H be a Hopf algebra. Recall that an H-*comodule algebra* is an algebra A equipped with a right H-comodule structure whose (coassociative, counital) *coaction*

$$\delta : A \to A \otimes H$$

is an algebra map. The subalgebra A^H of *coinvariants* of an H-comodule algebra A is defined by

$$A^H = \{a \in A \,|\, \delta(a) = a \otimes 1\}\,.$$

Given two H-comodule algebras A and A' with respective coactions δ and δ', an algebra map $f : A \to A'$ is an H*-comodule algebra map* if

$$\delta' \circ f = (f \otimes \mathrm{id}_H) \circ \delta\,.$$

We denote by Alg^H the category whose objects are H-comodule algebras and arrows are H-comodule algebra maps.

Let us give a few examples of comodule algebras.

Example 1.1. If $H = k$, then an H-comodule algebra is nothing but an ordinary (associative, unital) algebra.

Example 1.2. The algebra $H = k[G]$ of a *group* G is a Hopf algebra with coproduct, counit, and antipode given for all $g \in G$ by

$$\Delta(g) = g \otimes g\,, \quad \varepsilon(g) = 1\,, \quad S(g) = g^{-1}\,.$$

It is well-known (see [5, Lemma 4.8]) that an H-comodule algebra A is the same as a *G-graded algebra*

$$A = \bigoplus_{g \in G} A_g\,, \qquad A_g\, A_h \subset A_{gh}\,.$$

The coaction $\delta : A \to A \otimes H$ is given by $\delta(a) = a \otimes g$ for all $a \in A_g$ and $g \in G$. We have $A^H = A_e$, where e is the neutral element of G.

Example 1.3. Let G be a *finite group* and $H = k^G$ be the algebra of k-valued functions on a finite group G. This algebra can be equipped with a Hopf algebra structure that is dual to the Hopf algebra $k[G]$ above. An H-comodule algebra A is the same as a *G-algebra*, i.e., an algebra equipped with a left action of G on A by group automorphisms.

If we denote the action of $g \in G$ on $a \in A$ by ${}^g a$, then the coaction $\delta : A \to A \otimes H$ is given by

$$\delta(a) = \sum_{g \in G} {}^g a \otimes e_g\,,$$

where $\{e_g\}_{g \in G}$ is the basis of H consisting of the functions e_g defined by $e_g(h) = 1$ if $h = g$, and 0 otherwise.

The subalgebra of coinvariants of A coincides with the subalgebra of G-invariant elements: $A^H = A^G$.

Example 1.4. Any Hopf algebra H is an H-comodule algebra whose coaction coincides with the coproduct of H:

$$\delta = \Delta : H \to H \otimes H\,.$$

In this case the coinvariants of H are exactly the scalar multiples of the unit of H; in other words, $H^H = k\,1$.

2. Identities

2.1. **Polynomial identities.** Let A be an algebra. A *polynomial identity* for an algebra A is a polynomial $P(X, Y, Z, \ldots)$ in a finite number of noncommutative variables $X, Y, Z, \ldots$ such that

$$P(x, y, z, \ldots) = 0$$

for all $x, y, z, \ldots \in A$.

Examples 2.1. (a) The polynomial $XY - YX$ is a polynomial identity for any *commutative algebra.*

(b) If $A = M_2(k)$ is the *algebra of* 2×2*-matrices* with entries in k, then

$$(XY - YX)^2 Z - Z(XY - YX)^2$$

is a polynomial identity for A. (Use the Cayley-Hamilton theorem to check this.)

The concept of a polynomial identity first emerged in the 1920's in an article [6] on the foundation of projective geometry by Max Dehn, the topologist. The above polynomial identity for the algebra of 2×2-matrices appeared in 1937 in [22]. Today there is an abundant literature on polynomial identities; see for instance [8, 17].

For algebras graded by a group G there exists the concept of a graded polynomial identity (see [1, 3]). In this case we need to take a family of non-commutative variables $X_g, Y_g, Z_g, \ldots$ for each element $g \in G$. Given a G-graded algebra $A = \bigoplus_{g\in G} A_g$, a *graded polynomial identity* is a polynomial P in these indexed variables such that P vanishes upon any substitution of each variable X_g appearing in P by an element of the g-component A_g.

In general, we should keep in mind that in order to define polynomial identities for a class of algebras, we need to single out

(i) a suitable algebra of non-commutative polynomials and
(ii) a suitable notion of specialization for these polynomials.

The algebras of interest to us in this survey are comodule algebras over a Hopf algebra H. The non-commutative variables we wish to use will be indexed by the elements of some linear basis of H. Since in general a Hopf algebra does not have a natural basis, we find it preferable to use a more canonical construction, namely the tensor algebra over H, and to resort to a given basis only when we need to perform computations.

2.2. **Definition and examples of H-identities.** Let H be a Hopf algebra. We pick a copy X_H of the underlying vector space of H and we denote the identity map from H to X_H by $x \mapsto X_x$ for all $x \in H$.

Consider the *tensor algebra* $T(X_H)$ of the vector space X_H over the ground field k:

$$T(X_H) = \bigoplus_{r\geq 0} T^r(X_H)\,,$$

where $T^r(X_H) = X_H^{\otimes r}$ is the tensor product of r copies of X_H over k, with the convention $T^0(X_H) = k$. If $\{x_i\}_{i\in I}$ is some linear basis of H, then $T(X_H)$ is isomorphic to the algebra of non-commutative polynomials in the indeterminates X_{x_i} $(i \in I)$.

Beware that the product $X_x X_y$ of symbols in the tensor algebra is different from the symbol X_{xy} attached to the product of x and y in H; the former is of degree 2 while the latter is of degree 1.

The algebra $T(X_H)$ is an *H-comodule algebra* equipped with the coaction

$$\delta : T(X_H) \to T(X_H) \otimes H\ ; \quad X_x \mapsto X_{x_1} \otimes x_2\,.$$

Note that $T(X_H)$ is *graded* with all generators X_x in degree 1. The coaction preserves the grading, where $T(X_H) \otimes H$ is graded by

$$(T(X_H) \otimes H)_r = T^r(X_H) \otimes H$$

for all $r \geq 0$.

We now give the main definition of this section.

Definition 2.2. *Let A be an H-comodule algebra. An element $P \in T(X_H)$ is an H-identity for A if $\mu(P) = 0$ for all H-comodule algebra maps*

$$\mu : T(X_H) \to A\,.$$

To convey the feeling of what an H-identity is, let us give some simple examples.

Example 2.3. Let $H = k$ be the one-dimension Hopf algebra as in Example 1.1. An H-comodule algebra A is then the same as an algebra. In this case, $T(X_H)$ coincides with the polynomial algebra $k[X_1]$ and an H-comodule algebra map is nothing but an algebra map. Therefore, an element $P(X_1) \in T(X_H) = k[X_1]$ is an H-identity for A if and only if all $P(a) = 0$ for all $a \in A$. Since k is assumed to be infinite, it follows that there are no non-zero H-identities for A.

Example 2.4. Let $H = k[G]$ be a group Hopf algebra as in Example 1.2. We know that an H-comodule algebra is a G-graded algebra $A = \bigoplus_{g\in G} A_g$. Since $\{g\}_{g\in G}$ is a basis of H, the tensor algebra $T(X_H)$ is the algebra of non-commutative polynomials in the indeterminates X_g $(g \in G)$.

It is easy to check that an algebra map $\mu : T(X_H) \to A$ is an H-comodule algebra map if and only if $\mu(X_g) \in A_g$ for all $g \in G$. This remark allows us to produce the following examples of H-identities.

(a) Suppose that A is *trivially graded*, i.e., $A_g = 0$ for all $g \neq e$. Then any non-commutative polynomial in the indeterminates X_g with $g \neq e$ is killed by any H-comodule algebra map $\mu : T(X_H) \to A$. Therefore, such a polynomial is an H-identity for A.

(b) Suppose that the trivial component A_e is *central* in A. We claim that
$$X_g X_{g^{-1}} X_h - X_h X_g X_{g^{-1}}$$
is an H-identity for A for all $g, h \in G$. Indeed, for any H-comodule algebra map $\mu : T(X_H) \to A$, we have
$$\mu(X_g) \in A_g \quad \text{and} \quad \mu(X_{g^{-1}}) \in A_{g^{-1}}\,;$$
therefore, $\mu(X_g X_{g^{-1}}) = \mu(X_g)\,\mu(X_{g^{-1}})$ belongs to A_e, hence commutes with $\mu(X_h)$. One shows in a similar fashion that if g is an element of G of finite order N, then for all $h \in G$,
$$X_g^N X_h - X_h X_g^N$$
is an H-identity for A.

Example 2.5. Let H be an arbitrary Hopf algebra, and let A be an H-comodule algebra such that the subalgebra A^H of coinvariants is central in A (the twisted comodule algebras of § 3.1 satisfy the latter condition).

For $x, y \in H$ consider the following elements of $T(X_H)$:
$$P_x = X_{x_1}\, X_{S(x_2)} \quad \text{and} \quad Q_{x,y} = X_{x_1}\, X_{y_1}\, X_{S(x_2 y_2)}\,.$$
Then for all $x, y, z \in H$,
$$P_x\, X_z - X_z\, P_x \quad \text{and} \quad Q_{x,y}\, X_z - X_z\, Q_{x,y}$$
are H-identities for A. Indeed, P_x and $Q_{x,y}$ are coinvariant elements of $T(X_H)$; see [2, Lemma 2.1]. It follows that for any H-comodule algebra map $\mu : T(X_H) \to A$, the elements $\mu(P_x)$ and $\mu(Q_{x,y})$ are coinvariant, hence central, in A.

More sophisticated examples of H-identities will be given in § 5.

2.3. **The ideal of H-identities.** Let H be a Hopf algebra and A an H-comodule algebra. Denote the set of all H-identities for A by $I_H(A)$. By definition,

$$I_H(A) = \bigcap_{\mu \in \mathrm{Alg}^H(T(X_H),A)} \mathrm{Ker}\,\mu\,.$$

A proof of the following assertions can be found in [2, Prop. 2.2].

Proposition 2.6. *The set $I_H(A)$ has the following properties:*

(a) it is a graded ideal of $T(X_H)$, i.e.,

$$I_H(A)\,T(X_H) \subset I_H(A) \supset T(X_H)\,I_H(A)$$

and

$$I_H(A) = \bigoplus_{r \geq 0} \Big(I_H(A) \bigcap T^r(X_H) \Big)\,;$$

(b) it is a right H-coideal of $T(X_H)$, i.e.,

$$\delta\big(I_H(A)\big) \subset I_H(A) \otimes H\,.$$

Note that for any H-comodule algebra map $\mu : T(X_H) \to A$, we have $\mu(1) = 1$; therefore, the degree 0 component of $I_H(A)$ is always trivial:

$$I_H(A) \bigcap T^0(X_H) = 0\,.$$

If, in addition, there exists an injective H-comodule map $H \to A$, then the degree 1 component of $I_H(A)$ is also trivial:

$$I_H(A) \bigcap T^1(X_H) = 0\,.$$

Remark 2.7. Right from the beginning we required the ground field k to be infinite. This assumption is used for instance to establish that $I_H(A)$ is a graded ideal of $T(X_H)$. Let us give a proof of this fact in order to show how the assumption is used. Indeed, expand $P \in I_H(A)$ as

$$P = \sum_{r \geq 0} P_r$$

with $P_r \in T^r(X_H)$ for all $r \geq 0$. To prove that $I_H(A)$ is a graded ideal, it suffices to check that each P_r is in $I_H(A)$. Given a scalar $\lambda \in k$, consider the algebra endomorphism λ_* of $T(X_H)$ defined by $\lambda(X_x) = \lambda X_x$ for all $x \in H$; clearly, λ_* is an H-comodule map. If $\mu : T(X_H) \to A$ is an H-comodule algebra map, then so is $\mu \circ \lambda_*$. Since $P \in I_H(A)$, we have

$$\sum_{r \geq 0} \lambda^r \mu(P_r) = (\mu \circ \lambda_*)(P) = 0\,.$$

The A-valued polynomial $\sum_{r \geq 0} \lambda^r \mu(P_r)$ takes zero values for all $\lambda \in k$. By the assumption on k, this implies that its coefficients are all zero, i.e., $\mu(P_r) = 0$ for all $r \geq 0$. Since this holds for all $\mu \in \mathrm{Alg}^H(T(X_H), A)$, we obtain $P_r \in I_H(A)$ for all $r \geq 0$.

If the ground field is *finite*, then Definition 2.2 still makes sense, but the ideal $I_H(A)$ may no longer be graded. Indeed, let k be the finite field $\mathbb{F}_p$ and $H = k$. Then for $q = p^N$, the finite field $\mathbb{F}_q$ is an H-comodule algebra. In view of Example 2.3, the polynomial $X_1^q - X_1$ is an H-identity for $\mathbb{F}_q$, but clearly the homogeneous summands in this polynomial, namely X_1^q and X_1, are not H-identities.

2.4. The universal H-comodule algebra. Let A be an H-comodule algebra and $I_H(A)$ the ideal of H-identities for A defined above. Since $I_H(A)$ is a graded ideal of $T(X_H)$, we may consider the quotient algebra

$$\mathcal{U}_H(A) = T(X_H)/I_H(A).$$

The grading on $T(X_H)$ induces a grading on $\mathcal{U}_H(A)$. As $I_H(A)$ is a right H-coideal of $T(X_H)$, the quotient algebra $\mathcal{U}_H(A)$ carries an H-comodule algebra structure inherited from $T(X_H)$.

By definition of $\mathcal{U}_H(A)$, all H-identities for A vanish in $\mathcal{U}_H(A)$. For this reason we call $\mathcal{U}_H(A)$ the *universal H-comodule algebra attached to A*.

The algebra $\mathcal{U}_H(A)$ has two interesting subalgebras:

(i) The subalgebra $\mathcal{U}_H(A)^H$ of *coinvariants* of $\mathcal{U}_H(A)$.

(ii) The *center* $\mathcal{Z}_H(A)$ of $\mathcal{U}_H(A)$.

We now raise the following question. Suppose that the comodule algebra A is free as a module over the subalgebra of coinvariants A^H (or over its center); is $\mathcal{U}_H(A)$, or rather some suitable central localization of it, then free as a module over some localization of $\mathcal{U}_H(A)^H$ (or of $\mathcal{Z}_H(A)$)? An answer to this question will be given below (see Theorem 4.5) for a special class of comodule algebras, which we introduce in the next section.

3. Detecting H-identities

Fix a Hopf algebra H. We now define a special class of H-comodule algebras for which we can detect all H-identities.

3.1. Twisted comodule algebras. Recall that a *two-cocycle* α on H is a bilinear form $\alpha : H \times H \to k$ such that

$$\alpha(x_1, y_1)\,\alpha(x_2y_2, z) = \alpha(y_1, z_1)\,\alpha(x, y_2z_2)$$

for all $x, y, z \in H$. We assume that α is convolution-invertible and write α^{-1} for its inverse. For simplicity, we also assume that α is normalized, i.e.,

$$\alpha(x, 1) = \alpha(1, x) = \varepsilon(x)$$

for all $x \in H$.

Any Hopf algebra possesses at least one normalized convolution-invertible two-cocycle, namely the *trivial* two-cocycle α_0, which is defined by

$$\alpha_0(x, y) = \varepsilon(x)\,\varepsilon(y)$$

for all $x, y \in H$.

Let u_H be a copy of the underlying vector space of H. Denote the identity map from H to u_H by $x \mapsto u_x$ ($x \in H$). We define the *twisted algebra* ${}^\alpha H$ as the vector space u_H equipped with the associative product given by

$$u_x\, u_y = \alpha(x_1, y_1)\, u_{x_2y_2}$$

for all x, $y \in H$. This product is associative because of the above cocycle condition; the two-cocycle α being normalized, u_1 is the unit of ${}^\alpha H$.

The algebra ${}^\alpha H$ is an H-comodule algebra with coaction $\delta\colon {}^\alpha H \to {}^\alpha H \otimes H$ given for all $x \in H$ by

$$\delta(u_x) = u_{x_1} \otimes x_2 .$$

It is easy to check that the subalgebra of coinvariants of ${}^\alpha H$ coincides with $k\,u_1$, which lies in the center of ${}^\alpha H$.

Note that if $\alpha = \alpha_0$ is the trivial two-cocycle, then ${}^{\alpha}H = H$ is the H-comodule algebra of Example 1.4.

The twisted comodule algebras of the form ${}^{\alpha}H$ coincide with the so-called *cleft H-Galois objects*; see [16, Prop. 7.2.3]. It is therefore an important class of comodule algebras. We next show how we can detect H-identities for such comodule algebras.

3.2. **The universal comodule algebra map.** We pick a third copy t_H of the underlying vector space of H and denote the identity map from H to t_H by $x \mapsto t_x$ ($x \in H$). Let $S(t_H)$ be the *symmetric algebra* over the vector space t_H. If $\{x_i\}_{i\in I}$ is a linear basis of H, then $S(t_H)$ is isomorphic to the (commutative) algebra of polynomials in the indeterminates t_{x_i} ($i \in I$).

We consider the algebra $S(t_H) \otimes {}^{\alpha}H$. As a k-algebra, it is generated by the symbols $t_z u_x$ ($x, z \in H$) (we drop the tensor product sign $\otimes$ between the t-symbols and the u-symbols).

The algebra $S(t_H) \otimes {}^{\alpha}H$ is an H-comodule algebra whose $S(t_H)$-linear coaction extends the coaction of ${}^{\alpha}H$:

$$\delta(t_z u_x) = t_z u_{x_1} \otimes x_2 .$$

Define an algebra map $\mu_\alpha : T(X_H) \to S(t_H) \otimes {}^{\alpha}H$ by

$$\mu_\alpha(X_x) = t_{x_1} u_{x_2}$$

for all $x \in H$. The map μ_α possesses the following properties (see [2, Sect. 4]).

Proposition 3.1. *(a) The map $\mu_\alpha : T(X_H) \to S(t_H) \otimes {}^{\alpha}H$ is an H-comodule algebra map.*

(b) For every H-comodule algebra map $\mu : T(X_H) \to {}^{\alpha}H$, there is a unique algebra map $\chi : S(t_H) \to k$ such that

$$\mu = (\chi \otimes \mathrm{id}) \circ \mu_\alpha .$$

In other words, any H-comodule algebra map $\mu : T(X_H) \to {}^{\alpha}H$ can be obtained from μ_α by specialization. For this reason we call μ_α the *universal comodule algebra map* for ${}^{\alpha}H$.

Theorem 3.2. *An element $P \in T(X_H)$ is an H-identity for ${}^{\alpha}H$ if and only if $\mu_\alpha(P) = 0$; equivalently,*

$$I_H({}^{\alpha}H) = \ker(\mu_\alpha) .$$

This result is a consequence of Proposition 3.1. It allows us to detect the H-identities for any twisted comodule algebra: it suffices to check them in the easily controllable algebra $S(t_H) \otimes {}^{\alpha}H$. In § 5 we shall show how to apply this result in an interesting example.

Let us derive some consequences of Theorem 3.2. To simplify notation, we denote the ideal of H-identities $I_H({}^{\alpha}H)$ by I_H^{α}, the universal H-comodule algebra $\mathcal{U}_H({}^{\alpha}H)$ by $\mathcal{U}_H^{\alpha}$, and the center $\mathcal{Z}_H({}^{\alpha}H)$ of $\mathcal{U}_H^{\alpha}$ by $\mathcal{Z}_H^{\alpha}$.

Corollary 3.3. *(a) The map $\mu_\alpha : T(X_H) \to S(t_H) \otimes {}^{\alpha}H$ induces an injection of comodule algebras*

$$\overline{\mu}_\alpha : \mathcal{U}_H^{\alpha} \hookrightarrow S(t_H) \otimes {}^{\alpha}H .$$

(b) An element of $\mathcal{U}_H^\alpha$ belongs to the subalgebra $(\mathcal{U}_H^\alpha)^H$ of coinvariants if and only if its image under $\overline{\mu}_\alpha$ sits in the subalgebra $S(t_H) \otimes u_1$.

We also proved that an element of $\mathcal{U}_H^\alpha$ belongs to the center $\mathcal{Z}_H^\alpha$ if and only if its image under $\overline{\mu}_\alpha$ sits in the subalgebra $S(t_H) \otimes Z({}^\alpha H)$, where $Z({}^\alpha H)$ is the center of ${}^\alpha H$ (see [2, Prop. 8.2]). In particular, since u_1 is central in ${}^\alpha H$, it follows that all coinvariant elements of $\mathcal{U}_H^\alpha$ belong to the center $\mathcal{Z}_H^\alpha$.

We mention another consequence: it asserts that there always exist non-zero H-identities for any non-trivial finite-dimensional twisted comodule algebra.

Corollary 3.4. *If $2 \leq \dim_k H < \infty$, then $I_H^\alpha \neq \{0\}$.*

Proof. Suppose that $I_H^\alpha = \{0\}$. Then in view of $\mathcal{U}_H^\alpha = T(X_H)/I_H^\alpha$ and of Corollary 3.3, we would have an injective linear map

$$T^r(X_H) \hookrightarrow S^r(X_H) \otimes {}^\alpha H$$

for all $r \geq 0$. (Here $S^r(X_H)$ is the subspace of elements of degree r in $S(t_H)$.) Taking dimensions and setting $\dim_k H = n$, we would obtain the inequality

$$n^r \leq n \binom{r+n-1}{n-1},$$

which is impossible for large r. □

4. Localizing the universal comodule algebra

We now wish to address the question raised in § 2.4 in the case A is a twisted comodule algebra of the form ${}^\alpha H$, where H is a Hopf algebra and α is a normalized convolution-invertible two-cocycle on H.

4.1. **The generic base algebra.** Recall the symmetric algebra $S(t_H)$ introduced in § 3.2. By [2, Lemma A.1] there is a unique linear map $x \mapsto t_x^{-1}$ from H to the field of fractions $\operatorname{Frac} S(t_H)$ of $S(t_H)$ such that for all $x \in H$,

$$\sum_{(x)} t_{x_{(1)}} t_{x_{(2)}}^{-1} = \sum_{(x)} t_{x_{(1)}}^{-1} t_{x_{(2)}} = \varepsilon(x)\,1.$$

(The algebra of fractions generated by the elements t_x and t_x^{-1} ($x \in H$) is Takeuchi's free commutative Hopf algebra on the coalgebra underlying H; see [21].)

Examples 4.1. (a) If g is a *group-like* element, i.e., $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, then

$$t_g^{-1} = \frac{1}{t_g}.$$

(b) If x is a *skew-primitive* element, i.e., $\Delta(x) = g \otimes x + x \otimes h$ for some group-like elements g, h, then

$$t_x^{-1} = -\frac{t_x}{t_g t_h}.$$

For $x, y \in H$, define the following elements of the fraction field $\operatorname{Frac} S(t_H)$:

$$\sigma(x,y) = \sum_{(x),(y)} t_{x_{(1)}} t_{y_{(1)}} \alpha(x_{(2)}, y_{(2)}) t^{-1}_{x_{(3)}y_{(3)}}$$

and

$$\sigma^{-1}(x,y) = \sum_{(x),(y)} t_{x_{(1)}y_{(1)}} \alpha^{-1}(x_{(2)}, y_{(2)}) t^{-1}_{x_{(3)}} t^{-1}_{y_{(3)}},$$

where α^{-1} is the inverse of α.

The map $(x,y) \in H \times H \mapsto \sigma(x,y) \in \operatorname{Frac} S(t_H)$ is a two-cocycle with values in the fraction field $\operatorname{Frac} S(t_H)$.

Definition 4.2. *The generic base algebra is the subalgebra $\mathcal{B}_H^\alpha$ of* $\operatorname{Frac} S(t_H)$ *generated by the elements $\sigma(x,y)$ and $\sigma^{-1}(x,y)$, where x and y run over H.*

Since $\mathcal{B}_H^\alpha$ is a subalgebra of the field $\operatorname{Frac} S(t_H)$, it is a domain and the Krull dimension of $\mathcal{B}_H^\alpha$ cannot exceed the Krull dimension of $S(t_H)$, which is $\dim_k H$. Actually, it is proved in [11, Cor. 3.7] that if the Hopf algebra H is finite-dimensional, then the Krull dimension of $\mathcal{B}_H^\alpha$ is exactly equal to $\dim_k H$. More properties of the generic base algebra are given in [11].

Example 4.3. If $H = k[G]$ is the Hopf algebra of a group G and $\alpha = \alpha_0$ is the trivial two-cocycle, then the generic base algebra $\mathcal{B}_H^\alpha$ is the algebra generated by the Laurent polynomials

$$\left(\frac{t_g t_h}{t_{gh}}\right)^{\pm 1},$$

where g, h run over G. A complete computation for the (in)finite cyclic groups $G = \mathbb{Z}$ and $G = \mathbb{Z}/N$ was given in [10, Sect. 3.3].

4.2. **Non-degenerate cocycles.** We now restrict to the case when α is a *non-degenerate* two-cocycle, i.e., when the center of the twisted algebra ${}^\alpha H$ is one-dimensional. In this case, the center of ${}^\alpha H$ coincides with the subalgebra of coinvariants.

Recall the injective algebra map $\overline{\mu}_\alpha : \mathcal{U}_H^\alpha \to S(t_H) \otimes {}^\alpha H$ of Corollary 3.3. By this corollary and the subsequent comment, it follows that in the non-degenerate case the center $\mathcal{Z}_H^\alpha$ of $\mathcal{U}_H^\alpha$ coincides with the subalgebra $(\mathcal{U}_H^\alpha)^H$ of coinvariants, and we have

$$\mathcal{Z}_H^\alpha = (\mathcal{U}_H^\alpha)^H = \overline{\mu}_\alpha^{-1}(S(t_H) \otimes u_1).$$

The following result connects $\mathcal{Z}_H^\alpha$ to the generic base algebra $\mathcal{B}_H^\alpha$ introduced in § 4.1 (see [2, Prop. 9.1]).

Proposition 4.4. *If α is a non-degenerate two-cocycle on H, then $\overline{\mu}_\alpha$ maps $\mathcal{Z}_H^\alpha$ into $\mathcal{B}_H^\alpha \otimes u_1$.*

This result allows us to view the center $\mathcal{Z}_H^\alpha$ of $\mathcal{U}_H^\alpha$ as a subalgebra of the generic base algebra $\mathcal{B}_H^\alpha$. It follows from the discussion in § 4.1 that $\mathcal{Z}_H^\alpha$ is a domain whose Krull dimension is at most $\dim_k H$.

We may now consider the $\mathcal{B}_H^\alpha$-algebra

$$\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha .$$

It is an H'-comodule algebra, where $H' = \mathcal{B}_H^\alpha \otimes H$.

The following answers the question raised in § 2.4.

Theorem 4.5. *If H is a Hopf algebra and α is a non-degenerate two-cocycle on H such that $\mathcal{B}_H^\alpha$ is a localization of $\mathcal{Z}_H^\alpha$, then $\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha$ is a cleft H-Galois extension of $\mathcal{B}_H^\alpha$. In particular, there is a comodule isomorphism*

$$\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha \cong \mathcal{B}_H^\alpha \otimes H .$$

It follows that under the hypotheses of the theorem, a suitable central localization of the universal comodule algebra $\mathcal{U}_H^\alpha$ is free of rank $\dim_k H$ as a module over its center.

5. An example: the Sweedler algebra

We assume in this section that the characteristic of k is different from 2.

5.1. Presentation and twisted comodule algebras.

The *Sweedler algebra H_4* is the algebra generated by two elements x, y subject to the relations

$$x^2 = 1 , \quad xy + yx = 0 , \quad y^2 = 0 .$$

It is four-dimensional. As a basis of H_4, we take the set $\{1, x, y, z\}$, where $z = xy$.

The algebra H_4 carries the structure of a non-commutative, non-cocommutative Hopf algebra with coproduct, counit, and antipode given by

$$\begin{array}{rclrcl} \Delta(1) &=& 1 \otimes 1 , & \Delta(x) &=& x \otimes x , \\ \Delta(y) &=& 1 \otimes y + y \otimes x , & \Delta(z) &=& x \otimes z + z \otimes 1 , \\ \varepsilon(1) &=& \varepsilon(x) = 1 , & \varepsilon(y) &=& \varepsilon(z) = 0 , \\ S(1) &=& 1 , & S(x) &=& x , \\ S(y) &=& z , & S(z) &=& -y . \end{array}$$

The tensor algebra $T(H_4)$ is the free non-commutative algebra on the four symbols

$$E = X_1 , \quad X = X_x , \quad Y = X_y , \quad Z = X_z ,$$

whereas $S(t_{H_4})$ is the polynomial algebra on the symbols t_1, t_x, t_y, t_z.

Masuoka [13] (see also [7]) showed that any twisted H_4-comodule algebra as in § 3.1 has, up to isomorphism, the following presentation:

$${}^\alpha H_4 = k \left\langle \, u_x, u_y \, | \, u_x^2 = a u_1 , \; u_x u_y + u_y u_x = b u_1 , \; u_y^2 = c u_1 \, \right\rangle$$

for some scalars a, b, c with $a \neq 0$. To indicate the dependence on the parameters a, b, c, we denote ${}^\alpha H_4$ by $A_{a,b,c}$.

The center of $A_{a,b,c}$ consists of the scalar multiples of the unit u_1 for all values of a, b, c. In other words, all two-cocycles on H_4 are non-degenerate.

The coaction $\delta : A_{a,b,c} \to A_{a,b,c} \otimes H_4$ is determined by

$$\delta(u_x) = u_x \otimes x \quad \text{and} \quad \delta(u_y) = u_1 \otimes y + u_y \otimes x .$$

As observed in § 3.1, the coinvariants of $A_{a,b,c}$ consists of the scalar multiples of the unit u_1. Therefore, coinvariants and central elements of $A_{a,b,c}$ coincide.

5.2. **Identities.** In this situation, the universal comodule algebra map

$$\mu_\alpha : T(X_H) \to S(t_H) \otimes A_{a,b,c}$$

is given by

$$\mu_\alpha(E) = t_1 u_1 , \qquad \mu_\alpha(X) = t_x u_x ,$$
$$\mu_\alpha(Y) = t_1 u_y + t_y u_x , \qquad \mu_\alpha(Z) = t_x u_z + t_z u_1 .$$

Let us set

$$R = X^2 , \quad S = Y^2 , \quad T = XY + YX , \quad U = X(XZ + ZX) .$$

Lemma 5.1. *In the algebra $S(t_H) \otimes A_{a,b,c}$ we have the following equalities:*

$$\begin{aligned} \mu_\alpha(R) &= a t_x^2 u_1 , \\ \mu_\alpha(S) &= (a t_y^2 + b t_1 t_y + c t_1^2)\, u_1 , \\ \mu_\alpha(T) &= t_x (2a t_y + b t_1)\, u_1 , \\ \mu_\alpha(U) &= a t_x^2 (2 t_z + b t_x)\, u_1 . \end{aligned}$$

Proof. This follows from a straightforward computation. Let us compute $\mu_\alpha(S)$ as an example. We have

$$\begin{aligned} \mu_\alpha(S) &= \mu_\alpha(Y)^2 = (t_1 u_y + t_y u_x)^2 \\ &= t_y^2 u_x^2 + t_1 t_y (u_x u_y + u_y u_x) + t_1^2 u_y^2 \\ &= (a t_y^2 + b t_1 t_y + c t_1^2)\, u_1 \end{aligned}$$

in view of the definition of $A_{a,b,c}$. □

We now exhibit two non-trivial H_4-identities.

Proposition 5.2. *The elements*

$$T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R \quad \text{and} \quad ERZ - RXY - \frac{EU - RT}{2}$$

are H_4-identities for $A_{a,b,c}$.

Proof. It suffices to check that these two elements are killed by μ_α, which is easily done using Lemma 5.1. □

Since E, R, S, T, U are sent under μ_α to $S(t_H) \otimes u_1$, their images in $\mathcal{U}_H^\alpha$ belong to the center $\mathcal{Z}_H^\alpha$. We assert that after inverting the elements E and R, all relations in $\mathcal{Z}_H^\alpha$ are consequences of the leftmost relation in Proposition 5.2. More precisely, we have the following (see [2, Thm. 10.3]).

Theorem 5.3. *There is an isomorphism of algebras*

$$\mathcal{Z}_H^\alpha[E^{-1}, R^{-1}] \cong k[E, E^{-1}, R, R^{-1}, S, T, U]/(D_{a,b,c}) ,$$

where

$$D_{a,b,c} = T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R .$$

To prove this theorem, we first check that the generic base algebra $\mathcal{B}_H^\alpha$ (whose generators we know) is generated by $E, E^{-1}, R, R^{-1}, S, T, U$; this implies that $\mathcal{B}_H^\alpha$ is the localization

$$\mathcal{B}_H^\alpha = \mathcal{Z}_H^\alpha[E^{-1}, R^{-1}]$$

of $\mathcal{Z}_H^\alpha$. In a second step, we establish that all relations between the above-listed generators of $\mathcal{B}_H^\alpha$ follow from the sole relation $D_{a,b,c} = 0$.

Let us now turn to the universal comodule algebra $\mathcal{U}_H^\alpha$. By Proposition 5.2, we have the following relation in $\mathcal{U}_H^\alpha$, where we keep the same notation for the elements of $T(X_H)$ and their images in $\mathcal{U}_H^\alpha$:

$$(ER)Z = (R)XY + \left(\frac{EU - RT}{2}\right) \quad \text{in } \mathcal{U}_H^\alpha .$$

The elements in parentheses being central, it follows from the previous relation that if we again invert the central elements E and R, then Z is a linear combination of 1 and XY with coefficients in $\mathcal{B}_H^\alpha = \mathcal{Z}_H^\alpha[E^{-1}, R^{-1}]$. Noting that

$$YX = -XY + T \in -XY + \mathcal{Z}_H^\alpha \subset -XY + \mathcal{B}_H^\alpha ,$$

we easily deduce that after inverting E and R any element of $\mathcal{U}_H^\alpha$ is a linear combination of $1, X, Y, XY$ over $\mathcal{B}_H^\alpha$.

In [2] the following more precise result was established (see *loc. cit.*, Thm. 10.7). It answers positively the question of § 2.4.

Theorem 5.4. *The localized algebra $\mathcal{U}_H^\alpha[E^{-1}, R^{-1}]$ is free of rank 4 over its center $\mathcal{B}_H^\alpha = \mathcal{Z}_H^\alpha[E^{-1}, R^{-1}]$, and there is an isomorphism of algebras*

$$\mathcal{U}_H^\alpha[E^{-1}, R^{-1}] \cong \mathcal{B}_H^\alpha \langle \xi, \eta \rangle \,/\, \left(\xi^2 - R,\ \xi\eta + \eta\xi - T,\ \eta^2 - S\right) .$$

Note that the algebra $\mathcal{B}_H^\alpha$ coincides with the subalgebra of coinvariants of $\mathcal{U}_H^\alpha[E^{-1}, R^{-1}]$.

5.3. **An open problem.** To complete this survey, we state a problem who will hopefully attract the attention of some researchers.

Fix an integer $n \geq 2$ and suppose that the ground field k contains a primitive n-th root q of 1. Consider the Taft algebra H_{n^2}, which is the algebra generated by two elements x, y subject to the relations

$$x^n = 1 , \quad yx = qxy , \quad y^n = 0 .$$

This is a Hopf algebra of dimension n^2 with coproduct determined by

$$\Delta(x) = x \otimes x \quad \text{and} \quad \Delta(y) = 1 \otimes y + y \otimes x .$$

The twisted comodule algebras ${}^\alpha H_{n^2}$ have been classified in [7, 13]. (All two-cocycles of H_{n^2} are non-degenerate.)

Give a presentation by generators and relations of the generic base algebra $\mathcal{B}_{H_{n^2}}^\alpha$ and show that $\mathcal{B}_{H_{n^2}}^\alpha$ is a localization of $\mathcal{Z}_{H_{n^2}}^\alpha$. (By [11, Rem. 2.4 (c)] it is enough to consider the case where α is the trivial cocycle.)

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REFERENCES

[1] Aljadeff E., Haile D., Natapov M., Graded identities of matrix algebras and the universal graded algebra, *Trans. Amer. Math. Soc.* **362** (2010), 3125–3147.

[2] Aljadeff E., Kassel C., Polynomial identities and noncommutative versal torsors, *Adv. Math.* **218** (2008), 1453–1495.

[3] Bahturin Y. A., Zaicev M., Identities of graded algebras, *J. Algebra* **202** (1998), 634–654.

[4] Bichon J., Galois and bigalois objects over monomial non-semisimple Hopf algebras, *J. Algebra Appl.* **5** (2006), 653–680.

[5] Blattner R. J., Montgomery S., A duality theorem for Hopf module algebras, *J. Algebra* **95** (1985), 153–172.

[6] Dehn M., Über die Grundlagen der projektiven Geometrie und allgemeine Zahlsysteme, *Math. Ann.* **85** (1922), 184–194.

[7] Doi Y., Takeuchi M., Quaternion algebras and Hopf crossed products, *Comm. Algebra* **23** (1995), 3291–3325.

[8] Formanek E., *The polynomial identities and invariants of $n \times n$ matrices*, CBMS Conf. Series in Math., **78**, American Mathematical Society, Providence, RI, 1991.

[9] Kassel C., Quantum principal bundles up to homotopy equivalence, *The Legacy of Niels Henrik Abel, The Abel Bicentennial,* Oslo, 2002, O. A. Laudal, R. Piene (eds.), Springer-Verlag 2004, 737–748 (see also arXiv:math.QA/0507221).

[10] Kassel C., Generic Hopf Galois extensions, arXiv:0809.0638, to appear in *Proc. Workshop on Quantum Groups and Noncommutative Geometry,* Max Planck Institut für Mathematik, Bonn, 2007, M. Marcolli and D. Parashar (eds.).

[11] Kassel C., Masuoka A., Flatness and freeness properties of the generic Hopf Galois extensions, arXiv:0911.3719, to appear in the Proceedings of *Coloquio de Álgebras de Hopf, Grupos Cuánticos y Categorías Tensoriales,* Rev. Un. Mat. Argentina.

[12] Kassel C., Schneider H.-J., Homotopy theory of Hopf Galois extensions, *Ann. Inst. Fourier (Grenoble)* **55** (2005), 2521–2550.

[13] Masuoka A., Cleft extensions for a Hopf algebra generated by a nearly primitive element, *Comm. Algebra* **22** (1994), 4537–4559.

[14] Masuoka A., Cocycle deformations and Galois objects for some cosemisimple Hopf algebras of finite dimension, *New trends in Hopf algebra theory* (La Falda, 1999), 195–214, Contemp. Math., **267**, Amer. Math. Soc., Providence, RI, 2000.

[15] Masuoka A., Abelian and non-abelian second cohomologies of quantized enveloping algebras, *J. Algebra* **320** (2008), 1–47.

[16] Montgomery S., *Hopf algebras and their actions on rings,* CBMS Conf. Series in Math., **82**, Amer. Math. Soc., Providence, RI, 1993.

[17] Rowen L., *Polynomial identities in ring theory,* Pure and Applied Mathematics, **84**, Academic Press, Inc., New York-London, 1980.

[18] Schauenburg P., Hopf bi-Galois extensions, *Comm. Algebra* **24** (1996), 3797–3825.

[19] Schneider H.-J., Principal homogeneous spaces for arbitrary Hopf algebras, *Israel J. Math.* **72** (1990), 167–195.

[20] Sweedler M. E., *Hopf algebras,* W. A. Benjamin, Inc., New York, 1969.

[21] Takeuchi M., Free Hopf algebras generated by coalgebras, *J. Math. Soc. Japan* **23** (1971), 561–582.

[22] Wagner W., Über die Grundlagen der projektiven Geometrie und allgemeine Zahlsysteme, *Math. Ann.* **113** (1937), 528–567.

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