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Dynamical braided monoids and dynamical Yang-Baxter maps

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Abstract

By means of torsors (principal homogeneous spaces), we prove that dynamical braided monoids can produce dynamical Yang-Baxter maps.

1 Introduction

Finding solutions to the quantum Yang-Baxter equation [1, 21] is essential in the study of integrable systems [2, 8]. This quantum Yang-Baxter equation is exactly the braid relation in a suitable tensor category; for example, the usual quantum Yang-Baxter equation is the braid relation in the tensor category of vector spaces, and the quantum group [3, 7] is useful for the construction of solutions.

Lu, Yan, and Zhu [12] constructed Yang-Baxter maps [4, 20], solutions to the braid relation in the tensor category of sets, by means of braided groups [19]. Let S and B be groups whose unit elements are respectively denoted by 1_S and 1_B, and let σ be a map from $S \times B$ to $B \times S$.

Definition 1.1. A triple (S, B, σ) is a matched pair of groups [18], iff the map $\sigma : S \times B \ni (s, b) \mapsto (s \rightharpoonup b, s \leftharpoonup b) \in B \times S$ satisfies:

$$s \rightharpoonup (t \rightharpoonup b) = (st) \rightharpoonup b; \tag{1.1}$$
$$(st) \leftharpoonup b = (s \leftharpoonup (t \rightharpoonup b))(t \leftharpoonup b); \tag{1.2}$$
$$(s \leftharpoonup b) \leftharpoonup c = s \leftharpoonup (bc); \tag{1.3}$$
$$s \rightharpoonup (bc) = (s \rightharpoonup b)((s \leftharpoonup b) \rightharpoonup c); \tag{1.4}$$
$$1_S \rightharpoonup b = b; \tag{1.5}$$
$$s \leftharpoonup 1_B = s \quad (\forall s, t \in S, \forall b, c \in B). \tag{1.6}$$

The Cartesian product $B \times S$ is a group with the multiplication

$$(b, s)(c, t) = (b(s \rightharpoonup c), (s \leftharpoonup c)t) \quad ((b, s), (c, t) \in B \times S).$$

To be more precise, the unit element is $(1_B, 1_S)$, and the inverse of the element $(b, s) \in B \times S$ is $(s^{-1} \rightharpoonup b^{-1}, s^{-1} \leftharpoonup b^{-1})$.

Definition 1.2. A pair (G, σ) of a group G and a map $\sigma : G \times G \to G \times G$ is a braided group, iff:

(1) (G, G, σ) is a matched pair of groups;

(2) if $(y', x') = \sigma(x, y)$, then $y'x' = xy$ $(x, y, x', y' \in G)$.

In [12], Lu, Yan, and Zhu showed

Theorem 1.3. *If (G, σ) is a braided group, then σ satisfies the braid relation.*

$$(\sigma \times \mathrm{id}_G) \circ (\mathrm{id}_G \times \sigma) \circ (\sigma \times \mathrm{id}_G) = (\mathrm{id}_G \times \sigma) \circ (\sigma \times \mathrm{id}_G) \circ (\mathrm{id}_G \times \sigma).$$

We can rephrase the definition of the matched pair of groups by using category theory.

Let $I_{\mathbf{Set}}$ denote the set $\{e\}$ of one element. We write m_S and m_B for the multiplications of the groups S and B, respectively. We define the maps $\eta_S : I_{\mathbf{Set}} \to S$ and $\eta_B : I_{\mathbf{Set}} \to B$ by

$$\eta_S(e) = 1_S; \eta_B(e) = 1_B.$$

The above equations (1.1)-(1.6) are equivalent to:

$$(\mathrm{id}_B \times m_S) \circ (\sigma \times \mathrm{id}_S) \circ (\mathrm{id}_S \times \sigma) = \sigma \circ (m_S \times \mathrm{id}_B); \quad (1.7)$$
$$(m_B \times \mathrm{id}_S) \circ (\mathrm{id}_B \times \sigma) \circ (\sigma \times \mathrm{id}_B) = \sigma \circ (\mathrm{id}_S \times m_B); \quad (1.8)$$
$$(\mathrm{id}_B \times m_S) \circ (\sigma \times \mathrm{id}_S) \circ (\eta_S \times \mathrm{id}_{B \times S}) = l_{B \times S}; \quad (1.9)$$
$$(m_B \times \mathrm{id}_S) \circ (\mathrm{id}_B \times \sigma) \circ (\mathrm{id}_{B \times S} \times \eta_B) = r_{B \times S}. \quad (1.10)$$

Here, the maps $l_{B \times S} : I_{\mathbf{Set}} \times B \times S \to B \times S$ and $r_{B \times S} : B \times S \times I_{\mathbf{Set}} \to B \times S$ are defined by

$$l_{B \times S}(e, b, s) = (b, s); r_{B \times S}(b, s, e) = (b, s) \quad (I_{\mathbf{Set}} = \{e\}, b \in B, s \in S).$$

It is natural to try to solve the braid relation in another tensor category similarly.

The aim of this article is to make an analogy between the Yang-Baxter maps and dynamical Yang-Baxter maps (Definition 2.1) [14], solutions to the braid relation in a tensor category $\mathbf{Set}_H$ [15] defined in the next section. We construct the dynamical Yang-Baxter maps by means of dynamical braided monoids in Definition 4.2. Torsors [9, 11], also known as the principal homogeneous spaces, are important in this construction.

The organization of this article is as follows.

In Section 2, we briefly sketch a tensor category $\mathbf{Set}_H$. Section 3 explains monoids in $\mathbf{Set}_H$. After introducing dynamical braided monoids, our main results are stated and proved in Sections 4 and 5. The crucial fact is that the dynamical braided monoid satisfying (3.1) is exactly a torsor (See Proposition 5.6).

2 Tensor category $\mathbf{Set}_H$ and dynamical Yang-Baxter maps

This section explains the tensor category $\mathbf{Set}_H$ (cf. the tensor category $\mathcal{V}_{\mathfrak{h}}$ in [5, Section 3]), in which we will focus on the braid relation (For the tensor category, see [10, Chapter XI]).

Let H be a nonempty set. $\mathbf{Set}_H$ is a category whose object is a pair $(X, \cdot_X)$ of a nonempty set X and a map $\cdot_X : H \times X \ni (\lambda, x) \mapsto \lambda \cdot_X x \in H$ and whose morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ is a map $f : H \to \mathrm{Map}(X, Y)$ satisfying that

$$\lambda \cdot_Y f(\lambda)(x) = \lambda \cdot_X x \quad (\forall \lambda \in H, \forall x \in X). \tag{2.1}$$

To simplify notation, we will often use the symbol λx instead of $\lambda \cdot_X x$.

The identity id and the composition $\circ$ are defined as follows: for objects X, Y, Z and morphisms $f : X \to Y$, $g : Y \to Z$,

$$\mathrm{id}_X(\lambda)(x) = x \quad (\lambda \in H, x \in X); (g \circ f)(\lambda) = g(\lambda) \circ f(\lambda) \quad (\lambda \in H).$$

This $\mathbf{Set}_H$ is a tensor category: the tensor product $X \otimes Y$ of the objects $X = (X, \cdot_X)$ and $Y = (Y, \cdot_Y)$ is a pair $(X \times Y, \cdot)$ of the Cartesian product $X \times Y$ and the map $\cdot : H \times (X \times Y) \to H$ defined by

$$\lambda \cdot (x, y) = (\lambda \cdot_X x) \cdot_Y y \quad (\lambda \in H, (x, y) \in X \times Y); \tag{2.2}$$

the tensor product of the morphisms $f : X \to X'$ and $g : Y \to Y'$ is defined by $(f \otimes g)(\lambda)(x, y) = (f(\lambda)(x), g(\lambda x)(y))$ $(\lambda \in H, (x, y) \in X \times Y)$.

The other ingredients of the tensor category $\mathbf{Set}_H$ are: the associativity constraint $a_{XYZ}(\lambda)((x, y), z) = (x, (y, z))$; the unit $I = (\{e\}, \cdot_I)$, a pair of the set $\{e\}$ of one element and the map $\cdot_I$ defined by $\lambda \cdot_I e = \lambda$; the left and the right unit constraints $l_X(\lambda)(e, x) = x = r_X(\lambda)(x, e)$.

In what follows, the associativity constraint will be omitted.

Definition 2.1. A morphism $\sigma : X \otimes X \to X \otimes X$ of $\mathbf{Set}_H$ is a dynamical Yang-Baxter map [14, 15], iff σ satisfies the following braid relation in $\mathbf{Set}_H$.

$$(\sigma \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \sigma) \circ (\sigma \otimes \mathrm{id}_X) = (\mathrm{id}_X \otimes \sigma) \circ (\sigma \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \sigma). \quad (2.3)$$

Remark 2.2. (1) If H is a set of one element, the tensor category $\mathbf{Set}_H$ is exactly the tensor category **Set** consisting of nonempty sets, and the dynamical Yang-Baxter map is a Yang-Baxter map.

(2) The dynamical Yang-Baxter maps satisfying suitable conditions can produce bialgebroids, each of which gives birth to a tensor category of its dynamical representations [16]. Note that the definition of the tensor product in [16] is slightly different from that in this section.

3 Monoids in $\mathbf{Set}_H$

In this section, we introduce the monoid in $\mathbf{Set}_H$ (See [13, VII.3]).

Let X be an object of the tensor category $\mathbf{Set}_H$ and let $m_X : X \otimes X \to X$ and $\eta_X : I \to X$ be morphisms of $\mathbf{Set}_H$.

Definition 3.1. The triple (X, m_X, η_X) is a monoid, iff:

$$m_X \circ (m_X \otimes \mathrm{id}_X) = m_X \circ (\mathrm{id}_X \otimes m_X);$$
$$m_X \circ (\eta_X \otimes \mathrm{id}_X) = l_X;$$
$$m_X \circ (\mathrm{id} \otimes \eta_X) = r_X.$$

We explain a construction of the monoid in $\mathbf{Set}_H$, which is due to Mitsuhiro Takeuchi. Let X be an object of $\mathbf{Set}_H$. Suppose that

$$\forall \lambda, \lambda' \in H, \exists_1 x \in X \text{ such that } \lambda x = \lambda'. \quad (3.1)$$

We will denote by $\lambda \backslash\backslash \lambda'$ the unique element $x \in X$.

Proposition 3.2. *X satisfying* (3.1) *is a monoid, together with the morphisms m_X and η_X:*

$$m_X(\lambda)(x, y) = \lambda \backslash\backslash ((\lambda x) y); \eta_X(\lambda)(e) = \lambda \backslash\backslash \lambda \quad (\lambda \in H, x, y \in X, I = \{e\}).$$

Furthermore, this monoid structure is unique.

Proof. We give the proof only for the uniqueness of the morphism m_X. Suppose that $m_X : X \otimes X \to X$ is a morphism of $\mathbf{Set}_H$. It follows from (2.1) and (2.2) that $\lambda m_X(\lambda)(x, y) = \lambda(x, y) = (\lambda x) y$ $(\lambda \in H, x, y \in X)$. By taking (3.1) into account, $m_X(\lambda)(x, y)$ is uniquely determined. □

Example 3.3. The set H with the map $\lambda \cdot_H \lambda' = \lambda'$ $(\lambda, \lambda' \in H)$ is an object of $\mathbf{Set}_H$, and obviously satisfies (3.1); hence, $H = (H, \cdot_H)$ is a monoid.

4 Dynamical braided monoids

After introducing dynamical braided monoids in $\mathbf{Set}_H$, we show in this section that each dynamical braided monoid satisfying (3.1) gives birth to the dynamical Yang-Baxter map.

Let (X, m_X, η_X) be a monoid in the tensor category $\mathbf{Set}_H$. Suppose that a morphism $\sigma : X \otimes X \to X \otimes X$ of $\mathbf{Set}_H$ satisfies:

$$(\mathrm{id}_X \otimes m_X) \circ (\sigma \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \sigma) = \sigma \circ (m_X \otimes \mathrm{id}_X); \quad (4.1)$$

$$(m_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \sigma) \circ (\sigma \otimes \mathrm{id}_X) = \sigma \circ (\mathrm{id}_X \otimes m_X); \quad (4.2)$$

$$(\mathrm{id}_X \otimes m_X) \circ (\sigma \otimes \mathrm{id}_X) \circ (\eta_X \otimes \mathrm{id}_{X\otimes X}) = l_{X\otimes X}; \quad (4.3)$$

$$(m_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \sigma) \circ (\mathrm{id}_{X\otimes X} \otimes \eta_X) = r_{X\otimes X}. \quad (4.4)$$

We define the morphisms $m_{X\otimes X} : (X \otimes X) \otimes (X \otimes X) \to X \otimes X$ and $\eta_{X\otimes X} : I \to X \otimes X$ by:

$$m_{X\otimes X} = (m_X \otimes m_X) \circ (\mathrm{id}_X \otimes \sigma \otimes \mathrm{id}_X); \eta_{X\otimes X} = (\eta_X \otimes \eta_X) \circ l_I^{-1}.$$

A straightforward computation shows

Proposition 4.1. *$(X \otimes X, m_{X\otimes X}, \eta_{X\otimes X})$ is a monoid.*

Definition 4.2. (X, σ) is a dynamical braided monoid, iff the morphism σ satisfies (4.1)-(4.4).

Remark 4.3. (1) By taking (1.7)-(1.10) into account, the conditions (4.1)-(4.4) correspond to (1) in Definition 1.2, while (2) in Definition 1.2 corresponds to (2.1) for the morphism σ. If the monoid X satisfies (3.1), then $m_X(\lambda)(x, y) = \lambda\backslash((\lambda x)y)$ $(\lambda \in H, x, y \in X)$ because of Proposition 3.2, and (2.1) for the morphism σ is equivalent to that $m_X \circ \sigma = m_X$, which is similar to (2) in Definition 1.2.

(2) Let (X, m_X, η_X) and (Y, m_Y, η_Y) be a monoid in the tensor category $\mathbf{Set}_H$. Suppose that a morphism $\sigma : X \otimes Y \to Y \otimes X$ of $\mathbf{Set}_H$ satisfies:

$$(\mathrm{id}_Y \otimes m_X) \circ (\sigma \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \sigma) = \sigma \circ (m_X \otimes \mathrm{id}_Y);$$

$$(m_Y \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes \sigma) \circ (\sigma \otimes \mathrm{id}_Y) = \sigma \circ (\mathrm{id}_X \otimes m_Y);$$

$$(\mathrm{id}_Y \otimes m_X) \circ (\sigma \otimes \mathrm{id}_X) \circ (\eta_X \otimes \mathrm{id}_{Y\otimes X}) = l_{Y\otimes X};$$

$$(m_Y \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes \sigma) \circ (\mathrm{id}_{Y\otimes X} \otimes \eta_Y) = r_{Y\otimes X}.$$

We define the morphisms $m_{Y\otimes X} : (Y \otimes X) \otimes (Y \otimes X) \to Y \otimes X$ and $\eta_{Y\otimes X} : I \to Y \otimes X$ by:

$$m_{Y\otimes X} = (m_Y \otimes m_X) \circ (\mathrm{id}_Y \otimes \sigma \otimes \mathrm{id}_X); \eta_{Y\otimes X} = (\eta_Y \otimes \eta_X) \circ l_I^{-1}.$$

Then $(Y \otimes X, m_{Y\otimes X}, \eta_{Y\otimes X})$ is a monoid, which is called a matched pair of monoids.

The following theorem is an analogue of Theorem 1.3.

Theorem 4.4. *If a dynamical braided monoid* (X, σ) *satisfies* (3.1), *then* σ *is a dynamical Yang-Baxter map* (*Definition* 2.1).

We give a proof of this theorem in the next section.

5 Torsors (Principal homogeneous spaces)

This section is devoted to proving Theorem 4.4, in which the notion of a torsor [11, Section 4.2] plays an essential role.

Definition 5.1. A pair (M, μ) of a nonempty set M and a ternary operation $\mu : M \times M \times M \to M$ is called a torsor, iff μ satisfies:

$$\mu(u, v, v) = u = \mu(v, v, u); \tag{5.1}$$

$$\mu(\mu(u, v, w), x, y) = \mu(u, v, \mu(w, x, y)) \quad (\forall u, v, w, x, y \in M). \tag{5.2}$$

Remark 5.2. (1) A Mal'cev operation is a ternary operation satisfying (5.1) [9, Section 1]; moreover, an associative Mal'cev operation is a ternary operation satisfying (5.1) and (5.2). The torsor is also called a herd, a Schar (in German), a flock, and a heap [17, Section 1].

(2) For a pair (M, μ), the following conditions are equivalent (cf. [6, Section 2.1]):

(a) (5.1) and (5.2);

(b) (5.1) and (5.3).

$$\mu(\mu(u, v, w), x, y) = \mu(u, \mu(x, w, v), y) = \mu(u, v, \mu(w, x, y)) \quad (\forall u, v, w, x, y \in M). \tag{5.3}$$

In fact, (5.1) and (5.2) induce (5.3), because

$$\begin{aligned}
\mu(u, v, \mu(w, x, y)) &= \mu(u, v, \mu(w, x, \mu(\mu(x, w, v), \mu(x, w, v), y))) \\
&= \mu(u, v, \mu(\mu(w, x, \mu(x, w, v)), \mu(x, w, v), y)) \\
&= \mu(u, v, \mu(v, \mu(x, w, v), y)) \\
&= \mu(\mu(u, v, v), \mu(x, w, v), y) \\
&= \mu(u, \mu(x, w, v), y).
\end{aligned}$$

Thus, a pair (M, μ) satisfying (5.1) and (5.3) is exactly a torsor.

(3) The torsor (M, μ) is a principal homogeneous space [11, Section 4.2]. Let $\mu(a, b)$ $(a, b \in M)$ denote the map from M to itself defined by $\mu(a, b)(c) = \mu(a, b, c)$ $(c \in M)$. The set $G = \{\mu(a, b); a, b \in M\}$ is a subgroup of $\mathrm{Aut}(M)$, which makes M a G-principal homogeneous space. Conversely, the principal homogeneous space gives birth to a torsor.

Each group G produces a torsor. Define the ternary operation μ_G on G by

$$\mu_G(a, b, c) = ab^{-1}c \quad (a, b, c \in G). \tag{5.4}$$

The pair (G, μ) is a torsor.

Remark 5.3. Every torsor (M, μ) is isomorphic to (5.4) [17, Section 1.6]. We first fix any element $e \in M$. The nonempty set M, together with the binary operation

$$M \times M \ni (a, b) \mapsto \mu(a, e, b) \in M,$$

is a group [9, Section 1]; in fact, the unit element is e, and the inverse of the element a is $\mu(e, a, e)$. This group M gives birth to the torsor (5.4), which is isomorphic to (M, μ).

Let $H = (H, \cdot_H)$ denote the object of the category $\mathbf{Set}_H$ in Example 3.3. Here, $\lambda \cdot_H \lambda' = \lambda'$ $(\lambda, \lambda' \in H)$. Suppose that an object X of $\mathbf{Set}_H$ satisfies (3.1). We define the map $i : H \to \mathrm{Map}(H, X)$ by

$$i(\lambda)(u) = \lambda \backslash u \quad (\lambda, u \in H).$$

Proposition 5.4. *The map i is an isomorphism of $\mathbf{Set}_H$ from H to X.*

In fact, its inverse is as follows.

$$i^{-1}(\lambda)(x) = \lambda x \quad (\lambda \in H, x \in X).$$

Let $\sigma : X \otimes X \to X \otimes X$ be a morphism of $\mathbf{Set}_H$. By virtue of (2.1) for the morphism $i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i : H \otimes H \to H \otimes H$,

Proposition 5.5. *The second component of $(i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i)(\lambda)(u, v)$ $(\lambda, u, v \in H)$ is v.*

We define the ternary operation μ on the set H by the first component of $(i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i)(\lambda)(u, v)$; that is,

$$(i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i)(\lambda)(u, v) = (\mu(\lambda, u, v), v) \quad (\lambda, u, v \in H).$$

Proposition 5.6. *(H, μ) is a torsor, if and only if (X, σ) is a dynamical braided monoid.*

Proof. We first observe (4.1) is equivalent to that

$$\mu(u, v, \mu(v, w, x)) = \mu(u, w, x) \quad (\forall u, v, w, x \in H). \tag{5.5}$$

On account of Proposition 5.4, the morphism σ satisfies (4.1), if and only if

$$\begin{aligned}&(\mathrm{id}_H \otimes (i^{-1} \circ m_X \circ i \otimes i)) \circ ((i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i) \otimes \mathrm{id}_H)\\ &\quad \circ(\mathrm{id}_H \otimes (i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i))\\ &= (i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i) \circ ((i^{-1} \circ m_X \circ i \otimes i) \otimes \mathrm{id}_H).\end{aligned} \tag{5.6}$$

Because $(i^{-1} \circ m_X \circ i \otimes i)(\lambda)(u, v) = v$ $(\lambda, u, v \in H)$, (5.5) and (5.6) are equivalent.

Similar argument implies to: (4.2) is equivalent to that

$$\mu(\mu(u, v, w), w, x) = \mu(u, v, x) \quad (\forall u, v, w, x \in H); \tag{5.7}$$

(4.3) is equivalent to that $\mu(v, v, u) = u$ $(\forall u, v \in H)$; and (4.4) is equivalent to that $\mu(u, v, v) = u$ $(\forall u, v \in H)$.

An easy computation shows that (5.2) is equivalent to (5.5) and (5.7), if μ satisfies (5.1); in fact, (5.5) and (5.7) induce (5.2), because

$$\mu(\mu(u, v, w), x, y) = \mu(\mu(u, v, w), w, \mu(w, x, y)) = \mu(u, v, \mu(w, x, y)).$$

Hence, (H, μ) is a torsor, if and only if (X, σ) is a dynamical braided monoid. □

Proof of Theorem 4.4. Let (X, σ) be a dynamical braided monoid satisfying (3.1). From (3.1) and Proposition 5.6, (H, μ) is a torsor. If (H, μ) is a torsor, then the morphism $(i^{-1} \otimes i^{-1}) \circ \sigma \circ (i \otimes i) : H \otimes H \to H \otimes H$ satisfies the braid relation (2.3), and so does the morphism σ. Thus, σ is a dynamical Yang-Baxter map (Definition 2.1). □

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References

[1] Baxter, R.J.: Partition function of the eight-vertex lattice model. Ann. Physics **70** (1972), 193–228; Exactly solved models in statistical mechanics. Academic Press, Inc., London, 1982.

[2] Chari, V., Pressley, A.: A guide to quantum groups. Cambridge University Press, Cambridge, 1994.

[3] Drinfel'd, V.G.: Quantum groups. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[4] Drinfel'd, V. G.: On some unsolved problems in quantum group theory. Quantum groups (Leningrad, 1990), 1–8, Lecture Notes in Math., 1510, Springer, Berlin, 1992.

[5] Etingof, P., Varchenko, A.: Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups. Comm. Math. Phys. **196** (1998), no. 3, 591–640.

[6] Grunspan, C.: Quantum torsors. J. Pure Appl. Algebra **184** (2003), no. 2-3, 229–255.

[7] Jimbo, M.: A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation. Lett. Math. Phys. **10** (1985), no. 1, 63–69.

[8] Jimbo, M.: Yang-Baxter equation in integrable systems. World Scientific, Singapore, 1990.

[9] Johnstone, P. T.: The 'closed subgroup theorem' for localic herds and pregroupoids. J. Pure Appl. Algebra **70** (1991), no. 1-2, 97–106.

[10] Kassel, C.: Quantum groups. Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.

[11] Kontsevich, M.: Operads and motives in deformation quantization. Lett. Math. Phys. **48** (1999), no. 1, 35–72.

[12] Lu, J.-H., Yan, M., Zhu, Y.-C.: On the set-theoretical Yang-Baxter equation. Duke Math. J. **104** (2000), no. 1, 1–18.

[13] Mac Lane, S.: Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998.

[14] Shibukawa, Y.: Dynamical Yang-Baxter maps. Int. Math. Res. Not. **2005**, no. 36, 2199–2221; Dynamical Yang-Baxter maps with an invariance condition. Publ. Res. Inst. Math. Sci. **43** (2007), no. 4, 1157–1182.

[15] Shibukawa, Y.: Survey on dynamical Yang-Baxter maps. Proceedings of noncommutative structures in mathematics and physics (Brussels, Belgium, 2008), 238–243, The Royal Flemish Academy of Belgium for Sciences and Arts, 2010.
http://homepages.vub.ac.be/~scaenepe/proceedingsnomap.htm

[16] Shibukawa, Y., Takeuchi, M.: FRT construction for dynamical Yang-Baxter maps. J. Algebra **323** (2010), 1698–1728.

[17] Škoda, Z.: Quantum heaps, cops and heapy categories. Math. Commun. **12** (2007), no. 1, 1–9.

[18] Takeuchi, M.: Matched pairs of groups and bismash products of Hopf algebras. Comm. Algebra **9** (1981), no. 8, 841–882.

[19] Takeuchi, M.: Survey on matched pairs of groups—an elementary approach to the ESS-LYZ theory. Noncommutative geometry and quantum groups (Warsaw, 2001), 305–331, Banach Center Publ., **61**, Polish Acad. Sci., Warsaw, 2003.

[20] Veselov, A.: Yang-Baxter maps: dynamical point of view. Combinatorial aspect of integrable systems, 145–167, MSJ Mem., 17, Math. Soc. Japan, Tokyo, 2007.

[21] Yang, C.N.: Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. Phys. Rev. Lett. **19** (1967), 1312–1315.